The Ruijsenaars self-duality map as a mapping class symplectomorphism

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Abstract

This is a brief review[1] of the main results of our paper arXiv:1101.1759 that contains a complete global treatment of the compactified trigonometric Ruijsenaars-Schneider system by quasi-Hamiltonian reduction. Confirming previous conjectures of Gorsky and collaborators, we have rigorously established the interpretation of the system in terms of flat $SU(n)$ connections on the one-holed torus and demonstrated that its self-duality symplectomorphism represents the natural action of the standard mapping class generator S on the phase space. The pertinent quasi-Hamiltonian reduced phase space turned out to be symplectomorphic to the complex projective space equipped with a multiple of the Fubini-Study symplectic form and two toric moment maps playing the roles of particle-positions and action-variables that are exchanged by the duality map. Open problems and possible directions for future work are also discussed.

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1 Introduction

In his study of action-angle maps, Ruijsenaars [12] discovered an intriguing duality relation for both non-relativistic and relativistic Calogero type classical many-body systems associated to A_n root systems and rational, hyperbolic or trigonometric interaction potentials. In this paper our concern is a particular system of that kind, locally given by the trigonometric Hamiltonian (25) later on, which was invented and proved to be self-dual in [13]. Our principal goal is to give a self-contained but concise presentation of the main results of our detailed work [4], where we showed that the global variant of this system (called compactified trigonometric Ruijsenaars-Schneider III_b system) and its self-duality can be naturally understood by means of quasi-Hamiltonian reduction. This connects the system to the $SU(n)$ Chern-Simons theory on the one-holed torus, with a special boundary condition, and traces back its self-duality symplectomorphism to the standard duality generator $S \in SL(2,\mathbb{Z})$ of the mapping class group of the one-holed torus. Our results thus provide rigorous justification of conjectures put forward over a decade ago by Gorsky and his collaborators [8, 6] about the III_b system.

The plan of this contribution is as follows. In Section 2 we start with the definition of the concept of "Ruijsenaars duality". In particular, we shall discuss two alternative, equivalent definitions of self-duality. Necessary background information from quasi-Hamiltonian geometry is summarized next in Subsection 3.1, focusing on the example of the internally fused double that will be used subsequently. Then in Subsection 3.2 we explain how the mapping class group $SL(2,\mathbb{Z})$ acts on every reduced phase space arising from the double. Section 4 is devoted to expounding the definition of the compactified III_b system. The main results of [4] are presented in Section 5. The content of Section 5 and related further results are discussed in Section 6 together with an exposition of open problems.

2 The concept of Ruijsenaars duality

This concept is relevant for classical integrable many-body systems of "particles" moving in 1-dimension. Due to their physical interpretation and Liouville integrability, these systems possess "particle-positions" and "action-variables" that span two Abelian subalgebras in the Poisson algebra of observables. By definition, two such systems are in duality if there exists a symplectomorphism between their phase spaces that converts the particle-positions of system (i) into the action-variables of system (ii) and converts the action-variables of system (i) into the particle-positions of system (ii). In particular, one speaks of self-duality if the leading Hamiltonians of both systems (which underlie the many-body interpretation) have the same form. An alternative second definition of self-duality is to consider a single integrable many-body Hamiltonian system (M,Ω,H), and call it self-dual if there exists a symplectomorphism $\mathfrak{S}$ of the phase space (M,Ω) that converts the particle-positions into the action-variables and the action-variables into the particle-positions. Notice that the second definition is a special case of the first definition where the two systems in duality are two copies of the same system and their duality relation is provided by $\mathfrak{S}$.

If not clear from the context, we propose the full name of the above duality be "Ruijsenaars duality" or "duality in the sense of Ruijsenaars" (also known as action-angle duality).

Let us further discuss the relation between the above two definitions of (Ruijsenaars) self-duality. To do this, denote by $\mathcal{J}_k$ and $\mathcal{I}_k$ $(k = 1, ..., N)$ the particle-positions and action-variables for the system (M, Ω, H). It is required that there exists a dense open submanifold $M^{\rm loc} \subseteq M$ where the symplectic form Ω is equal to $\Omega^{\rm loc} = \sum_{k=1}^{N} d\theta_k \wedge d\mathcal{J}_k$, with conjugates θ_k of the $\mathcal{J}_k$. We can view $(\mathcal{J}, \theta)$ and $\mathcal{I}$ as maps from $M^{\rm loc}$ into $\mathbb{R}^{2N}$ and $\mathbb{R}^N$, and then have

$$H^{\rm loc} = \mathcal{H} \circ (\mathcal{J}, \theta) = h \circ \mathcal{I} \tag{1}$$

with some functions $\mathcal{H}$ and h, where the form of $\mathcal{H}$ underlies the many-body interpretation. Any global symplectomorphism $\mathfrak{S}$ takes H into the integrable Hamiltonian $\tilde{H} := H \circ \mathfrak{S}$. One has the relations $\Omega^{\rm loc} = \sum_{k=1}^{N} d\tilde{\theta}_k \wedge d\tilde{\mathcal{J}}_k$ and

$$\tilde{H}^{\rm loc} = H^{\rm loc} \circ \mathfrak{S} = \mathcal{H} \circ (\tilde{\mathcal{J}}, \tilde{\theta}) = h \circ \tilde{\mathcal{I}} \tag{2}$$

with $(\tilde{\mathcal{J}}, \tilde{\theta}) := (\mathcal{J}, \theta) \circ \mathfrak{S}$ and $\tilde{\mathcal{I}} := \mathcal{I} \circ \mathfrak{S}$. Thus $\tilde{H}^{\rm loc}$ has the same form in terms of the tilded-variables as $H^{\rm loc}$ in terms of the tilde-free variables. Now observe that the system (M, Ω, H) is in duality with $(M, \Omega, \tilde{H})$ if $\tilde{\mathcal{J}}$ is the same as $\mathcal{I}$ and $\tilde{\mathcal{I}}$ is the same as $\mathcal{J}$. Spelling this out in more detail: if (M, Ω, H) is self-dual in the sense of the second definition, then its dual pair $(M, \Omega, \tilde{H})$ is automatically manufactured and these two systems are in duality with respect to the identity map[2] on M. The full equivalence of our alternative definitions of self-duality is also not difficult to prove. In this paper we adopt the second definition.

To be precise, we note that in the statement "is the same as" above one must admit some some sign change or re-labeling of the indices of the variables. In fact, the *self-duality symplectomorphism* $\mathfrak{S}$ is usually not an involution but has order 4. As an illustration, consider the free system with Hamiltonian $H = p^2$ on the phase space $\mathbb{R}^2 = \{(q, p)\}$, whose particle-position and action-variable are q and p, respectively. The free system is trivially self-dual with self-duality symplectomorphism $\mathfrak{S} : (q, p) \mapsto (p, -q)$, and dual Hamiltonian $\tilde{H} = q^2$.

Ruijsenaars [12, 13] actually found three distinct dual pairs of systems and three self-dual systems. For example, the dual of the hyperbolic Sutherland system is the rational Ruijsenaars-Schneider system, and the rational Calogero system is self-dual. See the review [14] for the other cases. Incidentally, at the quantum mechanical level, all these systems are known to enjoy the related bispectral property [2], too. As was already mentioned, in this paper our concern will be the self-dual $\mathrm{III_b}$ system. For a detailed geometric treatment of a very different, not self-dual, case of the trigonometric Ruijsenaars duality, the reader may consult [3].

3 Generalities about the internally fused double D

The basic reference for Subsection 3.1 is [1]. The mapping class group action presented in Subsection 3.2 is also well-known to experts [1, 7, 9]; in its explicit description we follow [4].

[2] In general, identifying the phase spaces of any dual pair by the symplectomorphism that appears in the definition of the duality relation given at the beginning, one may always turn this symplectomorphism into the identity map. Thus the phase spaces of the systems in duality become models of a single phase space, (not accidentally) similar to two gauge slices serving as models of the single space of gauge orbits in a gauge theory.

3.1 Quasi-Hamiltonian systems on D and their reductions

Let G be a (connected and simply connected) compact Lie group and fix a positive definite invariant scalar product $\langle\ ,\ \rangle$ on its Lie algebra $\mathcal{G}$. Equip the Cartesian product

$$D := G \times G = \{(A, B) \,|\, A, B \in G\} \tag{3}$$

with the 2-form ω,

$$2\omega := \langle A^{-1}dA \stackrel{\wedge}{,} dBB^{-1}\rangle + \langle dAA^{-1} \stackrel{\wedge}{,} B^{-1}dB\rangle - \langle (AB)^{-1}d(AB) \stackrel{\wedge}{,} (BA)^{-1}d(BA)\rangle, \tag{4}$$

which is invariant under the G-action Ψ on D defined by

$$\Psi_g : (A, B) \mapsto (gAg^{-1}, gBg^{-1}), \qquad \forall g \in G. \tag{5}$$

Introduce the G-equivariant map $\mu : D \to G$ by the group commutator

$$\mu(A, B) := ABA^{-1}B^{-1}. \tag{6}$$

These data satisfy

$$d\omega = -\frac{1}{12}\mu^*\langle\vartheta, [\vartheta, \vartheta]\rangle, \quad \omega(\zeta_D, \cdot) = \frac{1}{2}\mu^*\langle\vartheta + \bar{\vartheta}, \zeta\rangle, \quad \forall\zeta \in \mathcal{G}, \tag{7}$$

$$\mathrm{Ker}(\omega_x) = \{\zeta_D(x) \,|\, \zeta \in \mathrm{Ker}(\mathrm{Ad}_{\mu(x)} + \mathrm{Id}_{\mathcal{G}})\}, \qquad \forall x \in D, \tag{8}$$

where ϑ and $\bar{\vartheta}$ denote, respectively, the $\mathcal{G}$-valued left- and right-invariant Maurer-Cartan forms on G and ζ_D generates the infinitesimal action of $\zeta \in \mathcal{G}$ on D. All this means [1] that (D, ω, μ) is a so-called quasi-Hamiltonian G-space with moment map μ. This quasi-Hamiltonian G-space is nicknamed the internally fused double of G.

According to the general theory [1], every G-invariant function $h \in C^\infty(D)^G$ induces a unique vector field v_h on D by requiring that $\omega(v_h, \cdot) = dh$ and $\mathcal{L}_{v_h}\mu = 0$. The vector field v_h is G-invariant and its flow preserves ω. In this way, (D, ω, μ, h) yields a quasi-Hamiltonian dynamical system. Although (D, ω) is not a symplectic manifold, one can also introduce an honest Poisson bracket on $C^\infty(D)^G$. Naturally, for G-invariant functions f and h the Poisson bracket is furnished by

$$\{f, h\} := \omega(v_f, v_h). \tag{9}$$

Generally speaking, quasi-Hamiltonian systems are of interest since they can be reduced to true Hamiltonian systems by a generalization of the Marsden-Weinstein symplectic reduction, and this can give convenient realizations of important Hamiltonian systems. To specialize to our case, let us choose a moment map value $\mu_0 \in G$ and denote its stabilizer with respect to the adjoint action by G_0. Then consider the space of G_0-orbits

$$P(\mu_0) := \mu^{-1}(\mu_0)/G_0, \tag{10}$$

where $\mu^{-1}(\mu_0) := \{x \in D \,|\, \mu(x) = \mu_0\}$. Denote by $\iota : \mu^{-1}(\mu_0) \to D$ the tautological injection and $p : \mu^{-1}(\mu_0) \to P(\mu_0)$ the obvious projection. Under favourable circumstances (where the meaning of "favourable" is the same as for usual symplectic reduction), there exists a standard

Hamiltonian system $(P(\mu_0), \hat{\omega}, \hat{h})$ such that the *symplectic* form $\hat{\omega}$ and the reduced Hamiltonian $\hat{h}$ satisfy the relations

$$p^*\hat{\omega} = \iota^*\omega, \quad p^*\hat{h} = \iota^* h. \tag{11}$$

The Hamiltonian vector field and the flow defined by $\hat{h}$ on $P(\mu_0)$ can be obtained by first restricting the quasi-Hamiltonian vector field v_h and its flow to $\mu^{-1}(\mu_0)$ and then applying the projection p. The Poisson brackets on $(P(\mu_0), \hat{\omega})$ are inherited from the Poisson brackets (9) of the G-invariant functions like in usual symplectic reduction.

Of course, the space of orbits $P(\mu_0)$ is not a smooth manifold in general. However, it always turns out to be a stratified symplectic space [9], which means that it is a disjoint union of symplectic manifolds of various dimensions glued together (in a specific manner).

The symplectic spaces obtained from quasi-Hamiltonian reduction always arise also from usual symplectic reduction of certain infinite-dimensional manifolds with respect to infinite-dimensional symmetry groups [1]. In particular, let Σ denote the torus with a hole (that is, with an open disc removed); often called the "one-holed torus". It is known that the moduli space (space of gauge equivalence classes) of flat principal G-connections on Σ whose holonomy along the boundary of the hole is constrained to the conjugacy class of μ_0 is a stratified symplectic space, which can be canonically identified with the quasi-Hamiltonian reduced phase space $P(\mu_0)$ in (10). It is also worth noting that this space supports two natural *Abelian Poisson algebras*. Namely, for any $\mathcal{H} \in C^\infty(G)^G$ let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the G-invariant functions on D given by

$$\mathcal{H}_1(A, B) := \mathcal{H}(A) \quad \text{and} \quad \mathcal{H}_2(A, B) := \mathcal{H}(B). \tag{12}$$

The two Abelian Poisson algebras on $P(\mu_0)$ are provided by

$$\mathcal{C}^a := \{\hat{\mathcal{H}}_1 \,|\, \mathcal{H} \in C^\infty(G)^G\}, \quad \mathcal{C}^b := \{\hat{\mathcal{H}}_2 \,|\, \mathcal{H} \in C^\infty(G)^G\}. \tag{13}$$

Note also that D itself can be identified as the space of flat connections on Σ modulo the "based gauge transformations" defined by maps $\eta \in C^\infty(\Sigma, G)$ for which $\eta(p_0) = e$ for a fixed point p_0 on the boundary of the removed disc. The matrices A and B represent the holonomies of the flat connections along the standard generators of the fundamental group $\pi_1(\Sigma, p_0)$.

3.2 Symplectic action of the mapping class group on $P(\mu_0)$

Let us consider the (orientation-preserving) mapping class group of the one-holed torus,

$$\mathrm{MCG}^+(\Sigma) \equiv \pi_0(\mathrm{Diff}^+(\Sigma)), \tag{14}$$

whose elements are equivalence classes of orientation-preserving diffeomorphisms up to homotopy. It is known that the mapping class groups acts by structure preserving smooth maps on every reduced phase space $P(\mu_0)$ (10), where "structure preserving" means symplectomorphism whenever $P(\mu_0)$ is a smooth manifold. The origin of the mapping class group action is especially clear in the setting of flat connections, where it arises from the pull-back of the connection 1-forms by diffeomorphisms. However, it is also possible to directly describe the mapping class group action on $P(\mu_0)$ by taking advantage of the quasi-Hamiltonian formalism.

For the one-holed torus there exists a (geometrically engendered) isomorphism

$$\mathrm{MCG}^+(\Sigma) \simeq SL(2, \mathbb{Z}). \tag{15}$$

The infinite discrete group $SL(2,\mathbb{Z})$ is generated by two elements S and T subject to the relations

$$S^2 = (ST)^3, \quad S^4 = 1. \tag{16}$$

As concrete matrices, one may take

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{17}$$

which actually represent the action of corresponding mapping classes on the standard basis of the homology group $H_1(\Sigma;\mathbb{Z}) \simeq \mathbb{Z}^2$. The mapping class of T is known as a Dehn twist and that of S as the standard orientation-preserving duality generator "exchanging" the standard homology cycles. By arguments detailed in [4, 7], it is natural to associate to S and T the following diffeomorphisms S_D and T_D of the double:

$$S_D(A,B) := (B^{-1}, BAB^{-1}), \qquad T_D(A,B) := (AB, B). \tag{18}$$

It is not difficult to check that

$$S_D^*\omega = \omega, \quad S_D \circ \Psi_g = \Psi_g \circ S_D, \quad \mu \circ S_D = \mu, \tag{19}$$

and similar relations hold for T_D as well, i.e., both S_D and T_D are automorphisms of the internally fused double. Moreover, one finds that S_D and T_D satisfy

$$S_D^2 = (S_D \circ T_D)^3, \quad S_D^4 = Q, \tag{20}$$

where Q is the central element of the group of automorphisms of the double given by

$$Q(A,B) = \Psi_{\mu(A,B)^{-1}}(A,B). \tag{21}$$

It is an immediate consequence of the above relations that S_D and T_D descend to maps S_P and T_P on any reduced phase space $P(\mu_0)$ (10), and these maps generate an $SL(2,\mathbb{Z})$ action on $P(\mu_0)$. Indeed, Q descends to the trivial identity map id_P on $P(\mu_0)$, and thus (20) implies the identities

$$S_P^2 = (S_P \circ T_P)^3, \quad S_P^4 = \mathrm{id}_P. \tag{22}$$

The resulting $SL(2,\mathbb{Z})$ action preserves the (stratified) symplectic structure on $P(\mu_0)$.

Finally, consider the action of S_P on the two Abelian Poisson algebras $\mathcal{C}^a$ and $\mathcal{C}^b$ displayed in (13). For any $\mathcal{H} \in C^\infty(G)^G$, define $\mathcal{H}^\sharp \in C^\infty(G)^G$ by

$$\mathcal{H}^\sharp(g) := \mathcal{H}(g^{-1}). \tag{23}$$

Then the following identities hold:

$$\hat{\mathcal{H}}_2 \circ S_P = \hat{\mathcal{H}}_1 \quad \text{and} \quad \hat{\mathcal{H}}_1 \circ S_P = \hat{\mathcal{H}}_2^\sharp, \quad \forall \mathcal{H} \in C^\infty(G)^G. \tag{24}$$

In this way, S_P exchanges the elements $\hat{\mathcal{H}}_2$ of $\mathcal{C}^b$ with the elements $\hat{\mathcal{H}}_1$ of $\mathcal{C}^a$.

4 Compactified Ruijsenaars-Schneider $\mathrm{III_b}$ system

In [13] Ruijsenaars studied, among others, a particular real form of the complex trigonometric Ruijsenaars-Schneider system whose Hamiltonian exhibits periodic dependence both on the particle-positions and on the conjugate momenta. This system is termed the $\mathrm{III_b}$ system, where the label "b" indicates the bounded nature of the underlying phase space. The $\mathrm{III_b}$ Hamiltonian given by (25) below is formally integrable since it admits the sufficient number of constants of motion in involution. However, true integrability holds only after compactifying the local phase space, whereby the Hamiltonian flows become complete. Here, we first summarize the definition of the local $\mathrm{III_b}$ system and then present its compactification. Although the content of this section can be found in [13], too, for the sake of readability we display all definitions in a self-contained manner.

The many-body interpretation of the $\mathrm{III_b}$ system is based on the Hamiltonian

$$H_y^{\mathrm{loc}}(\delta,\Theta) \equiv \sum_{j=1}^{n} \cos p_j \prod_{k\neq j}^{n} \left[1 - \frac{\sin^2 y}{\sin^2(x_j - x_k)}\right]^{\frac{1}{2}}, \tag{25}$$

where $\delta_j = e^{\mathrm{i}2x_j}$ $(j = 1, ..., n)$ are interpreted as the positions of n "particles" moving on the circle and the canonically conjugate momenta p_j encode the compact variables $\Theta_j = e^{-\mathrm{i}p_j}$; the index k in the product runs over $\{1, 2, ..., n\} \setminus \{j\}$. To guarantee the reality of H_y^{loc} on a non-trivial connected open domain, one must have $|y| < |x_j - x_k| < \pi - |y|$ for all $j \neq k$, and consistency then requires the real coupling constant $y \neq 0$ to satisfy

$$0 < |y| < \pi/n. \tag{26}$$

We impose the center of mass condition $\prod_{j=1}^{n} \delta_j = \prod_{j=1}^{n} \Theta_j = 1$, and parametrize the variables so that the local phase space of the system gets identified with

$$M_y^{\mathrm{loc}} \equiv \mathcal{P}_y^0 \times \mathbb{T}_{n-1}, \tag{27}$$

where $\mathbb{T}_{n-1}$ is the $(n-1)$-torus and $\mathcal{P}_y^0$ is the interior of the polytope

$$\mathcal{P}_y := \Big\{(\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1} \,\Big|\, \xi_j \geq |y|,\ \ j = 1, ..., n-1,\ \ \sum_{j=1}^{n-1} \xi_j \leq \pi - |y|\Big\}. \tag{28}$$

Using the $n \times n$ matrix $E_{j,j}$ having 1 in the jj position and the identity matrix $\mathbf{1}_n$, we introduce

$$H_k := E_{k,k} - E_{k+1,k+1}, \quad \lambda_k := \sum_{j=1}^{k} E_{j,j} - \frac{k}{n}\mathbf{1}_n, \quad k = 1, ..., n-1. \tag{29}$$

Then, for $\xi \in \mathcal{P}_y^0$ and $\tau = (\tau_1, ..., \tau_{n-1}) = (e^{\mathrm{i}\theta_1}, ..., e^{\mathrm{i}\theta_{n-1}}) \in \mathbb{T}_{n-1}$, we define the diagonal $SU(n)$ matrices

$$\delta(\xi) := \exp\big(-2\mathrm{i}\sum_{k=1}^{n-1} \xi_k \lambda_k\big), \quad \Theta(\tau) := \exp\big(-\mathrm{i}\sum_{k=1}^{n-1} \theta_k H_k\big). \tag{30}$$

The choice of $\mathcal{P}_y^0$ as the domain of the particle-positions ξ guarantees the positivity of the expressions under the square root in (25). In terms of the variables $(\xi,\tau)\in\mathcal{P}_y^0\times\mathbb{T}_{n-1}$, the symplectic form of the system reads

$$\Omega^{\mathrm{loc}}:=\frac{1}{2}\mathrm{tr}\left(\delta^{-1}d\delta\wedge\Theta^{-1}d\Theta\right)=\mathrm{i}\sum_{k=1}^{n-1}d\xi_k\wedge\tau_k^{-1}d\tau_k=\sum_{k=1}^{n-1}d\theta_k\wedge d\xi_k. \tag{31}$$

Note that for any diagonal matrix $\mathcal{D}$ (like δ,Θ etc), we apply the notation $\mathcal{D}=\mathrm{diag}(\mathcal{D}_1,...,\mathcal{D}_n)$.

The Hamiltonian (25) admits $(n-1)$ Poisson commuting constants of motion given by independent spectral invariants of the following $SU(n)$-valued local Lax matrix:

$$L^y_{\mathrm{loc}}(\xi,\tau)_{jl}:=\frac{e^{\mathrm{i}y}-e^{-\mathrm{i}y}}{e^{\mathrm{i}y}\delta_j(\xi)\delta_l(\xi)^{-1}-e^{-\mathrm{i}y}}W_j(\xi,y)W_l(\xi,-y)\Theta_l(\tau)\Delta_l(\tau)\Delta_j(\tau)^{-1}. \tag{32}$$

Here we use the positive functions

$$W_j(\xi,y):=\prod_{k\neq j}^{n}\left[\frac{e^{\mathrm{i}y}\delta_j(\xi)-e^{-\mathrm{i}y}\delta_k(\xi)}{\delta_j(\xi)-\delta_k(\xi)}\right]^{\frac{1}{2}}, \tag{33}$$

and $\Delta(\tau):=\mathrm{diag}(\tau_1,...,\tau_{n-1},1)$. The Hamiltonian (25) is recovered from the local Lax matrix as the real part of the trace

$$H_y^{\mathrm{loc}}(\delta(\xi),\Theta(\tau))=\mathrm{Re}\,\mathrm{tr}\left(L^y_{\mathrm{loc}}(\xi,\tau)\right). \tag{34}$$

Ruijsenaars [13] realized that the flows of H_y^{loc} and of its commuting family are not complete on M_y^{loc} , and then completed the local phase space in the way described below.

Let us consider the symplectic manifold $(\mathbb{C}P(n-1),\chi_0\omega_{\mathrm{FS}})$, where

$$\chi_0:=\pi-n|y|, \tag{35}$$

and ω_{FS} is the standard Fubini-Study symplectic form. It is convenient to identify the complex projective space $\mathbb{C}P(n-1)$ as the factor space $S^{2n-1}_{\chi_0}/U(1)$ with

$$S^{2n-1}_{\chi_0}=\left\{(u_1,...,u_n)\in\mathbb{C}^n\mid\sum_{k=1}^{n}|u_k|^2=\chi_0\right\}. \tag{36}$$

Let $\mathbb{C}P(n-1)_0$ be the open dense submanifold of $\mathbb{C}P(n-1)$ where none of the homogeneous coordinates can vanish. By utilizing the canonical projection $\pi_{\chi_0}:S^{2n-1}_{\chi_0}\to\mathbb{C}P(n-1)$, we define a diffeomorphism $\mathcal{E}:M_y^{\mathrm{loc}}\to\mathbb{C}P(n-1)_0$ by the formula

$$\mathcal{E}(\xi,\tau):=\pi_{\chi_0}(\tau_1\sqrt{\xi_1-|y|},...,\tau_{n-1}\sqrt{\xi_{n-1}-|y|},\sqrt{\xi_n-|y|}) \tag{37}$$

with $\xi_n:=\pi-\sum_{k=1}^{n-1}\xi_k$. By using that $\pi^*_{\chi_0}(\chi_0\omega_{\mathrm{FS}})=\mathrm{i}\sum_{k=1}^{n}d\bar{u}_k\wedge du_k$, one sees that $\mathcal{E}$ is a symplectomorphism

$$\mathcal{E}^*(\chi_0\omega_{\mathrm{FS}})=\Omega^{\mathrm{loc}}. \tag{38}$$

Thus we can identify $(M_y^{\mathrm{loc}},\Omega^{\mathrm{loc}})$ with the dense open submanifold $\mathbb{C}P(n-1)_0$ of the compact phase space $(\mathbb{C}P(n-1),\chi_0\omega_{\mathrm{FS}})$. The crucial fact is that, by means of this identification, the

local Lax matrix $L^y_{\rm loc}$ extends to a smooth (even real-analytic) matrix function on $\mathbb{C}P(n-1)$. This fact is actually not difficult to verify [13, 4]. From now on we denote the resulting "global Lax matrix" as L^y. Since $L^y \in C^\infty(\mathbb{C}P(n-1), SU(n))$ satisfies

$$L^y \circ \mathcal{E} = L^y_{\rm loc}, \tag{39}$$

it follows that all the smooth spectral invariants of $L^y_{\rm loc}$ (like the Hamiltonian (34)) extend to smooth functions on the compactified phase space $\mathbb{C}P(n-1)$. The corresponding Hamiltonian flows are automatically complete on $\mathbb{C}P(n-1)$, simply since every smooth vector field has complete flows on a compact manifold. By definition, the compactified $\mathrm{III_b}$ system is the integrable system on the phase space $(\mathbb{C}P(n-1), \chi_0\omega_{\rm FS})$ whose commuting Hamiltonians are generated by the Lax matrix L^y.

5 Self-duality of the $\mathrm{III_b}$ system from reduction

The compactified $\mathrm{III_b}$ system, encapsulated by the triple

$$(\mathbb{C}P(n-1), \chi_0\omega_{\rm FS}, L^y), \tag{40}$$

possesses two distinguished Abelian Poisson algebras of observables. The first Abelian algebra is generated by the "global particle-position variables" $\mathcal{J}_k$ defined by

$$\mathcal{J}_k \circ \pi_{\chi_0}(u) = |u_k|^2 + |y|, \qquad k = 1, ..., n-1. \tag{41}$$

The terminology is justified by the identity $\mathcal{J}_k(\mathcal{E}(\xi, \tau)) = \xi_k$. The $\mathcal{J}_k$ are the components of the toric moment map

$$\mathcal{J} := (\mathcal{J}_1, ..., \mathcal{J}_{n-1}) : \mathbb{C}P(n-1) \to \mathbb{R}^{n-1} \tag{42}$$

that generates the so-called rotational action of the torus $\mathbb{T}_{n-1}$ on $(\mathbb{C}P(n-1), \chi_0\omega_{\rm FS})$. Its image is the closed polytope $\mathcal{P}_y$ (28). The other distinguished Abelian algebra is spanned by the action-variables furnished by certain spectral functions of the global Lax matrix L^y.

In the rest of this section we take

$$G := SU(n), \qquad \langle X, Y\rangle := -\frac{1}{2}\mathrm{tr}\,(XY), \quad \forall X, Y \in \mathcal{G}. \tag{43}$$

Define the polytope $\mathcal{P}_0$ similarly to (28) and also define $\delta(\xi)$ like in (30) for any $\xi \in \mathcal{P}_0$. It is well-known that any $g \in G$ is conjugate to a matrix $\delta(\xi)$ for a unique $\xi \in \mathcal{P}_0$, and g is regular (has n distinct eigenvalues) if and only if the corresponding ξ belongs to the interior $\mathcal{P}_0^0$ of $\mathcal{P}_0$. Therefore we can uniquely define a G-invariant (i.e. conjugation invariant) function Ξ_k on G by requiring that

$$\Xi_k(\delta(\xi)) = \xi_k, \quad \forall \xi \in \mathcal{P}_0, \quad k = 1, ..., n-1. \tag{44}$$

The "spectral function" Ξ_k is continuous on G and its restriction to the dense open submanifold of regular elements, $G_{\rm reg}$, belongs to $C^\infty(G_{\rm reg})^G$.

It was shown in [13], and follows readily from our Theorem 1 given below, that the global Lax matrix L^y takes values in $G_{\rm reg}$ and the functions

$$\mathcal{I}_k := \Xi_k \circ L^y \tag{45}$$

can serve as action-variables of the compactified $\mathrm{III}_{\mathrm{b}}$ system. In fact, these functions Poisson commute and their Hamiltonian flows are 2π-periodic. The image of the toric moment map

$$\mathcal{I} := (\mathcal{I}_1, ..., \mathcal{I}_{n-1}) : \mathbb{C}P(n-1) \to \mathbb{R}^{n-1} \tag{46}$$

is the same polytope $\mathcal{P}_y$ as the image of moment map $\mathcal{J}$.

One can check that the spectral functions satisfy

$$\Xi_k^\sharp = \Xi_{n-k}, \tag{47}$$

where we applied the definition (23). Thus, if we define the spectral Hamiltonians α_k and β_k on D by

$$\alpha_k(A,B) := \Xi_k(A) \quad \text{and} \quad \beta_k(A,B) := \Xi_k(B), \tag{48}$$

then (18) implies the identities $\beta_k \circ S_D = \alpha_k$ and $\alpha_k \circ S_D = \beta_{n-k}$. Although they are not globally C^∞, α_k and β_k descend to "reduced spectral Hamiltonians" $\hat{\alpha}_k$ and $\hat{\beta}_k$ on any reduced phase space $P(\mu_0)$ obtained from the double. As special cases of (24), with the $SL(2,\mathbb{Z})$ generator S_P they satisfy

$$\hat{\beta}_k \circ S_P = \hat{\alpha}_k \quad \text{and} \quad \hat{\alpha}_k \circ S_P = \hat{\beta}_{n-k}, \quad \forall k = 1, ..., n-1. \tag{49}$$

Having the necessary preliminaries at hand, the principal result of our paper [4] can be summarized as follows.

Theorem 1. *For the particular moment map value*

$$\mu_0 = \mathrm{diag}(e^{2\mathrm{i}y}, ..., e^{2\mathrm{i}y}, e^{2(1-n)\mathrm{i}y}), \qquad 0 < |y| < \pi/n, \tag{50}$$

the "constraint surface" $\mu^{-1}(\mu_0)$ lies in $G_{\mathrm{reg}} \times G_{\mathrm{reg}}$ and the reduced phase space $(P(\mu_0), \hat{\omega})$ is a smooth manifold symplectomorphic to $(\mathbb{C}P(n-1), \chi_0\omega_{\mathrm{FS}})$. The maps

$$\hat{\alpha} := (\hat{\alpha}_1, ..., \hat{\alpha}_{n-1}) : P(\mu_0) \to \mathbb{R}^{n-1} \quad \textit{and} \quad \hat{\beta} : (\hat{\beta}_1, ..., \hat{\beta}_{n-1}) : P(\mu_0) \to \mathbb{R}^{n-1} \tag{51}$$

are toric moment maps generating two effective Hamiltonian actions of $\mathbb{T}_{n-1}$ on $(P(\mu_0), \hat{\omega})$. The images of both $\hat{\alpha}$ and $\hat{\beta}$ yield the polytope $\mathcal{P}_y$ (28), and there exists a symplectomorphism

$$f_\beta : \mathbb{C}P(n-1) \to P(\mu_0) \tag{52}$$

that satisfies

$$\hat{\beta}_k \circ f_\beta = \mathcal{J}_k \quad \textit{and} \quad \hat{\alpha}_k \circ f_\beta = \mathcal{I}_k, \qquad \forall k = 1, ..., n-1. \tag{53}$$

Combining Theorem 1 with the generalities reviewed in Subsection 3.2, we obtain the following important result.

Corollary 1. *The symplectomorphisms $f_\beta^{-1} \circ S_P \circ f_\beta$ and $f_\beta^{-1} \circ T_P \circ f_\beta$ generate an $SL(2,\mathbb{Z})$ action on the compactified $\mathrm{III}_{\mathrm{b}}$ phase space $(\mathbb{C}P(n-1), \chi_0\omega_{\mathrm{FS}})$. The mapping class duality symplectomorphism*

$$\mathfrak{S} := f_\beta^{-1} \circ S_P \circ f_\beta \tag{54}$$

acts by exchanging the particle-positions $\mathcal{J}_k$ with the action-variables $\mathcal{I}_k$ according to

$$\mathcal{J}_k \circ \mathfrak{S} = \mathcal{I}_k, \quad \textit{and} \quad \mathcal{I}_k \circ \mathfrak{S} = \mathcal{J}_{n-k}, \qquad \forall k = 1, ..., n-1. \tag{55}$$

For the sake of completeness, let us also present the explicit formula of our map f_β. For this, we introduce a unitary matrix $g_y(\xi)$ for each $\xi \in \mathcal{P}_y^0$ by

$$\begin{aligned} &g_y(\xi)_{jn} := -g_y(\xi)_{nj} := v_j(\xi, y), \quad \forall j = 1, ..., n-1, \quad g_y(\xi)_{nn} := v_n(\xi, y), \\ &g_y(\xi)_{jl} := \delta_{jl} - \frac{v_j(\xi, y) v_l(\xi, y)}{1 + v_n(\xi, y)}, \quad \forall j, l = 1, ..., n-1, \end{aligned} \tag{56}$$

where $v_j(\xi, y) := \left[\frac{\sin y}{\sin ny}\right]^{\frac{1}{2}} W_j(\xi, y)$ using (33).

Theorem 2. *Applying the previous notations, the map $f_0 : \mathbb{C}P(n-1)_0 \to P(\mu_0)$ defined by*

$$(f_0 \circ \mathcal{E})(\xi, \tau) := p\left(g_y(\xi)^{-1} \Delta(\tau) L^y_{\mathrm{loc}}(\xi, \tau) \Delta(\tau)^{-1} g_y(\xi), g_y(\xi)^{-1} \delta(\xi) g_y(\xi)\right) \tag{57}$$

is a diffeomorphism from $\mathbb{C}P(n-1)_0$ onto a dense open submanifold of $P(\mu_0)$. This map is symplectic, $f_0^* \hat{\omega} = \chi_0 \omega_{\mathrm{FS}}$, and it extends to a global diffeomorphism $f_\beta : \mathbb{C}P(n-1) \to P(\mu_0)$.

The map f_β that extends f_0 automatically has the properties mentioned in the Theorem above. The statement that f_0 is symplectic and that it extends to a global diffeomorphism were quite non-trivial to prove. In [4][3] the extended map f_β was also given explicitly by making use of a covering of $\mathbb{C}P(n-1)$ by n coordinate patches and giving f_β explicitly on each patch.

To conclude this section, we remind that an integrable many-body system is self-dual in the sense of Ruijsenaars if there exists a symplectomorphism that exchanges its particle-position variables with the action-variables. Hence the message of equation (55) is that *our mapping class symplectomorphism $\mathfrak{S}$ (54) qualifies as a self-duality symplectomorphism in the sense of Ruijsenaars.* In fact, we have also checked that $\mathfrak{S}$ coincides precisely with the self-duality symplectomorphism of the III_b system constructed originally by a very different (non-geometric, direct) method in [13].

6 Further results and open problems

This section contains a collection of remarks concerning the results of [4] and open problems.

First of all, let us recall that every quasi-Hamiltonian reduction of the internally fused double represents the moduli space of flat connections on the one-holed torus Σ with fixed conjugacy class of the holonomy around the hole. This is also the classical phase space of the Chern-Simons field theory on the three-dimensional manifold $[0, 1] \times \Sigma$ with corresponding boundary condition. Therefore, our results outlined in the previous section prove the Chern-Simons interpretation of the III_b system and that of its self-duality, confirming the conjectures of Gorsky and his collaborators [8, 6].

[3]The correspondence $L^y_{\mathrm{loc}}(\xi, \tau) \equiv \Delta(\tau)^{-1} L^{\mathrm{loc}}_y(\delta(\xi), \rho(\tau)^{-1}) \Delta(\tau)$ between the respective notations should be noted for those wish to see the details in [4].

In addition to the coupling constant, y, a second parameter, Λ, can be introduced into the III_b system by replacing the symplectic form (31) by $\Lambda\Omega^{\mathrm{loc}}$. This parameter, which is important at the quantum mechanical level, can be incorporated into the reduction approach by taking the invariant scalar product on $su(n)$ to be $-\frac{\Lambda}{2}\mathrm{tr}$ instead of (43). The quantum mechanics of the III_b system was studied by van Diejen and Vinet [15], who diagonalized the relevant commuting difference operators using Macdonald polynomials; see also our note [5] where we reproduced the joint spectrum of the action-variables by a simple argument. The Hilbert space of the Chern-Simons theory can be always equipped with a representation of the mapping class group [16], and it could be interesting to elaborate this representation in the specific case of the III_b system by building on the work [15].

Ruijsenaars [13] also considered an anti-symplectic involution $\mathfrak{R}$ on $\mathbb{C}P(n-1)$ that enjoys

$$\mathcal{J}_k \circ \mathfrak{R} = \mathcal{I}_k, \qquad \mathcal{I}_k \circ \mathfrak{R} = \mathcal{J}_k, \qquad k = 1, ..., n-1, \tag{58}$$

and is given by $\mathfrak{R} = \hat{C} \circ \mathfrak{S}$ where $\hat{C}$ is the complex conjugation involution. We have shown [4] that $\mathfrak{R}$ arises from the map R_D of the double of $SU(n)$ defined by

$$R_D := \varrho_D \circ S_D^2, \qquad \varrho_D(A, B) := (\bar{B}, \bar{A}), \qquad \forall (A, B) \in D. \tag{59}$$

Although R_D is not quite an automorphism of D, it descends to a map R_P on any reduced phase space $P(\mu_0)$ with diagonal constant matrix μ_0. (If μ_0 and μ_0' are conjugate then $P(\mu_0)$ and $P(\mu_0')$ are naturally equivalent, and therefore one may take μ_0 diagonal without loss of generality.) The involution R_P reverses the sign of the induced Poisson structure on $P(\mu_0)$, and together with S_P and T_P it generates a $GL(2, \mathbb{Z})$ action on $P(\mu_0)$.

Let $\mathcal{Z}$ be the center of the group G. Notice that $\mathcal{Z} \times \mathcal{Z}$ acts on the internally fused double $D = G \times G$ by the automorphisms

$$(z_1, z_2) : (A, B) \mapsto (z_1 A, z_2 B), \qquad \forall (z_1, z_2) \in \mathcal{Z} \times \mathcal{Z}. \tag{60}$$

This action descends to the reduced phase space $P(\mu_0)$, and in the special case $G = SU(n)$ and μ_0 (50) it gives rise to the $\mathbb{Z}_n \times \mathbb{Z}_n$ action on $\mathbb{C}P(n-1)$ used in some considerations in [13].

The reader is invited to study [4] for further results, which include for example the factorization of S_D as a product of three Dehn twist automorphisms of the double, where the Dehn twist automorphisms themselves are realized in terms of certain quasi-Hamiltonian flows.

It could be worthwhile to explore the structure of the stratified symplectic spaces $P(\mu_0)$ in general, and to possibly uncover new integrable systems on them. Some sort of trigonometric spin Ruijsenaars-Schneider systems are expected to arise in this way, which might be integrable analogously to spin Sutherland systems [11].

Finally, the most intriguing open problem stems from the fact that a reduction treatment of the self-dual hyperbolic Ruijsenaars-Schneider system (the one which is related for example to sine-Gordon solitons) is still missing. Presently we do not know what master phase space should give this system upon reduction. Is it possible to construct such a master phase space? Of course, there exist other important variants of the Ruijsenaars-Schneider system (BC_n case [10], elliptic systems) that should be further studied as well.

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