RANDOM WALKS ON THE CIRCLE AND DIOPHANTINE APPROXIMATION

Convergence of Markov chains to stationary distribution

Card mixing: How many shuffles to uniformity?

Aldous (1983): Cutoff at $\frac{3}{2}\log_2 n$ steps

New York Times (January 9, 1990): *In Shuffling Cards, 7 Is Winning Number*

Random walk on circle: Moving forward or backward with angle $\pm\alpha$, α irrational

Convergence speed depends on **rational approximation properties of** α

$S_n = k\alpha$, $|k| \le n$ Assume $\alpha = \frac{p}{q} + O(q^{-100})$ $S_n = k\frac{p}{q} + O(kq^{-100})$

Su (1998): If α is quadratic irrational $(\alpha = r + s\sqrt{t})$, then

$$C_1 n^{-1/2} \le |\sup_x P(S_n < x) - x| \le C_2 n^{-1/2}$$

Quadratic irrationals: **bad rational approximation** (worst case: $\frac{\sqrt{5}+1}{2}$)

Nonrandom analogue: $x_n = \{n\alpha\}$, α irrational

Empirical measure

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I_{(0,x)}(x_k)$$

$$D_N(x_k) = \sup_{0 \le a \le 1} \left| \frac{1}{N} \sum_{k=1}^{N} I_{[0,a)}(x_k) - a \right| = \sup_{0 \le x \le 1} |F_N(x) - x|$$

Sierpinski, Weyl, Hardy, Littlewood, Ostrowski, Khinchin (1910-)

The magnitude of $D_N(\{k\alpha\})$ depends on the continued fraction digits of α

$\sum_{k=1}^{N} \{k\alpha\}$: number of lattice points in a triangle

Dirichlet circle problem:

Number of lattice points in circle $\{(x, y) : x^2 + y^2 \le R\} = \pi R^2 + O(R^\gamma)$

Subsequence problem: $D_N(\{n_k\alpha\}) = ?$ for general subsequences (n_k)

Try to study **random sequences** (n_k), e.g. $n_{k+1} - n_k = 1$ or 2 with probability $1/2 - 1/2$, when $\{n_k\alpha\}$ is a random walk on circle

Diophantine approximation theory

Dirichlet (1842) For any irrational α, there exist infinitely many fractions p/q such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}$$

Khinchin (1924) For (Lebesgue) almost all α we have

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2 \log q}$$

for infinitely many fractions p/q and this is sharp.

Roth (1955) For any algebraic α there exist only finitely many p/q with

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{2+\varepsilon}}$$

Diophantine type

$$t(\alpha) = \sup\left\{c : \left|\alpha - \frac{p}{q}\right| < \frac{A}{q^{c+1}} \text{ for infinitely many } p/q\right\}$$

Strong type

$$\left|\alpha - \frac{p}{q}\right| < \frac{A}{q^{c+1}}$$

holds for infinitely many p/q for large A and finitely many p/q for small A.

Applications of Diophantine approximation

Thue (1909) Let $P(x,y)$ be an irreducible, homogeneous polynomial with integer coefficients and with degree ≥ 3. If A is any integer, then the equation $P(x,y) = A$ has only finitely many integer solutions.

Siegel (1942) Stability of fix point algorithms depends on rational approximation of α in $z_0 = e^{2\pi i \alpha}$

Poincaré (1890): Sur le problème des trois corps et les équations de la dynamique, Acta Math. 13, 1-271.

Jupiter $\omega_1 = 299.1$", Saturn $\omega_2 = 12.5$", $2\omega_1 - 5\omega_2 \approx 0$.

$$\sum_{m,n \neq 0} a_{m,n} \frac{e^{i(m\omega_1 + n\omega_2)t}}{m\omega_1 + n\omega_2}.$$

Arnold (1963) Small denominators and problems of stability of motions in classical and celestial mechanics. Uspehi Math. Nauk 18 (1963), 27-163.

Results (B & Borda 2018)

$X_1, X_2, \ldots$ i.i.d. integer valued nondegenerate random variables, α irrational, $S_n = \sum_{k=1}^n X_k$,

$$\Delta_n = \sup_x |P(\{S_n\alpha\} < x) - x|$$

Theorem. If $EX_1^2 < \infty$, S_n is unimodal and α is of strong type γ, then

$$\Delta_n = O(n^{-1/(2\gamma)}), \qquad \Delta_n = \Omega(n^{-1/(2\gamma)}).$$

The upper bound remains valid without $EX_1^2 < \infty$.

$\gamma = 1$: **badly approximable number** α (bounded digits in continued fraction)

Theorem. Let $0 < \beta < 2$ and assume

$$P(|X_1| > t) \sim ct^{-\beta} \quad \text{and} \quad \lim_{x\to\infty} P(X_1 \geq x)/P(|X_1| \geq x) \quad \text{exists.}$$

If S_n is unimodal and α has strong type γ, then

$$\Delta_n = O(n^{-1/(\beta\gamma)}), \qquad \Delta_n = \Omega(n^{-1/(\beta\gamma)}).$$

Berry-Esseen problem for ordinary i.i.d. sums:

$$\sup_x \left| P\left(\frac{S_n}{n^{1/\beta}} < x\right) - G_\beta(x) \right| = O(n^{1-2/\beta})$$

Empirical measure of speed:

$$D_N(x_k) = \sup_{0\le a\le 1} \left| \frac{1}{N} \sum_{k=1}^{N} I_{[0,a)}(x_k) - a) \right| = \sup_{0\le a\le 1} |F_N(a) - a|$$

Theorem. For any nondegenerate i.i.d. sequence (X_n) and any irrational α we have

$$D_N(\{S_k\alpha\}) = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{a.s.}$$

This bound is attained if

$$P(|X_1| > x) \sim \frac{1}{\log x}$$

Comparison with deterministic case:

For any irrational α

$$D_N(\{k\alpha\}) = \Omega\left(\frac{\log N}{N}\right)$$

and this bound is attained for $\alpha = \frac{\sqrt{5}+1}{2}$

Theorem. Assume $P(X_1 = 1) = P(X_1 = 2) = 1/2$ and let α have strong Diophantine type γ.

(i) If $1 \le \gamma \le 2$, then

$$D_N = O\left(\sqrt{\frac{\log\log N}{N}}\log N\right), \quad D_N = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{a.s.}$$

(ii) If $\gamma > 2$, then

$$D_N = O\left(\left(\frac{\log\log N}{N}\right)^{1/\gamma}\right), \quad D_N = \Omega\left(\frac{1}{N^{1/\gamma}}\right) \quad \text{a.s.}$$

Change of weak dependence to strong dependence at $\gamma = 2$ $\quad |\alpha - p/q| \ll 1/q^3$

Theorem. Let α have strong Diophantine type γ and assume

$$P(|X_1| > t) \sim ct^{-\beta} \quad \text{and} \quad \lim_{x\to\infty} P(X_1 \ge x)/P(|X_1| \ge x) \quad \text{exists.}$$

(1) If $\gamma \le 2/\beta$, then

$$D_N = O\left(\sqrt{\frac{\log\log N}{N}}\log N\right), \quad D_N = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{a.s.}$$

(ii) If $\gamma > 2/\beta$, then

$$D_N = O\left(\left(\frac{\log\log N}{N}\right)^{1/(\beta\gamma)}\right), \quad D_N = \Omega\left(\frac{1}{N^{1/(\beta\gamma)}}\right) \quad \text{a.s.}$$