Darling–Kac theorem in the semistable case

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Joint work with Dalia Terhesiu (Exeter).

Outline

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Markov renewal chain

$(f_k)_{k\geq 0}$ probability distribution $\sum_{k=0}^{\infty} f_k = 1$. Markov renewal chain $(X_n)_{n\geq 0}$, $X_n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$p_{\ell,k} := \mathbf{P}(X_{n+1} = k | X_n = \ell) = \begin{cases} f_k, & \ell = 0, \\ 1, & k = \ell - 1,\ \ell \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

X_n recurrent, with unique invariant measure

$$\pi_n = \pi_0 \sum_{i=n}^{\infty} f_i, \quad n \geq 1, \quad \text{and } \pi_0 > 0.$$

X is null recurrent (i.e. the invariant measure is infinite) iff $\sum_{k=1}^{\infty} k f_k = \infty$.

Return times and occupation times

$X_0 = 0$, and $0 = S_0 < S_1 < S_2 < \dots$ consecutive return times to 0.

$$S_n = \tau_1 + \tau_2 + \dots + \tau_n, \quad n \geq 1,$$

where $\tau, \tau_1, \tau_2, \dots$ are iid random variables, with distribution $\mathbf{P}(\tau = k) = f_{k-1}$, $k \geq 1$.
Occupation time of 0, i.e. the number of visits to 0 up to time $n - 1$:

$$T_n = \sum_{j=0}^{n-1} I_{X_j=0}, \quad n \geq 1.$$

Duality

$$T_n \geq m \iff S_{m-1} \leq n - 1,$$

number of visits to the state 0 before time n is at least m if and only if the $(m-1)$st return takes place before time n.

Stable laws, domain of attraction

V is *stable*, if there exist $X, X_1, X_2, \ldots$ iid, $a_n > 0, c_n \in \mathbb{R}$, such that

$$\frac{1}{a_n} \sum_{i=1}^{n} X_i \xrightarrow{\mathcal{D}} V.$$

$F \in D(\alpha)$ iff $1 - F(x) = \ell(x) x^{-\alpha}$.

Regular variation

$\ell : (0, \infty) \to (0, \infty)$ is slowly varying if for every $\lambda > 0$

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

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f is regularly varying with parameter $-\alpha$, $f \in \mathcal{RV}_{-\alpha}$ if $f(x) = \ell(x) x^{-\alpha}$.

Domain of attraction

$(f_j)_{j\geq 0} \in D(\alpha)$, $\alpha < 1$; that is,

$$\sum_{j\geq n} f_j = \ell(n) n^{-\alpha},$$

for a slowly varying ℓ. Then

$$\frac{\sum_{i=1}^{n} X_i}{n^{1/\alpha}\ell_1(n)} \xrightarrow{\mathcal{D}} Z_\alpha,$$

where $n^{1/\alpha}\ell_1(n)$ is the asymptotic inverse of $n^\alpha/\ell(n)$, Z_α is α-stable.

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Darling–Kac theorem

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$$\begin{aligned}
\mathbf{P}(T_n \geq a(n)x) &= \mathbf{P}(S_{a(n)x-1} \leq n-1) \\
&= \mathbf{P}\left(\frac{S_{a(n)x-1}}{b(a(n)x-1)} \leq \frac{n-1}{b(a(n)x-1)}\right) \\
&\to \mathbf{P}(Z_\alpha \leq x^{-1/\alpha}) \\
&= \mathbf{P}(M_\alpha \geq x).
\end{aligned}$$

$M_\alpha \stackrel{\mathcal{D}}{=} (Z_\alpha)^{-\alpha}$ Mittag-Leffler distribution. Hence

$$\frac{T_n}{a(n)} \stackrel{\mathcal{D}}{\to} M_\alpha.$$

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Kruglov, Mejzler, Pillai, Shimizu, Grinevich, Khokhlov
Dodunekova, Berkes, Csáki, Megyesi, Györfi, K
Meerschaert, Scheffler, Kern, Wedrich
Sato, Watanabe, Yamamuro

Semistable laws

V is *stable*, if there exist $X, X_1, X_2, \ldots$ iid, $a_n > 0$, such that

$$\frac{1}{a_n} \sum_{i=1}^{n} X_i \xrightarrow{\mathcal{D}} V.$$

W is *semistable*, if there exist $X, X_1, X_2, \ldots$ iid, $a_n > 0$, n_k geometrically increasing $(= c^k)$, such that

$$\frac{1}{a_{n_k}} \sum_{i=1}^{n_k} X_i \xrightarrow{\mathcal{D}} W.$$

Characteristic function

Characteristic function of a nonnegative semistable random variable V:

$$\mathbf{E}e^{\mathrm{i}tV} = \exp\left\{\mathrm{i}ta + \int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R(x)\right\},$$

where $a \geq 0$
$M : (0, \infty) \to (0, \infty)$ logarithmically periodic $M(c^{1/\alpha}x) = M(x)$
$-R(x) := M(x)/x^\alpha$ is nonincreasing for $x > 0$, $\alpha \in (0, 1)$.

Domain of geometric partial attraction

Grinevich, Khokhlov (1995); Megyesi (2000)
$X, X_1, X_2, \ldots$ iid $F(x) = \mathbf{P}(X \leq x)$. $V = V(R)$ semistable

$$\mathbf{E}e^{\mathrm{i}tV} = \exp\left\{ \int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R(x) \right\}, \quad -R(x) = \frac{M(x)}{x^\alpha}.$$

$X \in D_{gp}(G)$ if $\exists k_n, A_n$

$$\frac{\sum_{i=1}^{k_n} X_i}{A_{k_n}} \xrightarrow{\mathcal{D}} V.$$

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$F \in D_g(V)$ iff $1 - F(x) = \ell(x)M(x)x^{-\alpha}$.

Outline

Circular convergence

u_n *converges circularly* to $u \in (c^{-1}, 1]$, $u_n \overset{cir}{\to} u$, if $u \in (c^{-1}, 1)$ and $u_n \to u$ in the usual sense, or $u = 1$ and u_n has limit points 1, or c^{-1}, or both.

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For $x > 0$ (large) we define the position parameter as

$$\gamma_x = \gamma(x) = \frac{x}{c^n}, \quad \text{where } c^{n-1} < x \le c^n.$$

$$c^{-1} = \liminf_{x\to\infty} \gamma_x < \limsup_{x\to\infty} \gamma_x = 1.$$

Limits on subsequences

$$\mathbf{E}e^{\mathrm{i}tV} = \exp\left\{\int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R(x)\right\}, \quad -R(x) = \frac{M(x)}{x^\alpha}$$

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Theorem (Csörgő & Megyesi (2002))

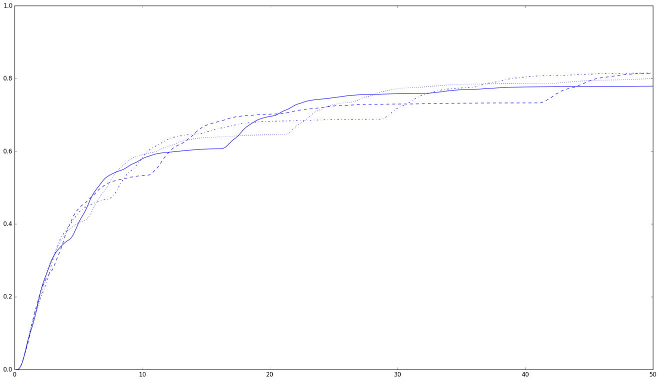
$$\frac{\sum_{i=1}^{n_r} X_i}{n_r^{1/\alpha}\ell_1(n_r)} \xrightarrow{\mathcal{D}} V_\lambda \quad \text{as } r \to \infty,$$

whenever $\gamma_{n_r} \stackrel{cir}{\to} \lambda$. *Here*

$$\mathbf{E}e^{\mathrm{i}tV_\lambda} = \exp\left\{\int_0^\infty (e^{\mathrm{i}tx} - 1)\mathrm{d}R_\lambda(x)\right\}, \quad R_\lambda(x) = -\frac{M(\lambda^{1/\alpha}x)}{x^\alpha}.$$

Possible limits

$$G_\lambda(x) = \mathbf{P}(V_\lambda \leq x)$$



Merging

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$$\gamma_x = \gamma(x) = \frac{x}{c^n}, \quad \text{where } c^{n-1} < x \le c^n.$$

Theorem (Csörgő & Megyesi (2002))

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbf{P}\left(\frac{S_n}{n^{1/\alpha}\ell_1(n)} \le x\right) - \mathbf{P}(V_{\gamma_n} \le x)\right| = 0.$$

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Return times and occupation times

$X_0 = 0$, and $0 = S_0 < S_1 < S_2 < \ldots$ return times to 0.

$$S_n = \tau_1 + \tau_2 + \ldots + \tau_n, \quad n \geq 1,$$

$\tau, \tau_1, \tau_2, \ldots$ are iid random variables, with distribution $\mathbf{P}(\tau = k) = f_{k-1}$, $k \geq 1$.
Occupation time of 0:

$$T_n = \sum_{j=0}^{n-1} I_{X_j = 0}, \quad n \geq 1.$$

$$T_n \geq m \iff S_{m-1} \leq n - 1,$$

$a_n^{1/\alpha} \ell_1(a_n) \sim n$

Theorem (K & Terhesiu (2018))

If $\gamma(a_{n_r}) \overset{cir}{\to} \lambda \in (c^{-1}, 1]$*, then for any* $x > 0$

$$\lim_{r\to\infty} \mathbf{P}(S_{n_r}/a_{n_r} \le x) = \mathbf{P}\left((V_{h_\lambda(x)})^{-\alpha} \le x \right) =: H_\lambda(x),$$

where

$$h_\lambda(x) = \frac{\lambda x}{c^{\lceil \log_c(\lambda x) \rceil}}.$$

More generally, the following merging result holds

$$\lim_{n\to\infty} \sup_{x>0} \left| \mathbf{P}(S_n \ge a_n x) - \mathbf{P}(V_{\gamma(a_n x)} \le x^{-1/\alpha}) \right| = 0.$$

Proof

$$
\begin{aligned}
\mathbf{P}(T_n \geq a_n x) &= \mathbf{P}(S_{\lceil a_n x \rceil - 1} \leq n - 1) \\
&= \mathbf{P}\left(\frac{S_{\lceil a_n x \rceil - 1}}{(a_n x)^{1/\alpha} \ell_1(a_n x)} \leq \frac{n-1}{(a_n x)^{1/\alpha} \ell_1(a_n x)} \right) \\
&\sim \mathbf{P}(V_{\gamma(a_n x)} \leq x^{-1/\alpha})
\end{aligned}
$$

Outline

Behavior at infinity

$$\mathbf{P}\left(\left(V_{h_\lambda(x)} \right)^{-\alpha} \le x \right) =: H_\lambda(x),$$

Theorem (K – Terhesiu)

For x large enough, there exist $\kappa_1 > \kappa_2 > 0$ (independent of x) such that

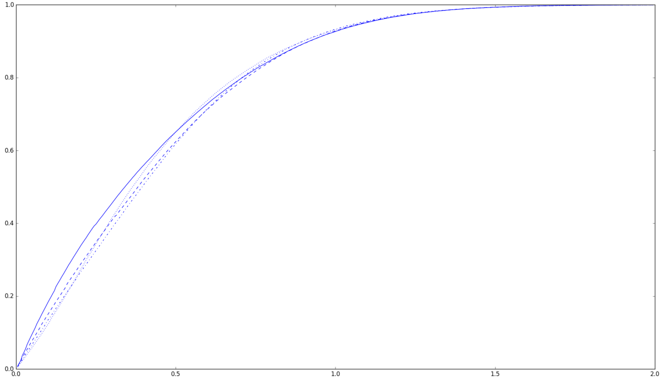
$$\exp\left\{ -\kappa_1 x^{\frac{1}{1-\alpha}} \right\} \le \overline{H}_\lambda(x) = 1 - H_\lambda(x) \le \exp\left\{ -\kappa_2 x^{\frac{1}{1-\alpha}} \right\}.$$

If M is continuous, then for any $\lambda \in (c^{-1}, 1]$

$$H'_\lambda(0) = \lim_{x \downarrow 0} \frac{H_\lambda(x)}{x} = M\left(\lambda^{1/\alpha} \right).$$

On the distribution function

H function



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