STOCHASTIC COMPACTNESS OF TWO SIDED-EXIT TIMES

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Dedicated to the memory of Sándor Csörgő.

ABSTRACT

We characterize stochastic compactness of the two sided exit times of partial sums and Lévy processes at "large times", i.e., as $t \to \infty$, and "small times", i.e., as $t \searrow 0$, as well as examine the continuity properties of the subsequential distributions of the two sided exit times.

This talk is based on work in progress.

PARTIAL SUMS

Let $\xi, \xi_1, \xi_2, \ldots,$ be i.i.d. nondegenerate random variables (rvs) with cumulative distribution function (cdf) F and for each integer $n \geq 1$ denote their partial sum by

$$S_n = \sum_{i=1}^{n} \xi_i.$$

LEVY PROCESS

Consider a Lévy process $(X_t)_{t\geq 0}$, having nondegenerate infinitely divisible (inf. div.) characteristic function (cf)

$$Ee^{\mathrm{i}\theta X_t} = e^{t\Psi(\theta)}, \quad \theta \in \mathbb{R},$$

where $\Psi(\theta) =$

$$-\frac{1}{2}\sigma^2\theta^2 + \mathrm{i}\gamma\theta + \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\leq 1\}}\right) \Pi(\mathrm{d}x),$$

$\gamma \in R$, $\sigma^2 \geq 0$, and Π is a measure on $\mathbb{R}$ with

$$\int_{\mathbb{R}\setminus\{0\}} (x^2 \wedge 1)\Pi(\mathrm{d}x) < \infty.$$

We say that X_t has canonical triplet (γ, σ^2, Π).

LEVY TAIL FUNCTIONS

Introduce the Lévy tail functions for $x > 0$

$$\overline{\Pi}^{+}(x) = \Pi\{(x, \infty)\},\ \overline{\Pi}^{-}(x) = \Pi\{(-\infty, -x)\},$$

$$\text{and } \overline{\Pi}(x) = \overline{\Pi}^{+}(x) + \overline{\Pi}^{-}(x),$$

and the truncated mean and variance functions defined for $x > 0$ by

$$\nu(x) = \gamma + \int_{1<|y|\leq x} y\Pi(\mathrm{d}y)$$

$$\text{and} \quad V(x) = \sigma^2 + \int_{0<|y|\leq x} y^2\Pi(\mathrm{d}y).$$

FELLER CLASS FOR RVs

We shall say that a sequence of partial sums $\{S_n\}_{n\geq 1}$ of i.i.d. ξ rv with cdf F is in the *Feller class* (*stochastically compact*) if there exist norming and centering constants $B(n) > 0$, $A(n)$ such that every subsequence $\{n_k\}$ of $\{n\}$ contains a further subsequence $n_{k'} \to \infty$ with

$$\frac{S_{n_{k'}} - A(n_{k'})}{B(n_{k'})} \xrightarrow{\mathrm{D}} Y',$$

where Y' is a finite nondegenerate rv, a.s. (The prime on Y' signifies that in general it depends on the choice of the subsequence.) We shall write this as "$S_n \in FC$", also written "$F \in FC$".

If the centering function $A(n)$ can be chosen to be identically equal to zero, we shall say that S_n is in the centered Feller class at infinity, written "$S_n \in FC_0$", also written "$F \in FC_0$".

FELLER CLASS AT INFINITY

We shall say that a Lévy process X_t, $t \geq 0$, is in the *Feller class* at infinity (*stochastically compact* at infinity) if there exist nonstochastic functions $B(t) > 0$, $A(t)$ such that every sequence $t_k \to \infty$ contains a subsequence $t_{k'} \to \infty$ with

$$\frac{X_{t_{k'}} - A(t_{k'})}{B(t_{k'})} \xrightarrow{\mathrm{D}} Y', \tag{F}$$

where Y' is a finite nondegenerate rv, a.s. We shall write this as "$X_t \in FC$ at ∞".

If the centering function $A\,(t)$ can be chosen to be identically equal to zero, we shall say that X_t is in the centered Feller class at infinity, written "$X_t \in FC_0$ at ∞".

FELLER CLASS AT ZERO

We shall say that a Lévy process X_t, $t \geq 0$, is in the *Feller class* at zero if there exist nonstochastic functions $B(t) > 0$, $A(t)$ such that every sequence $t_k \downarrow 0$ contains a subsequence $t_{k'} \downarrow 0$ for which (F) holds.

We shall write this as "$X_t \in FC$ at 0".

In this situation it is assumed that whenever $\sigma^2 = 0$

$$\overline{\Pi}\,(0+) = \infty.$$

If the centering function $A\,(t)$ can be chosen to be identically equal to zero, we shall say that X_t is in the centered Feller class at zero, written "$X_t \in FC_0$ at 0".

OBSERVATION

The rv Y' in (F) is infinitely divisible, having a cf of the form $E\exp(i\theta Y') =$

$$\exp\left[-\frac{1}{2}A\theta^2 + \mathrm{i}\gamma\theta\right]$$

$$\times \exp\left[\int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}x\theta} - 1 - \mathrm{i}\theta x \mathbf{1}\{|x| \le 1\}\right)\pi(\mathrm{d}x)\right],$$

where $A \ge 0$, θ, $\gamma \in \mathbb{R}$ and

$$\int_{\mathbb{R}\setminus\{0\}} \left(|x|^2 \wedge 1\right)\pi(\mathrm{d}x) < \infty.$$

It turns out that Y' has an infinity differentiable density.

PRUITT (1983) RESULT

By applying a result of Pruitt (1983), one can show that whenever $S_n \in FC$, respectively, $S_n \in FC_0$, then each of its subsequential limit rv Y' defines a Lévy process X_t such that

$$X_1 \stackrel{\mathrm{D}}{=} Y'$$

and X_t is both in FC (at infinity) and in FC (at zero), respectively, in FC_0 (at infinity) and FC_0 (at zero).

In fact each of the subsequential limit rvs Y' has an infinity differentiable density.

FELLER CONDITION

The classic Feller (1966) condition for $S_n \in FC$ is

$$\limsup_{y\to\infty} \frac{y^2 P\left(|\xi| > y\right)}{E\left(\xi^2 \mathbf{1}_{\{|\xi|\le y\}}\right)} < \infty. \tag{FC}$$

Here is an additional useful characterization of $S_n \in FC$.

QUANTILE CONDITION

In the course of developing their quantile-empirical process approach to the asymptotic distribution of partial sums of i.i.d. rvs, Csörgő, Haeusler and Mason (1988) show that $S_n \in FC$ (namely the Feller condition (FC) holds) if and only if for all $\lambda > 0$

$$\limsup_{s \searrow 0} \frac{\sqrt{s}\left\{\left|F^{-1}(\lambda s)\right| + \left|F^{-1}(1-\lambda s)\right|\right\}}{\sigma(s)} < \infty,$$

where F^{-1} is the *inverse* or *quantile function* of F defined to be, for each $0 < s < 1$,

$$F^{-1}(s) = \inf\left\{x : F(x) \geq s\right\},$$

and for $0 < s < 1/2$,

$$\sigma^2(s) = \int_s^{1-s}\int_s^{1-s} (u \wedge v - uv)\, F^{-1}(\mathrm{d}u)\, F^{-1}(\mathrm{d}v).$$

CENTERED FELLER CLASS FOR RVs

Clearly $S_n \in FC_0$ if and only if $S_n \in FC$ and

$$\limsup_{n\to\infty} \left| A(n) / B(n) \right| < \infty.$$

Maller (1979) (see also Giné and Mason (1998) and Griffin and Maller (1999) proved that $S_n \in FC_0$ if and only if

$$\limsup_{y\to\infty} \frac{y^2 P(|\xi| > y) + y \left| E\left(\xi \mathbf{1}_{\{|\xi|\le y\}}\right)\right|}{E\left(\xi^2 \mathbf{1}_{\{|\xi|\le y\}}\right)} < \infty.$$

There is also a quantile version of this condition.

FELLER CLASS AT LARGE TIME

The following theorem is from Maller and Mason (2009).

Theorem 1 *Let X be a nondegenerate inf. div. rv having cf $e^{\Psi(\theta)}$, where Ψ is defined as above, and let X_t be a Lévy process with $X_1 \stackrel{\mathrm{D}}{=} X$.*
(i) We have $X_t \in FC$ at infinity if and only if

$$\limsup_{y\to\infty} y^2\overline{\Pi}(y)/V(y) < \infty.$$

(ii) We have $X_t \in FC_0$ at infinity if and only if

$$\limsup_{y\to\infty} \left(y^2\overline{\Pi}(y) + y\,|\nu(y)|\right)/V(y) < \infty.$$

FELLER CLASS AT SMALL TIME

Here is the corresponding result at small time proved in Maller and Mason (2010).

Theorem 2 *Let X_t be a Lévy process having cf $e^{t\Psi(\theta)}$, where Ψ is defined in above, and whenever $\sigma^2 = 0$, assume that $\overline{\Pi}(0+) = \infty$.*

(i) We have $X_t \in FC$ at zero if and only if

$$\limsup_{y \searrow 0} y^2 \overline{\Pi}(y) / V(y) < \infty.$$

(ii) We have $X_t \in FC_0$ at zero if and only if

$$\limsup_{y \searrow 0} \left(y^2 \overline{\Pi}(y) + y\, |\nu(y)| \right) / V(y) < \infty.$$

TWO SIDED EXIT TIME

In the random walk case, let for $0 \le t < \infty$

$$X(t) = \sum_{0 \le i \le t} \xi_i$$

and in the Lévy process case let $X(t) = X_t$. Define for any $r > 0$ the two sided exit time

$$T(r) = \inf \{t > 0 : |X(t)| > r\}.$$

From results in Pruitt (1981) and Doney and Maller (2002) we can infer that for a suitable function h and constants $a_1 > 0$ and $a_2 > 0$ for all $r > 0$

$$\frac{a_1}{h(r)} \le ET(r) \le \frac{a_2}{h(r)}, \qquad \text{(T)}$$

THE FUNCTION h

In the Lévy process case

$$h\,(x) = \frac{x\,|\nu(x)| + U(x)}{x^2},$$

where

$$U(x) = x^2\overline{\Pi}\,(x) + V(x).$$

STOCHASTIC COMPACTNESS OF $\left|X_{T(r)}/r\right|$

From now on for ease of presentation we shall restrict ourselves to the Lévy process at 0 case and write

$$X_{T(r)} = X\left(T\left(r\right)\right).$$

We shall say that

$$\left|X_{T(r)}/r\right|$$

is stochastically compact (SC) at zero if for every positive sequence $r_k \searrow 0$ there exists a subsequence of $\{s_j\}$ of $\{r_k\}$ such that $\left|X_{T(s_j)}/s_j\right|$ converges in distribution to a nondegenerate rv.

NECESSARY & SUFFICIENT CONDITION

A clean necessary and sufficient condition for this due to Maller unpublished notes is that $X_t \in FC_0$ at zero and X_t is not in the domain of partial attraction of the normal distribution.

The latter means that there does not exist a sequence $t_k \searrow 0$ and positive norming sequence $B(t_k) > 0$ such that $X_{t_k}/B(t_k)$ converges in distribution to a standard normal rv.

For the random walk version of this necessary and sufficient condition see Griffin and Maller (1999).

QUESTION

Suppose for a sequence $r_k \searrow 0$

$$\left|X_{T(r_k)}\right| / r_k \xrightarrow{\text{D}} Y. \tag{Y}$$

When does Y have a cdf F that is absolutely continuous on $[a, \infty)$ for any $a > 1$?

STABLE EXAMPLE

Whenever $(X_t)_{t>0}$ is a subordinator in the domain of attraction of a stable law of index $0 < \alpha < 1$, as $t \searrow 0$, then

$$\left|X_{T(r)}\right| / r \to_d Y, \text{ as } r \to \infty,$$

where for $y > 1$

$$F(y) = P\{Y \leq y\}$$
$$= \frac{\sin(\alpha\pi)}{\pi} \int_1^y (x-1)^{-\alpha} x^{-1} \mathrm{d}x.$$

This can be deduced from the arguments on page 361 of Bingham, Goldie and Teugels (1987).

A USEFUL BOUND

If we can show that when (Y) holds, that for each $1 < a$ there is a $0 < C(a) < \infty$ such that for any $d > a > 0$, uniformly in $a \le c < d$

$$\limsup_{k\to\infty} P\left\{\left|\frac{X_{T(r_k)}}{r_k}\right| \in (c,d]\right\} / (d-c) \le C(a),$$

we are done.

Write for $u > 0$ and $|v| < u$

$$\Delta(u,v) = \overline{\Pi}^{+}(u-v) + \overline{\Pi}^{-}(u+v).$$

Using results in Pruitt (1981) and Doney and Maller (2002) one can show that

$$P\left\{\left|\frac{X_{T(r_k)}}{r_k}\right| \in (c,d]\right\}$$

$$\le \sup_{|y|\le 1} \left(\Delta(r_k c, r_k y) - \Delta(r_k d, r_k y)\right) ET(r_k),$$

which by using (T) is

$$\le \frac{a_2}{h(r_k)} \sup_{|y|\le 1} \left(\Delta(r_k c, r_k y) - \Delta(r_k d, r_k y)\right)$$

$$= \frac{a_2 r_k^2 \sup_{|y|\le 1} \left(\Delta(r_k c, r_k y) - \Delta(r_k d, r_k y)\right)}{r_k |\nu(r_k)| + U(r_k)}.$$

ASSUMPTIONS

Suppose that (Y) holds, then there exist positive continuous decreasing functions $\overline{\pi}^+$and $\overline{\pi}^-$ on $(0,\infty)$ and a sequence of positive constants $\{t_k\}_{k\geq 1}$ such that for all $u>0$

$$t_k\overline{\Pi}^+(r_k u)\to\overline{\pi}^+(u) \text{ and } t_k\overline{\Pi}^-(r_k u)\to\overline{\pi}^-(u) \quad \text{(A1)}$$

and

$$\limsup_{k\to\infty}(t_k h(r_k))^{-1}=:\gamma<\infty. \quad \text{(A2)}$$

Further assume that for each $a>1$, $\overline{\pi}^+$ and $\overline{\pi}^-$are Lipschitz on $[a-1,\infty)$ with Lipschitz constants $D^+(a)$ and $D^-(a)$, respectively. In particular this holds whenever $-\overline{\pi}^+$and $-\overline{\pi}^-$ have strictly decreasing positive derivatives on $(0,\infty)$, say φ^+ and φ^-.

In this case one can choose $D^+(a)=\varphi^+(a-1)$ and $D^-(a)=\varphi^-(a-1)$.

CONCLUSION

The convergence in (A1) is uniform on $[a-1,\infty)$ for any $a>1$. Thus by (A2) for all $a \le c < d$

$$\limsup_{k\to\infty}\left(\frac{t_k}{t_k h\left(r_k\right)}\right)\sup_{|y|\le 1}\left(\Delta\left(r_k c, r_k y\right)-\Delta\left(r_k d, r_k y\right)\right)$$

$$\le \gamma \sup_{|y|\le 1}\left[\overline{\pi}^{+}\left(c-y\right)-\overline{\pi}^{+}\left(d-y\right)\right]$$

$$+\gamma \sup_{|y|\le 1}\left[\overline{\pi}^{-}\left(c+y\right)-\overline{\pi}^{-}\left(d+y\right)\right],$$

which by the Lipschitz assumption is

$$\le \gamma\left(D^{+}\left(a\right)+D^{-}\left(a\right)\right)\left(d-c\right)=:D\left(a\right)\left(d-c\right).$$

CONJECTURE

Whenever $X_t \in FC_0$ at 0 then for each $1 < a$ there is a constant $D(a) > 0$ such that for all $a \le c < d$

$$\limsup_{r \searrow 0} \frac{r^2 \sup_{|y| \le 1} \left(\Delta(rc, ry) - \Delta(rd, ry) \right)}{r\,|\nu(r)| + U(r)}$$

$$\le D(a)\,(d - c)\,.$$

In this case we can choose in (A2)

$$t_k = r_k^2 / U(r_k)\,.$$

ST. PETERSBURG GAME PROCESS

Consider the St. Petersburg game type Lévy tail functions for $x > 0$

$$\overline{R}^{+}(x) = R\{(x, \infty)\} = 2^{-\lfloor \log_2(x) \rfloor},$$

$$\overline{R}^{-}(x) = R\{(-\infty, -x)\} = 2^{-\lfloor \log_2(x) \rfloor},$$

and

$$\overline{R}(x) = \overline{R}^{+}(x) + \overline{R}^{-}(x).$$

Notice that $\overline{R}(0+) = \infty$. Let $(X_t)_{t \geq 0}$ be the symmetric St. Petersburg Lévy process with cf, $\exp(t\Psi(\theta))$, where due to symmetry of X_1,

$$\Psi(\theta) = \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}\theta x} - 1\right) R(\mathrm{d}x), \quad \theta \in \mathbb{R}.$$

It is readily checked that $X_t \in FC_0$ at zero.

COUNTEREXAMPLE

The symmetric St. Petersburg process provides a counterexample to the conjecture. Define the norming function

$$b\,(t) = 2^{\lfloor \log_2(t) \rfloor},\ t > 0.$$

It can be shown that each subsequential limit law of $X_t/b\,(t)$ has a cf of the form $\Psi\,(\lambda\theta)$, where $\lambda \in [1, 2]$.

If the conjecture were true, $\overline{R}$ would be continuous on $(0, \infty)$. However it clearly is not, even though each such subsequential rv has an infinitely differentiable density.

SELF–DECOMPOSIBLE DISTRIBUTIONS

A distribution function F is said to be in the class of self-decomposable distributions (SD), also called the class $\mathcal{L}$, if there exists a sequence of independent rvs $\{Z_n\}_{n\geq 1}$ and constants $b_n > 0$ and c_n such that $b_n Z_n + c_n$ converges in distribution to F and

$$\max_{1\leq k\leq n} |b_n Z_n| \xrightarrow{\mathrm{P}} 0.$$

The distribution of a rv W is SD if and only if its cf is of the form $E\exp(i\theta W) =$

$$\exp\left[-\frac{1}{2}A_W\theta^2 + \mathrm{i}\gamma_W\theta\right]\times$$

$$\exp\left[\int_{\mathbb{R}\backslash\{0\}}\left(e^{\mathrm{i}x\theta} - 1 - \mathrm{i}\theta x\mathbf{1}\left\{|x|\leq 1\right\}\right)\frac{k(x)}{|x|}\mathrm{d}x\right],$$

where $A_W \geq 0$, θ, $\gamma_W \in \mathbb{R}$, $k(x) \geq 0$,

$$\int_{\mathbb{R}\backslash\{0\}}\left(|x|^2 \wedge \mathbf{1}\right)\frac{k(x)}{|x|}\mathrm{d}x < \infty,$$

and $k(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

FACTS

A Lévy process $(X_t)_{t\geq 0}$ is said to be a SD Lévy process if X_1 has a cf of the above form.

If we also assume that $X_t \in \mathrm{FC}_0$ at 0 and $\overline{\Pi}(0+) = \infty$ if $A_W > 0$ or $X(t) \in \mathrm{FC}_0$ at ∞ then each subsequential limit rv W of

$$X_t/b(t)$$

with the appropriate norming $b(t)$, is also SD.

A FAMILY OF EXAMPLES

Let $(X_t)_{t\geq 0}$ be a SD Lévy process not in the domain of partial attraction of a standard normal rv at zero. Assume that $X_t \in \mathrm{FC}_0$ at 0 and $\overline{\Pi}\,(0+) = \infty$. X_t has cf

$$Ee^{\mathrm{i}\theta X_t} = e^{t\Psi(\theta)}, \quad \theta \in \mathbb{R},$$

where

$$\Psi(\theta) = \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}x\theta} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\leq 1\}}\right) \frac{k(x)}{|x|}\mathrm{d}x,$$

$k\,(x) \geq 0$, $\int_{\mathbb{R}\setminus\{0\}} \left(|x|^2 \wedge \mathbf{1}\right) \frac{k(x)}{|x|} dx < \infty$, and $k(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

Here X_t has Lévy measure

$$\overline{\Pi}(z) = \overline{\Pi}^{+}(z) + \overline{\Pi}^{-}(z),\ z > 0,$$

where

$$\overline{\Pi}^{\pm}(z) = \int_{(z,\infty)} \frac{k(\pm x)}{x}\mathrm{d}x.$$

FELLER CLASS FACT

Maller and Mason (2018), assume that $(X_t)_{t\geq 0}$ is a Lévy process without a normal component, then whenever

$$\lim_{\lambda\to\infty}\limsup_{x\searrow 0}\frac{\overline{\overline{\Pi}}(\lambda x)}{\overline{\overline{\Pi}}(x)}<1,$$

$X_t \in FC$ at 0.

SELF–DECOMPOSABLE APPLICATION

Let $(X_t)_{t\geq 0}$ is be SD Lévy process without a normal component. Set for $x \in (0, \infty)$

$$m\,(x) = k(x) + k(-x).$$

We see that m is a decreasing function on $(0, \infty)$. Assuming that

$$\lim_{\lambda\to\infty} \limsup_{x\searrow 0} \frac{m\,(\lambda x)}{m\,(x)} < 1,$$

we get that $X_t \in FC$ at 0.

EXAMPLE

Let $(X_t)_{t\geq 0}$ is be SD Lévy process without a normal component such that for some $0<\alpha<2$

$$m(x)=L(x)\,x^{-\alpha},\ x>0,$$

where $L(x)$ is slowly varying at zero. Then

$$\frac{m(\lambda x)}{m(x)}\to\lambda^{-\alpha},\text{ as } x\to\infty,\ \text{ and as } x\searrow 0,$$

which implies that $X_t\in FC$ at 0.

Whenever $(X_t)_{t>0}$ is a subordinator in FC_0 at 0 in the domain of attraction of a stable law of index $0<\alpha<1$ as $t\searrow 0$, the above procedure works to verify the absolute continuity of the distribution F of the rv Y in the STABLE EXAMPLE.

It also works in the case when $(X_t)_{t\geq 0}$ is symmetric and in the domain of attraction of a stable law of index $0<\alpha<2$ as $t\searrow 0$.