Stability problems for second-order linear differential equations with random coefficients

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Stability properties of solutions of linear second order differential equations with random coefficients

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ABSTRACT

The equation

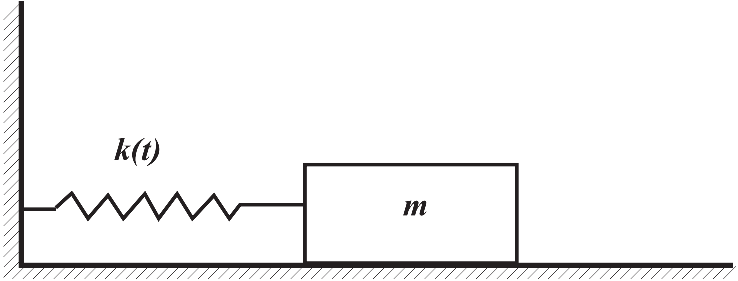
$$x'' + a^2(t)x = 0, \qquad a(t) := a_k > 0 \quad \text{if } t_{k-1} \leqslant t < t_k \quad (k \in \mathbb{N})$$

is considered where $\{a_k\}_{k=1}^{\infty}$ is given and $\{t_k\}_{k=1}^{\infty}$ is a random sequence. Sufficient conditions are proved which guarantee either stability or instability for the zero solution. Stability means that all solutions almost surely tend to zero as $t \to \infty$. By instability we mean that the sequence of the expected values of the amplitudes of every solution tends to infinity as $k \to \infty$. It turns out that $a_k \nearrow \infty$ $(k \to \infty)$ implies stability for all absolutely continuous distributions and for the "overwhelming majority" of the singular distributions. The instability theorem is applied to the problem of random swinging, when $\{a_k\}_{k=1}^{\infty}$ is periodic with two different terms (Meissner's equation) and $\{t_k - t_{k-1}\}_{k=1}^{\infty}$ are independent identically distributed random variables. The application gives conditions for stochastic parametric resonance.

1. Introduction

For a given function $a : [0, \infty) \to (0, \infty)$, consider the non-autonomous equation

A model from mechanics



force of elasticity (Hooke): $-k(t)x \quad (k(t) > 0)$

$$m\ddot{x} + k(t)x = 0$$

m, k are given; $x = x(t) = ?$

Oscillations

The model equation:

$$\ddot{x} + a^2(t)x = 0$$

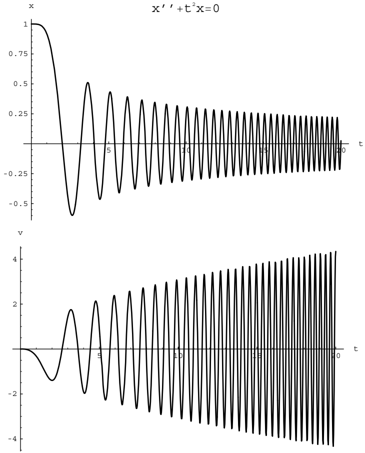
$\sqrt{a^2(t)} = a(t) > 0$ — varying frequency
the equation is not integrable
We investigate two cases:

I. $a(t) \nearrow \infty \; (t \to \infty)$

II. $a(t)$ is periodic

I. $\ddot{x} + a^2(t)x = 0, \qquad a(t) \nearrow \infty\ (t \to \infty)$

the amplitudes of the deviation $x(t)$ are decreasing
the amplitudes of the velocity $x'(t)$ are increasing



I. $\ddot{x} + a^2(t)x = 0, \qquad a(t) \nearrow \infty \ (t \to \infty)$

M. Biernacki (1933): What conditions guarantee

$$\lim_{t\to\infty} x(t) = 0$$

for all solutions x?

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G. Armellini, L. Tonelli, G. Sansone (1936): If a is smooth and $\ln a(t)$ tends to ∞ "regularly" $(t \to \infty)$, then ($\forall$ sol.x):

$$\lim_{t\to\infty} x(t) = 0.$$

(regularly $\sim$ the increase of $a(t)$ to ∞ cannot be localized to a set of small measure)

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(regularly $\sim$ the increase of $a(t)$ to ∞ cannot be localized to a set of small measure)

The condition of regularity cannot be dropped; the first counterexample:
A. S. Galbright, E. J. McShane, G. B. Parish, *Proc. Natl. Acad. Sci. USA*, 53(1965).

I. $\ddot{x} + a^2(t)x = 0, \qquad a(t) \nearrow \infty \ (t \to \infty)$

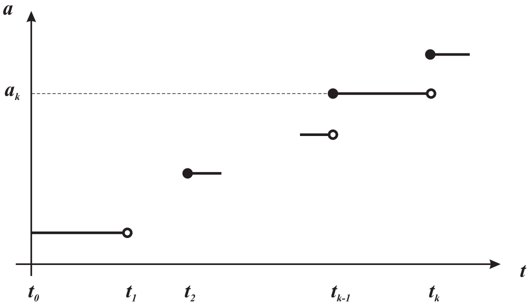
Conjecture:

It is true "generically" that $\forall\, x$

$$\lim_{t\to\infty} x(t) = 0.$$

"generically"$\sim$ apart from exceptional cases

$a(t) \nearrow \infty \ (t \to \infty)$ $a(t)$ step function



$$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N}).$$

The A-T-S Theorem cannot be applied.

$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N}),$ $a_k \nearrow \infty \ (k \to \infty)$

PROBABILISTIC APPROACH

Let $\tau_k = t_k - t_{k-1}$ $(k = 1, 2, \ldots)$ be independent, not necessarily identically distributed random variables.

$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N})$, $a_k \nearrow \infty \ (k \to \infty)$

PROBABILISTIC APPROACH

Let $\tau_k = t_k - t_{k-1}$ $(k = 1, 2, \ldots)$ be independent, not necessarily identically distributed random variables.

Probblem:

What is the probability of the event that for every solution x of the equation

$$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N})$$

the property $\lim_{t\to\infty} x(t) = 0$ holds?

Kolmogorov's $0 - 1$ law implies that this probability equals either 0 or 1.

Conjecture:

The probability above equals 1. *(generically=almost sure)*

$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N}),$ $a_k \nearrow \infty\ (k \to \infty)$

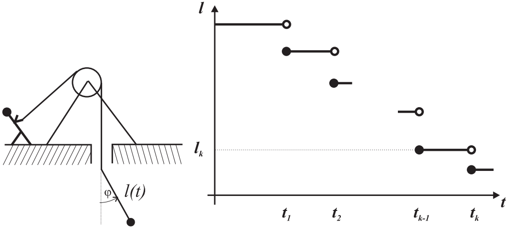
Theorem 1. (L.H.—L. Stachó, 1998). *Suppose that* $\tau_k = t_k - t_{k-1}$ *are independent identically distributed random variables of the uniform distribution on interval* $[0, 1]$.
Then it is almost sure that every solution x *has the property*

$$\lim_{t\to\infty} x(t) = 0.$$

L. H., *Acta Sci. Math.* 68(2002): the monotony of $\{a_k\}$ can be weakened.

$\ddot{x} + a_k^2 x = 0 \ (t_{k-1} \leq t < t_k); \ \ a_k \nearrow \infty$

Example: pendulum with varying length



$$\ddot{\varphi} + \frac{g}{\ell_k}\varphi = 0 \quad (t_{k-1} \leq t < t_k; \ k \in \mathbb{N})$$

Problem: Let the sequence $(\ell_k)_{k=1}^{\infty}$ be given. Suppose that the deviation $\varphi(t)$ cannot be observed, so the sequence $\{t_k\}$ is chosen "at random". What is the probability that one can lift the weight, i.e., $\lim_{t\to\infty} \varphi(t) = 0$ is satisfied for all motions?

Corollary: *If* $\tau_k = t_k - t_{k-1}$ *are independent random variables uniformly distributed on the same interval* $[0, 1]$, *then this probability equals* 1.

$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N})$, $a_k \nearrow \infty\ (k \to \infty)$

$$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k,\ k \in \mathbb{N})$$

$\{\tau_k = t_k - t_{k-1}\}_{k=1}^{\infty}$ independent r.v.

$F_k(x) = P(\tau_k \le x)$: the distribution function of τ_k

$\phi_k(s) := \int_0^\infty e^{isx} dF_k(x)$: the characteristic function of τ_k

$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N})$, $a_k \nearrow \infty\ (k \to \infty)$

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Theorem 2 (S. Csörgő, L. H.). *Suppose that $a_k \nearrow \infty$ $(k \to \infty)$. If*

$$\limsup_{k\to\infty} |\phi_k(2a_k)| < 1,$$

then the property

$$\lim_{t\to\infty} x(t) = 0$$

holds almost surely (i.e., with probability 1) for all solutions of the equation.

$\ddot{x} + a_k^2 x = 0 \quad (t_{k-1} \le t < t_k;\ k \in \mathbb{N}),$ $a_k \nearrow \infty\ (k \to \infty)$

Corollary. *Suppose that* τ_k, $k = 1, 2, \ldots$ *are independent, identically distributed random variables with characteristic function* ϕ*. If*

(F) $$\limsup_{s\to\infty} |\phi(s)| < 1,$$

then for arbitrary $(a_k)_{k=1}^{\infty}$ *the property*

$$\lim_{t\to\infty} x(t) = 0$$

holds **almost surely** *for all solutions of the equation.*

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$$\lim_{t\to\infty} x(t) = 0$$

holds **almost surely** *for all solutions of the equation.*

(F): Cramér's Continuity Condition
It is satisfied for *all* continuous random variables and for *"overwhelming majority"* of singular distributions, which means that our Conjecture is considerably established by the last Corollary.

Example 1. Let $\tau_k = t_k - t_{k-1}$ be an independent r.v. uniformly distributed on the interval $[0, T_k]$ $(k = 1, 2, \ldots)$.

$$\text{ch. f.: } |\phi_k(s)| = \frac{\sqrt{2}\sqrt{1 - \cos T_k s}}{T_k s} \quad (s \ge 0)$$

$$a_k := \sqrt{\frac{g}{\ell_k}} \to \infty \quad (k \to \infty)$$

The stability condition:

$$\limsup_{k\to\infty} |\phi_k(2a_k)| = \limsup_{k\to\infty} \frac{|\sin T_k a_k|}{T_k a_k} < 1,$$

Corollary. *Ha*

$$\liminf_{k\to\infty} \left\{ \frac{T_k}{2} \right\} = \liminf_{k\to\infty} \{E(\tau_k)\} > 0,$$

then $\lim_{t\to\infty} x(t) = 0$ **almost surely** *holds for all solutions.*

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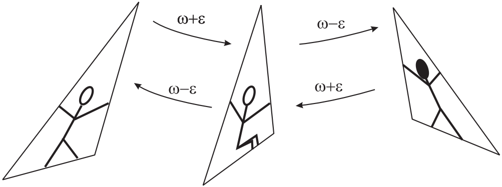
Application to the problem of lifting a weight by the use of a rope and a pulley: DO NOT HURRY!

II. $a(t)$ is periodic (parametric resonance)



$\sqrt{\frac{g}{\ell}} = \omega + \varepsilon$ $\sqrt{\frac{g}{\ell}} = \omega - \varepsilon$ $\sqrt{\frac{g}{\ell}} = \omega + \varepsilon$

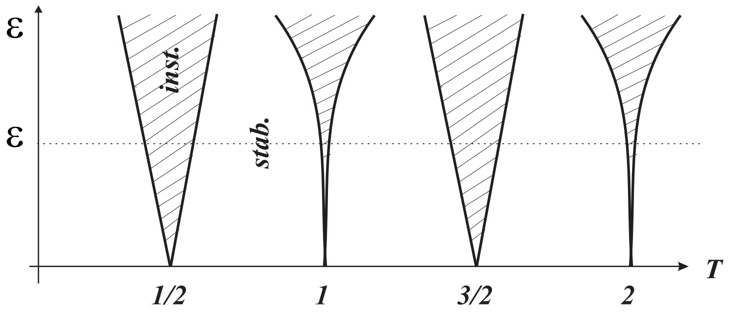
$$\ddot{x} + a^2(t)x = 0 \quad (a(t) \text{ periodikus})$$

Hill-Meissner's equation [G. W. Hill (1886), E. Meissner (1918)]

$$\ddot{x} + a^2(t)x = 0$$

$$a(t) := \begin{cases} \omega + \varepsilon & \text{ha } 2kT \le t < (2k+1)T \\ \omega - \varepsilon & \text{ha } (2k+1)T \le t < 2(k+1)T \end{cases}$$

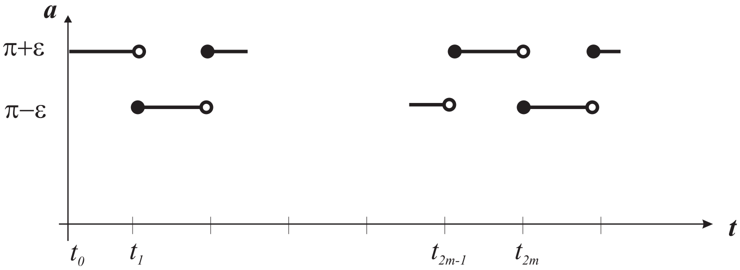
Stability map ("Arnold tongues") $(\omega = \pi)$



INSTABILITY - PARAMETRIC RESONANCE

$$\ddot{x} + a^2(t)x = 0$$

$$a(t) = \begin{cases} \pi + \varepsilon & \text{if } t_{2k-1} \le t < t_{2k} \\ \pi - \varepsilon & \text{if } t_{2k} \le t < t_{2k+1} \end{cases}$$



$(\tau_k = t_k - t_{k-1})_{k=1}^{\infty}$ are independent, identically distributed random variables with expected value T. (In Meissner Equation: $\tau_k = T$ $(k \in \mathbb{N})$.)
ϕ: the characteristic function of τ_k

Stochastic Hill-Meissner equation (random swinging)

Problem: *Let $\varepsilon > 0$ be given. For which distributions and for which values of T does the property*

$$\textbf{almost surely } (\forall sln.\, x) \quad \limsup_{t\to\infty} |x(t)| = \infty$$

hold?

(stochastic parametric resonance)

What is the map of the **almost sure** instability on the plain $\varepsilon - T$?

Theorem 3 (S. Csörgő, L. H.). *If*

$$\begin{aligned} \beta &= \beta(\varepsilon, T, \phi) \\ &:= -(\pi^2 + \varepsilon^2)\left\{|\phi(2(\pi+\varepsilon))| + |\phi(2(\pi-\varepsilon))|\right\} \\ &\quad + 2\varepsilon\pi\{1 + |\phi(2(\pi+\varepsilon))||\phi(2(\pi-\varepsilon))|\} > 0, \end{aligned}$$

then

$$\lim_{n\to\infty} E\left\{x^2(t_n) + \frac{(x'(t_n))^2}{a_{n+1}}\right\} = \infty.$$

$$a(t) = \begin{cases} \pi + \varepsilon & \text{if } t_{2k-1} \le t < t_{2k} \\ \pi - \varepsilon & \text{if } t_{2k} \le t < t_{2k+1} \end{cases}, \quad \tau_k := t_k - t_{k-1}$$

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then

$$\lim_{n\to\infty} E\left\{x^2(t_n) + \frac{(x'(t_n))^2}{a_{n+1}}\right\} = \infty.$$

Stochastic Parametric Resonance

Problem of random swinging: Which r.v.'s τ_n (i.e., which ϕ's) and which T's satisfy the condition of this Theorem for $\varepsilon \to 0+0$?

A necessary condition is: $\phi(2\pi) = 0$.

Stochastic instability, example

1. $\tau_k = \text{Uniform}([0, 2T]),\ E(\tau_k) = T$

Stochastic instability, example

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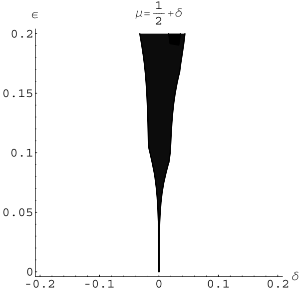
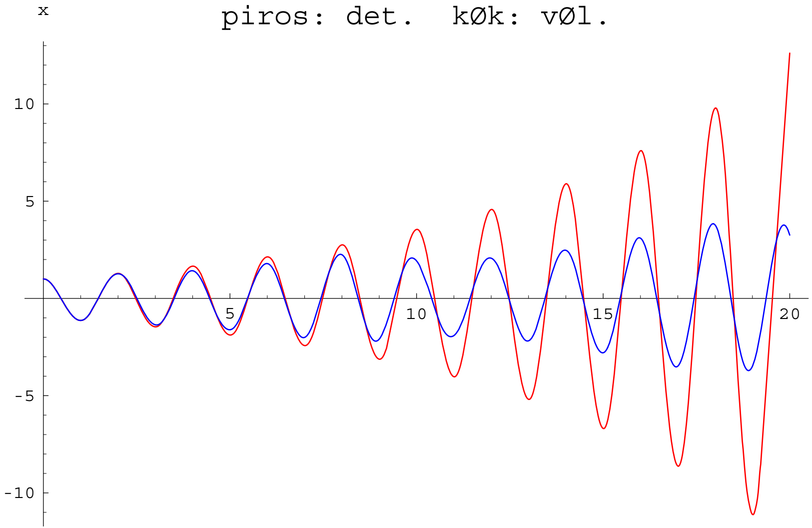
$$|\phi(s)| = \frac{|\sin(sT)|}{|sT|}$$

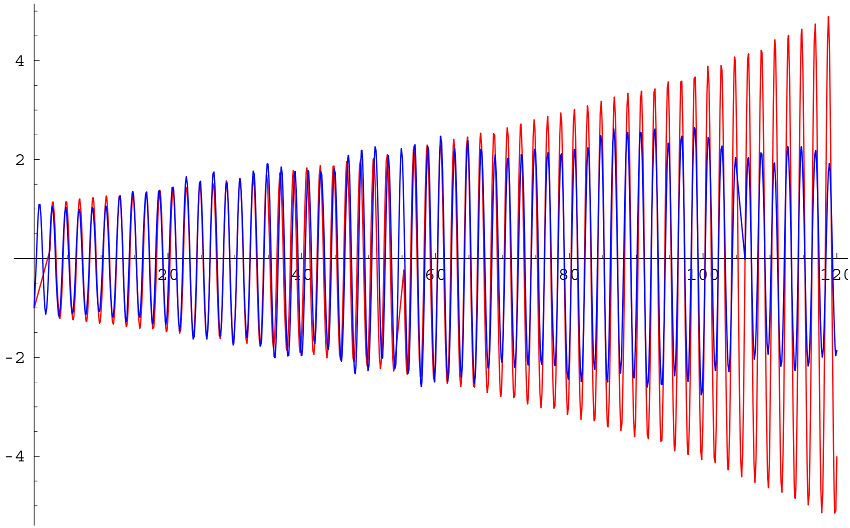
$$|\phi(2\pi)| = \frac{|\sin(2\pi T)|}{|2\pi T|} = 0, \quad T = j\frac{1}{2} \quad (j \in \mathbb{N})$$



Stochastic instability, computer simulation



Stochastic instability, computer simulation



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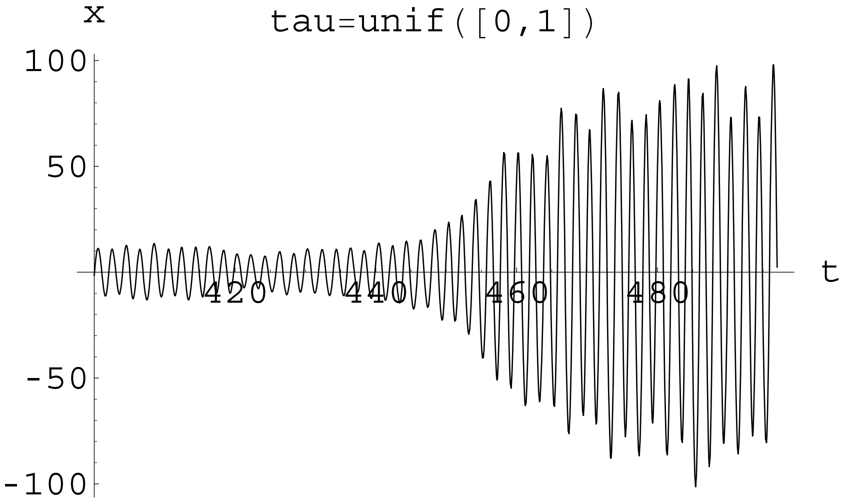

THANK YOU VERY MUCH FOR YOUR ATTENTION!