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WEIGHTED GRIDS IN COMPLEX JORDAN* TRIPLES

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Weighted grids are linearly independent sets $\{g_w:\ w\in W\}$ of signed tripotents in Jordan* triples indexed by figures W in real vector spaces such that $\{g_u g_v g_w\} \in \mathbb{C} g_{u-v+w} (= 0 \ \ if \ \ u-v+w \notin W)$. They arise naturally as systems of weight vectors of certain abelian families of Jordan* derivations. Based on Neher's grid theory, a classification of association free non-nil weighted grids is given. As a first step beyond the setting of classical grids, the complete list of complex weighted grids of pairwise associated signed tripotents indexed by $\mathbb{Z}^2$ is established.

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1. Introduction

Complex Jordan* triples are ternary algebras over the complex field $\mathbb{C}$ (i.e. complex vector spaces equipped with an operation of three variables) whose operation $(x,y,z) \mapsto \{xyz\}$ is linear in its first and third variables, conjugate linear in the second one and it satisfies the so-called Jordan identity (see (J) in Section 2). These structures seem to be the appropriate technical means in describing the geometry of symmetric complex manifolds: while the classical approach of E. and H. Cartan based upon the study of the Lie structure of the complete holomorphic vector fields encounters enormous difficulties not yet overcome in infinite dimensions, the Jordan theoretic approach initiated by the school of M. Koecher in the late '50-es lead to W. Kaup's far reaching Riemann mapping theorem [7] stating that any bounded symmetric domain in a Banach space is holomorphically equivalent to the unit ball of some topological Jordan* triple. For a typical example, the unit ball of any C^*-algebra is symmetric in Cartan's sense and the corresponding Jordan* triple product is expressed in classical terms as $\{abc\} = \frac{1}{2}ab^*c + \frac{1}{2}cb^*a$.

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One of the most powerful algebraic tools for the investigation of the structure of Jordan* triples is the concept of grids. Heuristically, these objects are aimed to play an analogous role for general Jordan theory as the system of matrices with a unique non-vanishing entry for the C^*-algebra $\mathrm{Mat}(n,n,\mathbb{C})$ of all complex $n\times n$ matrices. As a perhaps final step of a long development [11,9,12,10,13], Neher defines grids in his monograph [14] as maximal families of pairwise Peirce compatible non-associated positive tripotents satisfying some standardizing requirements. In particular, grids are families $\{e_i :\ i\in I\}$ such that for some matrix $\big(\pi_{ik}\big)_{i,k\in I}$, with entries $0,1,2$ called the Peirce matrix we have $\{e_ie_ie_k\} = \frac{1}{2}\pi_{ik}e_k$ and $\pi_{ik}=2=\pi_{ki} \iff i=k$ $(i,k\in I)$. Grids in this sense are completely classified up to isomorphism and association [14].

In this paper we propose a tool, the concept of *weighted grid*s which includes standard grids but enables to consider systems containing also associated couples of non-nil tripotents and nil tripotents as well. Given a subset W of some real vector space and a Jordan* triple E with triple product $\{\ldots\}$, by a *weighted grid* in E with *weight figure* W we mean an indexed system $\{g_w :\ w\in W\}\subset E$ consisting of linearly independent elements such that

$$\{g_ug_vg_w\} \in \mathrm{Span} g_{u-v+w}$$

whenever the parallelogram $\{u,v,w,u-v+w\}$ is contained in W and $\{g_ug_vg_w\} = 0$ else. It is implicitly established in [15, 3.7] that any standard grid $\{e_i :\ i\in I\}$ can be regarded as a weighted grid where for weight figure we can take the set $\big\{\pi_{\bullet k} :\ k\in I\big\}$ of the columns $\pi_{\bullet k} := \big(\pi_{ik}\big)_{i\in I}$ of the corresponding Peirce matrix which is linearly equivalent to the 1-part of some 3-graded root system. Keeping this example in mind, we mention some essential differences between the concepts of classical and weighted grids we are aimed to study more in details in this paper.

As soon as couples of pairwise associated tripotents occur in some weighted grid, the weight figure should necessarily contain infinite arithmetic sequences (while root systems admit arithmetic sequences of length at most 3). In contrast with the fact that, weighted grids of (positive) tripotents over an infinite arithmetic sequence span a (up to isomorphism) unique rather trivial Jordan* triple, in Section 6 we shall see that there are infinitely many pairwise non-isomorphic Jordan* triples spanned by weighted grids of positive tripotents over the weight figure $\mathbb{Z}^2$.

By multiplying the elements of a weighted grid with suitable positive constants, we may assume without loss of generality that its elements are positive, negative or nil tripotents (i.e. elements e such that $\{eee\} = \pm e, 0$). To our knowledge, signed tripotents were only considered in the classical grid theory of Hilbert triples [14, p.147]. However, Hilbert triples are hermitifiable by a positive definite inner product and hence the positive, negative and nil parts of their complete grids span orthogonal ideals, thus the presence of tripotents of different signs is relatively uninteresting in this case. As far as non-nil tripotents are concerned, there is often a shortcut way to a theory involving purely positive tripotents: from a non-nil weighted grid $G := \{g_w :\ w\in W\}$ in E we can pass to the set $\widetilde{G} := \{g_w \oplus (\mathrm{sgn(g_w)} g_w) :\ w\in W\}$

consisting of positive tripotents in $\widetilde{E} := E \oplus E$ equipped with a lifted triple product (3.1). In Section 3 we describe the precise condition for the distribution of the signs of the tripotents in G in terms of its weight figure W in order $\widetilde{G}$ be a weighted grid or which is the same $\mathrm{Span}(\widetilde{G})$ be a subtriple of $\widetilde{E}$. Actually this condition is fulfilled in the semisimple case where no associated couples appear in G. Therefore semisimple weighted grids can be constructed from standard grids of positive tripotents by the aid of a sign transformed triple product (investigated in more general context in Section 3) once we know the distribution of signs in terms of the weight figure. As it follows for grid theory of positive tripotents, in the semisimple case the weight figure is necessarily the affine image of the 1-part of some 3-graded root system. By an inspection of the geometry of 3-graded root systems, in Section 5 we show that $\mathrm{sgn}(g_w) = (-1)^{\langle \psi, w \rangle}$ $(w \in W)$ for a suitable linear functional ψ in the semisimple case with non-degenerate weight figure W.

Beyond the semisimple setting, in Section 4 we study the natural generalizations of elementary COG configurations [15, 2.1]. Except for parallelograms of four associated tripotents, the new ones turn out to be relatively harmless in the sense that, by Proposition 4.3, they generate subtriples whose structure can be derived by sign transformation from a (unique) triple spanned by positive tripotents. A major part of our paper will be occuped by the classification of non-nil weighted grids of associated tripotents over $\mathbb{Z}^2$ (Theorem 6.14). This classification provides infinitely many non-isomorphic weighted grids G without being $\mathrm{Span}(\widetilde{G})$ a subtriple in $\widetilde{E}$. Moreover (see Remark 6.16), as a limit object of Jordan* triples spanned by non-nil weighted grids we can also obtain a Jordan* triple with non-trivial triple product which is spanned by a weighted grid over $\mathbb{Z}^2$ consisting of nil tripotents. Thus even nil tripotents cannot simply be disregarded in weighted grid theory, though we investigate here merely *grid triples* i.e. Jordan* triples spanned by weighted grids of non-nil (but signed and possibly associated) tripotents.

Another chief aim of our paper is to give a self-contained description of the backgrounds in Lie representation theory of the concept of weighted grids. We restrict ourselves also here to the complex case mainly for the reason of being able to show the connections with the holomorphic geometry of symmetric manifolds and circular domains [6,5,1,16]. Our heuristic starting point in Section 2 is the observation (Theorem 2.4) that the weight spaces of abelian families of derivations with certain maximality properties, which we call *M-families* (for def. see 2.3), of a complex Jordan* triple are automatically 1-dimensional or trivial subtriples. Weighted grids turn out to be sets of joint eigenvectors of M-families in subtriples indexed with the carrying weights. A given set G may give rise to several weight figures that is to several different index systems in real vector spaces making G a weighted grid. It is remarkable that all the possible weight figures for a non-nil weighted grid G are the linear images of a universal one (called *non-degenerate weight figure* for G) which can be constructed by means of the derivations of the subtriple spanned by G. It is a challenge for later studies that, unlike in the semisimple case, the affine shape of the non-degenerate weight figure does not determine the structure of the spanned subtriple up to a plain sign transformation.

2. Weights and grids

For the sake of a simpler terminology, henceforth throughout this work, by a *Jordan* triple* we mean a a *complex* Jordan* triple i.e. a complex vector space E over the field $\mathbb{C}$ of complex numbers which is equipped with an operation $(x,y,z)\mapsto\{xyz\}$ of three variables such that the triple product $\{xyz\}$ is symmetric bilinear in its outer variables x,z, conjugate linear in the inner variable y and the commutators of the linear operators $a\square b:\ z\mapsto\{abz\}$ satisfy the Jordan identity

$$(J)\quad \begin{array}{l}[a\square b, x\square y]=\{abx\}\square y - x\square\{yab\}\quad \text{that is}\\ \{ab\{xyz\}\}=\{\{abx\}yz\}-\{x\{bay\}z\}+\{xy\{abz\}\}\qquad (a,b,x,y,z\in E)\ .\end{array}$$

This axiom means that the operator space $E\square E:=\mathrm{Span}\{a\square b:\ a,b\in E\}$ forms a *Lie subalgebra* in $\mathcal{L}(E)$ the space of all $\mathbb{C}$-linear operators $E\to E$.

A J^**-derivation* of a Jordan* triple E is an operator $D\in\mathcal{L}(E)$ such that

$$D\{xyz\}=\{(Dx)yz\}-\{x(Dy)z\}+\{xy(Dz)\}\qquad (x,y,z\in E)\ .$$

We shall write $\mathrm{Der}_*(E)$ for the $\mathbb{R}$-linear manifold of all J^*-derivations of the triple E. In particular, $a\square a\in\mathrm{Der}_*(E)\quad(a\in E)$. Moreover, by the Jordan identity,

$$(2.1)\qquad [D,a\square b]=(Da)\square b-a\square(Db)\qquad \big(a,b\in E\,,\ D\in\mathrm{Der}_*(E)\big)\ .$$

In general, if V is any vector space, $\mathcal{A}\subset\mathcal{L}(V)$ is a non-empty family of linear operators and $w:\mathcal{A}\to\mathbb{C}$, we denote the subspace of all *joint* $\mathcal{A}$*-eigenvectors* with eigenfunctional w by

$$V_w:=\big\{x\in V:\ Ax=w(A)x\quad (A\in\mathcal{A})\big\}\ .$$

The function $w:\mathcal{A}\to\mathbb{C}$ is called an $\mathcal{A}$*-weight* if $V_w\neq 0$. We use the notations

$$W(\mathcal{A}):=\{w(:\mathcal{A}\to\mathbb{C}):\ V_w\neq 0\}\ ,\quad W_{\mathbb{R}}(\mathcal{A}):=\{w\in W(\mathcal{A}):\ \mathrm{range}(w)\subset\mathbb{R}\}\ ,$$
$$V(\mathcal{A}):=\mathrm{Span}_{w\in W_{\mathbb{R}}(\mathcal{A})}V_w\ .$$

One of the basic tools in describing the geometry of weight figures is the following immediate consequence of the Jordan identity.

Lemma 2.2. *Let E be a Jordan* triple and $\emptyset\neq\mathcal{D}\subset\mathrm{Der}_*(E)$. Then the real $\mathcal{D}$-weights satisfy*

$$\{E_uE_vE_w\}\subset E_{u-v+w}\qquad (u,v,w\in W_{\mathbb{R}}(\mathcal{D}))\ .$$

In particular, $E(\mathcal{D})$ and all weight spaces E_w $(w\in W_{\mathbb{R}}(\mathcal{D}))$ are subtriples in E.

Definition 2.3. Let E be a Jordan*-triple and $\mathcal{D}$ a subset (not necessarily an $\mathbb{R}$-linear submanifold) in $\mathrm{Der}_*(E)$. We say that $\mathcal{D}$ is an *M-family* in $\mathrm{Der}_*(E)$ if for every $a\in E$, $a\square a\in\mathcal{D}$ whenever $\mathcal{D}a\subset\mathbb{R}a$.

Remark 2.4. By (2.1), maximal commutative subsets of $\{a\Box a:\ a\in E\}$ or $\mathrm{Span}_{\mathbb{R}}\{a\Box a:\ a\in E\}$ or $\mathrm{Der}_*(E)$ are M-families. Each M-family is included in some maximal abelian $\mathbb{R}$-linear subspace of $\mathrm{Der}_*(E)$.

Theorem 2.5. *Let E be a Jordan*-triple and $\mathcal{D}$ an M-family in $\mathrm{Der}_*(E)$. Then the weight spaces E_w with $w\in W_{\mathbb{R}}(\mathcal{D})$ are 1-dimensional or trivial subtriples (i.e. subtriples with vanishing triple product).*

Proof. Let $w:\mathcal{D}\to\mathbb{R}$ be an arbitrarily fixed weight. The linear extension $\widehat{w}:\widehat{\mathcal{D}}\to\mathbb{C}$ of w to $\widehat{\mathcal{D}}:=\mathrm{Span}(\mathcal{D})$ is well-defined and is a $\widehat{\mathcal{D}}$-weight with $E_w=\big\{a\in E:\ Da=\widehat{w}(D)a\ \ (D\in\widehat{\mathcal{D}})\big\}$. By assumption $a\Box a\in\mathcal{D}$ whenever $a\in E_w$. Moreover, since the mapping $(a,b)\mapsto a\Box b$ is sesquilinear and since E_w is a complex subspace, $4a\Box b=\sum_{\theta^4=1}\theta(a+\theta b)\Box(a+\theta b)\in\mathcal{D}\ \ (a,b\in E_w)$. However, then

$$\{abc\}=(a\Box b)c=\widehat{w}(a\Box b)c\qquad(a,b,c\in E_w)\ .$$

Hence we conclude $\dim(E_w)=1$ unless E_w is trivial. Indeed, if $a\in E$ with $0\neq\{aaa\}=w(a\Box a)a$ then $w(a\Box a)b=\{aab\}=\{baa\}=w(b\Box a)a$ for any $b\in E_w$.

Definition 2.6. An indexed set $G=\{g_w:\ w\in W\}$ in E is called a *weighted grid* with *weight figure* W if G is linearly independent and closed under the triple product in the following sense [a] More strictly, by the term the *indexed set* $G:=\{g_w:\ w\in W\}$ we mean a bijection $w\mapsto g_w$ between set W of indices and the collection of the elements of G. Without danger of confusion, we refer with G also to the range of the map $w\mapsto g_w$. By saying W is a *weight figure for* G we mean the existence of a bijection $w\mapsto g_w$ of W and G making G into a weighted grid.:

$$\{g_ug_vg_w\}\in\mathbb{C}g_{u-v+w}\quad(u,v,w,u{-}v{+}w\in W),\qquad\{g_ug_vg_w\}{=}0\quad(u,v,w,u{-}v{+}w\not\in W).$$

An element $0\neq e\in E$ is a *positive* [resp. *negative, nil*] *tripotent* if $\{eee\}=\varepsilon e$ for $\varepsilon=1$[resp. $0,-1$]. We call the value $\mathrm{sgn}(e):=[\varepsilon:\ \{eee\}=\varepsilon e]$ the *sign* of the tripotent e.

Corollary 2.7. *Let $\mathcal{D}$ be a maximal commutative subalgebra in $\mathrm{Der}_*(E)$. If each nil weight space of $\mathcal{D}$ is 1-dimensional then any basis $G=\{g_w:\ w\in W(\mathcal{D})\}$ of the subtriple $E(W(\mathcal{D})$ with $g_w\in E_w\ \ (w\in W(\mathcal{D}))$ is a weighted grid consisting of non-zero multiples of tripotents.*

It is well-known [2] that for any fixed $c\in E$ the Jordan* triple $\big(E,\{\ldots\}\big)$ becomes a commutative Jordan algebra when equipped with the c-product $x\bullet^c y:=\{xcy\}$. Moreover, we have the following expressions (direct proof see e.g. [4, p.263]) for the c-multiplication operators $R_c(a):=a\Box c$,

$$R_c(\{aca\})R_c(a)=\frac{2}{3}R_c(a)^3+\frac{1}{3}R_c(\{ac\{aca\}\})\qquad(a,c\in E)\ .$$

[a] *

In particular if $\{eee\} = \lambda e$ we get

$$(e\,\Box\, e)(e\,\Box\, e - \lambda/2\,\mathrm{id}_E)(e\,\Box\, e - \lambda \mathrm{id}_E) = 0\ .$$

Hence the elements of a weighted grid $G = \{g_w :\ w \in W\}$ have the following Peirce compatibility property:

$$(g_u \,\Box\, g_u) g_v \in \{0, \lambda_u/2, \lambda_u\} g_v \qquad (u, v \in W)$$

with the coefficients $\lambda_u := [\lambda :\ \{g_u g_u g_u\} = \lambda g_u]$. These latter are necessarily real for the following more general reason.

Lemma 2.8. *Elements with* $\{eee\} = \lambda e \neq 0$ *can only belong to eigensubspaces with real eigenvalues of* J^* *-derivations.*

Proof. Let $D \in \mathrm{Der}_*(E)$ with $De = \alpha e$. Then $0 = D[\{eee\} - \lambda e] = 2\{(De)ee\} - \{e(De)e\} - \alpha\lambda e = (2\alpha - \overline{\alpha} - \alpha)\lambda e$. Thus $\alpha - \overline{\alpha} = 0$ if $\lambda \neq 0$.

Corollary 2.9. *Weighted grids consist of multiples of tripotents.*

Theorem 2.10. *Let* G *be a family of non-nil tripotents such that the set* $\mathbb{C}G$ *is closed under the triple product. Then* G *can be equipped with the structure of a weighted grid if and only if*

$$\mathcal{D}_G := \{D \in \mathrm{Der}_*(F) :\ Dg_w \in \mathbb{R} g_w \quad (w \in W)\}$$

is a maximal abelian family in $\mathrm{Der}_*(F)$ *for the subtriple* $F := \mathrm{Span}(G)$ *. If* $G = \{g_w :\ w \in W\}$ *is a weighted grid then its weight figure* W *is a linear image of* $W(\mathcal{D}_G)$ *and* W *is linearly isomorphic to* $W(\mathcal{D}_G)$ *if and only if any* J^**-derivation* $D \in \mathcal{D}_G$ *has the form* $Dg_w = \phi(w) g_w \quad (w \in W)$ *for a suitable linear functional* $\phi : \mathrm{Span}_{\mathbb{R}} W \to \mathbb{R}$ *.*

Proof. If $\mathcal{D}_G$ is a maximal commutative family in $\mathrm{Der}_*(F)$ then, by Theorem 2.5, we can regard G as the weighted grid with the indexing $G = \{f_w :\ w \in W_{\mathbb{R}}(\mathcal{D}_G)\}$ where $f_u := [g \in G :\ Dg = w(D)g] \quad (w \in W_{\mathbb{R}}(\mathcal{D}_G))$.

Assume $G = \{g_w :\ w \in W\}$ is a weighted grid of non-nil tripotents. Consider the factor space

$$\widehat{U} := \mathrm{Span}_{\mathbb{R}} G/F_0\ , \qquad F_0 := \mathrm{Span}_{\mathbb{R}}\big\{g_u - g_v + g_w - g_{u-v+w} :\ \{g_u g_v g_w\} \neq 0\big\}$$

and let $U := \mathrm{Span}_{\mathbb{R}}(W)$. Since the vectors $g_w \quad (w \in W)$ are linearly independent,

$$w = P(g_w) \qquad\qquad (w \in W)$$

for some $\mathbb{R}$-linear $P : F \to U$. Trivially $P(g_u - g_v + g_w) = g_u - g_v + g_w = g_{u-v+w} = P g_{u-v+w}$ whenever $\{g_u g_v g_w\} \neq 0$. Since

$$F_0 = \mathrm{Span}_{\mathbb{R}}\big\{g_u - g_v + g_w - g_{u-v+w} :\ \{g_u g_v g_w\} \neq 0\big\}\ ,$$

we have $P(F_0) = 0$. Hence the factor mapping

$$\widehat{P} := P/F_0 : (\textstyle\sum_w \alpha_w g_w) + F_0 \mapsto \sum_w \alpha_w g_w$$

is well defined on $\widehat{U}$ and $\widehat{P}(\widehat{W}) = W$ for $\widehat{W} := \{g_w + F_0 :\ w \in W\}$. Observe that $\widehat{W}$ can be regarded as a weight figure for G in the sense that $\{h_{\widehat{w}} :\ \widehat{w} \in \widehat{W}\}$ is a weighted grid isomorphic to $\{g_w :\ w \in W\}$ where $h_{g_w+F_0} := g_w \ \ (w \in W)$.

Suppose $\mathcal{D}_G = \{D_\phi :\ \phi \in \mathcal{L}(\mathrm{Span}(W), \mathbb{R})\}$ where $D_\phi := \big[D \in \mathcal{L}(F) :\ Dg_w = \phi(w)g_w \ \ (w \in W)\big]$. We show that $\widehat{W}$ is a linear image of W in this case.

Indeed, the correspondence $\{w \mapsto g_w + F_0\}$ has an $\mathbb{R}$-linear extension if and only if $\sum_{w\in W} \alpha_w g_w + F_0 = F_0$ i.e. if $\widehat{\phi}\big(\sum_{w\in W} \alpha_w g_w + F_0\big) = 0 \ \ (\widehat{\phi} \in \mathcal{L}(\widehat{U}, \mathbb{R}))$ whenever $\sum_{w\in W} \alpha_w w = 0$ with $\alpha_w \in \mathbb{R} \ \ (w \in W)$. Let $\{\alpha_w :\ w \in W\}$ a system of coefficients such that $\sum_{w\in W} \alpha_w w = 0$. Then $\phi\big(\sum_{w\in W} \alpha_w w\big) = 0$ and hence $D_\phi \sum_{w\in W} \alpha_w g_w = 0$ for all $\phi \in \mathcal{L}(\mathrm{Span(W)}, \mathbb{R})$. Thus, by assumption, $D \sum_{w\in W} \alpha_w g_w = 0 \ \ (D \in \mathcal{D}_G)$. Since also $\mathcal{D}_G = \{D_{\widehat{\phi}} :\ \widehat{\phi} \in \mathcal{L}(\widehat{U}, \mathbb{R})\}$ where $D_{\widehat{\phi}} := \big[D \in \mathcal{L}(F) :\ Dg_w = \widehat{\phi}(g_w + F_0)g_w \ \ (w \in W)\big]$, we have $D_{\widehat{\phi}} \sum_{w\in W} \alpha_w g_w = 0$ i.e. $\sum_{w\in W} \alpha_w \widehat{\phi}(g_w + F_0) = 0 \ \ (\widehat{\phi} \in \mathcal{L}(\widehat{U}, \mathbb{R})$ Thus $\sum_{w\in W} \alpha_w (g_w + F_0) = 0$ in $\widehat{U}$ what we had to prove.

We complete the proof of the theorem with the following remark. For each derivation $D \in \mathcal{D}_G$ the evaluation mapping $\delta_D : u \mapsto u(D)$ is $\mathbb{R}$-linear $\mathrm{Span}_{\mathbb{R}} W(\mathcal{D}_G \to \mathbb{R}$ such that $Dg = \delta_D(w_g)g \ \ (g \in G)$ where $w_g \in \mathcal{L}(\mathcal{D}_G, \mathbb{R})$ denotes the weight $D \mapsto [\alpha \in \mathbb{R} :\ Dg = \alpha g]$. Hence $\widehat{W}$ is a linear image of $W(\mathcal{D}_G)$, too.

Definition 2.11. Henceforth throughout the whole work we assume (without loss of generality) all weighted grids considered consist of positive negative or nil tripotents. We say that the weight figure W of the weighted grid $G = \{g_w :\ w \in W\}$ is *non-degenerate* if for any $D \in \mathcal{D}_G (:= \{D \in \mathrm{Der}_*(\mathrm{Span}_{\mathbb{C}} G) :\ Dg_w \in \mathbb{R} g_w \ (w \in W)\})$ there exists a linear functional $\phi : \mathrm{Span}_{\mathbb{R}} W \to \mathbb{R}$ such that $Dg_w = \phi(w)g_w$. We shall use the term *non-nil weighted grid* for weighted grids of non-nil tripotents. Two non-nil weighted grids $\{h_w :\ w \in W\}$ and $\{g_w :\ w \in W\}$ are said to be *equivalent* if $h_w \in \mathbb{T}\, g_w \ \ (w \in W)$ with the standard notation $\mathbb{T} := \{\tau \in \mathbb{C} : |\tau| = 1\}$. We shall call Jordan* triples spanned by non-nil weighted grids shortly *grid triples* .

Remark 2.12. Theorem 2.10 establishes the existence of non-degenerate weight figures for any weighted grid.

Given a weighted grid $G := \{g_w :\ w \in W\}$, a linear mapping D with $Dg_w = \lambda_w g_w,\ \lambda_w \in \mathbb{R} \ \ (w \in W)$ belongs to $\mathcal{D}_G$ if and only if

$$\lambda_u - \lambda_v + \lambda_w = \lambda_{u-v+w} \quad (u, v, w, u - v + w \in W,\ \{g_u g_v g_w\} \neq 0).$$

Therefore if W is a non-degenerate weight figure for G then any mapping $L_0 : W \to H$ with the property

$$L_0(u) - L_0(v) + L_0(w) = L_0(u - v + w) \quad (u, v, w, u - v + w \in W,\ \{g_u g_v g_w\} \neq 0)$$

extends linearly to $\mathrm{Span}_{\mathbb{R}} W$. In particular any other weight figure of G is the linear image of any non-degenerate weight figure of G. Furthermore, the intersection of a non-degenerate weight figure W of $G=\{g_w:\ w\in W\}$ with an affine subspace U is also a non-degenerate weight figure (for $\{g_w:\ w\in W\cap U\}$).

Example 2.13. We say that E is a *Lorentz triple* with splitting S if E is equipped with a complex Hilbert space structure by a scalar product $\langle\ ,\ \rangle$ with respect to which $S\in\mathcal{L}(E)$ is an orthogonal reflection (i.e. $S=S^*,\ S^2=1$) such that for each eigenvector $e\neq 0$ of the reflection S we have $e^3(=\{eee\})\neq 0$ and the operator $e\,\square\, e$ is non-negative with respect to the indefinite inner product $\langle x,y\rangle^S:=\langle Sx,y\rangle$ (i.e. $\langle S(e\,\square\, e)x,x\rangle\geq 0$ whenever $Se=\pm e$ and $x\in E$).

Let E be a finite dimensional Lorentz triple with splitting S. Choose a maximal Abelian family $\mathcal{D}$ of $\{e\,\square\, e\ :\ Se\in\{\pm e\}\}$. Consider any weight vector $a\in E_w$ where $w:\mathcal{D}\to\mathbb{R}$. Given any $e\in E$ with $e\,\square\, e\in\mathcal{D}$, $0\leq\langle S(e\,\square\, e)x,x\rangle=\langle x,S(e\,\square\, e)x\rangle\ \ (x\in E)$. This means $S(e\,\square\, e)=[S(e\,\square\, e)]^*=(e\,\square\, e)^*S$, whence $e\,\square\, e=S^2e\,\square\, e=(e\,\square\, e)^*S^2=(e\,\square\, e)^*$ and $S(e\,\square\, e)=(e\,\square\, e)S$ whenever $e\,\square\, e\in\mathcal{D}$. Therefore the weight space E_w is invariant by S. In particular $a_1,a_2\in E_w$ where $a_k:=\frac{1}{2}[a+(-1)^kSa]$. Since $Sa_k=(-1)^ka_k$ and, by (2.1), $[D,a_k\,\square\, a_k]=[w(D)-\overline{w(D)}]a_k\,\square\, a_k=0$ for $D\in\mathcal{D}$, necessarily $a_k\,\square\, a_k\in\mathcal{D}$. On the other hand, $0\leq\langle S(a_k\,\square\, a_k)a_\ell,a_\ell\rangle=\langle(a_k\,\square\, a_k)a_\ell,Sa_\ell\rangle=(-1)^\ell w(a_k\,\square\, a_k)\langle a_\ell,a_\ell\rangle$ $(k,\ell=1,2)$. This is possible only if $w(a_1\,\square\, a_1)=0$ or $w(a_2\,\square\, a_2)=0$. By definition, $0\neq a_k=(-1)^kSa_k$ implies $0\neq a_k^3=w(a_k\,\square\, a_k)a_k\ \ (k=1,2)$. Therefore $a_1=0$ or $a_2=0$ whence $a=a_k$ and $a\,\square\, a=a_k\,\square\, a_k\in\mathcal{D}$ with $k=1$ or $k=2$. Consequently $\mathcal{D}$ is an abelian M-family consisting of $\langle\ ,\ \rangle$-self-adjoint operators. In particular, since $\dim(E)<\infty$, we have $E=\oplus_{w\in W_{\mathbb{R}}(\mathcal{D})}E_w$. By Theorem 2.5 and its corollary, the summands E_w consist of multiples of $\{\pm 1\}$-tripotents. Thus we got the following description.

Any finite dimensional Lorentz triple is spanned by a weighted grid consisting of $\{\pm 1\}$ *-tripotents which are pairwise orthogonal eigenvectors of the splitting reflection with respect to the underlying inner product.*

Remark 2.14. Non-degenerate Hilbert triples in the sense of [14] are Lorentz triples with the trivial splitting $S=1$. Any Hilbert triple is the orthogonal direct sum of ideals spanned by $\{\pm 1,0\}$-tripotents with the same sign. In general this is not the case for Lorentz triples. The complex Lorentz 2-space $H^{(1,1)}$ is $\mathbb{C}^2$ with the indefinite inner product $\langle\ ,\ \rangle^S$ where $Sx:=(-x_1,x_2)$ and $\langle x,y\rangle:=x_1\overline{y_1}-x_2\overline{y_2}$. The operation $\{xyz\}:=\frac{1}{2}\langle x,y\rangle^Sz+\frac{1}{2}\langle^Sz,y\rangle x$ makes $H^{(1,1)}$ a Lorentz triple with splitting S. This triple has only trivial ideals, as an easy consequence of the fact that the unit vectors $e^1:=(1,0)$, $e^2:=(0,1)$ are tripotents of opposite signs forming a weighted grid over the non-degenerate weight figure $\{1,2\}$ (as subset of $\mathbb{R}$) such that $\{e^ke^ke^\ell\}\neq 0\ \ (k,\ell=1,2)$.

Example 2.15. Consider the space $T(\mathbb{R}):=\mathrm{Span}_{n\in\mathbb{Z}}\,\chi^n$ of all complex trigonometric polynomials on $\mathbb{R}$ where χ^n is the function $\chi^n(\theta):=\mathrm{e}^{in\vartheta}\ \ \vartheta\in\mathbb{R})$. The triple product $\{fgh\}:=f\overline{g}h$ makes $T(\mathbb{R})$ a Jordan* triple. Each character χ^n is a positive tripotent and *the basis* $B := \{\chi^n : n \in \mathbb{Z}\}$ *is a non-nil weighted grid of* $T(\mathbb{R})$ *over the non-degenerate weight figure* $\mathbb{Z} \oplus 1$ *(as subset of* $\mathbb{R} \oplus \mathbb{R}$*).* Indeed $\{\chi^k \chi^\ell \chi^m\} = \chi^{k-\ell+m}$ $(k,\ell,m \in \mathbb{Z})$. The weight figure W of any other weighted grid structure $B = \{b_w : w \in W\}$ can be written as $W = \{w(k) : k \in \mathbb{Z}\}$ where $b_{w(k)} = \chi^k$. Necessarily $w(k-\ell+m) = w(k) - w(\ell) + w(m)$ $(k,\ell,m \in \mathbb{Z})$, whence W is an affine image of $\mathbb{Z}$ and a linear image of $\mathbb{Z} \oplus 1$.

Notice that the operator $D := d/d\vartheta$ is a J^*-derivation on $T(\mathbb{R})$ with $D\chi^n = n\chi^n$ $(n \in \mathbb{Z})$ which is not inner in the sense that $D \notin \mathrm{Span}_{k,\ell} \chi^k \square \chi^\ell$.

Remark 2.16. It is natural to ask if alone the closeness of $\mathbb{C}G$ under the triple product entails automatically the existence of a weighted grid structure on G in the previous theorem. The answer is negative:
Let G be a real orthonormed basis in a finite dimensional spin factor E. Then $\{abc\} = \big[c\ (a=b);\ 0\ \ (a \neq b \neq c \neq a),\ \ -b\ (a = c \neq b)\big]$ $(a,b,c \in G)$. Thus G is a family of (equivalent positive) tripotents and $\mathbb{C}G$ is closed under the triple product. However, G cannot carry the structure $G = \{g_w : w \in W\}$ of a weighted grid. Namely, in the latter case $u \neq v$ would imply $\{g_u g_v g_u\} = -g_v$ and hence the contradiction $u - v + u = v$.

3. Sign transformations

Next we investigate how the triple product on weighted grids with signed tripotents can be retrieved from that of classical Jordan* triples admitting only positive tripotents. Throughout the whole section E is an arbitrarily fixed Jordan* triple with the triple product $\{\ldots\}$.

According to [9, 1.14 and 1.15], $\widetilde{E} := E \oplus E$ with the twisted triple product

$$\{(x_1 \oplus x_2)(y_1 \oplus y_2)(z_1 \oplus z_2)\} := \{x_1 y_2 z_1\} \oplus \{x_2 y_1 z_2\} \tag{3.1}$$

becomes a Jordan* triple such that for any (signed) tripotent $g \in E$,

$$\widetilde{g}^3 = \widetilde{g} \qquad \text{where} \quad \widetilde{g} := g \oplus (\mathrm{sgn}(g) g) .$$

Proposition 3.2. *Suppose* $S : E \to E$ *is a linear mapping. Then, by writing* $\mathcal{N}(E) := \{e \in E : \{xex\} = 0 \ \ (x \in E)\}$,

(i) $\widetilde{H} := \{x \oplus (Sx) : x \in E\}$ *is a subtriple of* $\widetilde{E}$ *if and only if* $S\{x(Sy)z\} = \{(Sx)y(Sz)\}$ $(x,y,z \in E)$*;*

(ii) $[xyz] := \{x(Sy)z\}$ $(x,y,z \in E)$ *is Jordan* triple-product on* E *if and only if* $S\{x(Sy)z\} - \{(Sx)y(Sz)\} \in \mathcal{N}(E)$ $(x,y,z \in E)$.

Proof. (i) is straightforward. To see (ii), observe that the Jordan identity for $[\ldots]$ can be stated in terms of S as

$$\{a(Sb)\{x(Sy)x\}\} = 2\{\{a(Sb)x\}(Sy)x\} - \{x(S\{b(Sa)y\})x\} .$$

Extracting the left hand side by the Jordan identity of $\{\dots\}$, we get

$$\{a(Sb)\{x(Sy)x\}\} = 2\{\{a(Sb)x\}(Sy)x\} - \{x\{(Sb)a(Sy)\}x\} .$$

A comparison of both right hand sides yields (ii).

Definition 3.3. For an involution automorphism S of E we call the triple product $\{\dots\}_S$ defined by

$$\{xyz\}_S := \{x(Sy)z\} \qquad (x,y,z \in F)$$

the *sign transformation* of $\{\dots\}$ by means of S.

Remark 3.4. 1) By the equivalence in (ii), the operation $\{\dots\}_S$ is indeed a Jordan* triple-product if $S^2 = \mathrm{id}$ and $S\{xyz\} = \{(Sx)(Sy)(Sz)\}$ $(x,y,z \in E)$. The background of the terminology "sign transformation" is the fact that given any tripotent $e \in E$ with $Se = \varepsilon e$, $\{eee\}_S = \varepsilon \mathrm{sgn}(e)e$ that is e is a tripotent wrt. the triple product $\{\dots\}_S$ and its sign is the ε-multiple of that wrt. $\{\dots\}$.

2) An involution automorphism T of $(E,\{\dots\})$ is an involution automorphism of $(E,\{\dots\}_S)$ for any involution automorphism S of $(E,\{\dots\})$ commuting with T with $\{\dots\}_{ST} = (\{\dots\}_S)_T$.

3) It is straightforward to see that a linear operator $S : F \to F$ with $Sg_w = \varepsilon_w g_w$, $\varepsilon_w \in \{\pm 1\}$ $(w \in W)$, is a Jordan* triple-automorphism of F if and only if

(S) $\qquad \varepsilon_{u-v+w} = \varepsilon_u \varepsilon_v \varepsilon_w \qquad$ whenever $\{g_u g_v g_w\} \neq 0 \qquad (u,v,w \in W)$.

Notice that this formula is independent of the signs of the elements of G.

4) $\mathcal{N}(\mathrm{Span}(G)) = 0$ if G is a non-nil weighted grid . Indeed, given $e := \sum_v \alpha_v g_v$ with $\alpha_u \neq 0$, for suitable coefficients $\gamma_v \in \mathbb{C}$ we have $\{g_u e g_u\} = \sum_v \overline{\alpha_v} \gamma_v g_{2u-v}$ where $\gamma_u = [\gamma : \{g_u g_u g_u\} = \gamma g_u] \neq 0$.

Corollary 3.5. *Let* $G := \{g_w : w \in W\}$ *be a non-nil weighted grid and let* $\varepsilon : W \to \{\pm 1\}$. *Then there exists a triple product* $[\dots]$ *on the subtriple* $\mathrm{Span}(G)$ *such that* $[g_u g_v g_w] = \varepsilon_v \{g_u g_v g_w\}$ $(u,v,w \in W)$ *if and only if the sign condition* (S) *holds. In particular the triple product* $\{\dots\}$ *on* $\mathrm{Span}(G)$ *is the sign transformed form of some triple product* $[\dots]$ *such that* $\mathrm{sgn}_{[\dots]}(g_w) = 1$ $(w \in W)$ *if and only if*

$$\prod_{k=1}^4 \mathrm{sgn}(g_w) = 1 \text{ if } u_1,u_2,u_3 \in W,\ u_4 = u_1 - u_2 + u_3,\ \{g_{u_1} g_{u_2} g_{u_3}\} \neq 0 .$$

Remark 3.6. In the context of 3.5, $\widetilde{G} := \{\widetilde{g_w} : w \in W\}$ is a weighted grid in $\widetilde{E}$ if and only if the linear extension S of the mapping $g_w \mapsto \mathrm{sgn}(g_w)g_w$ is an involution automorphism of the subtriple F. In the latter case F equipped with the triple product $\{\dots\}_S$ is isomorphic to $\mathrm{Span}_{\mathbb{C}} \widetilde{G}$ (with the triple product of $\widetilde{E}$) by the first coordinate projection.

Example 3.7. The structure of a subtriple generated by two associated non-nil tripotents in a non-nil weighted grid can be described in terms of a sign transformed form of the triple 2.14 of trigonometric polynomials as follows.

Theorem 3.8. *Assume* $h_0, h_1 \in E$ *with* $(h_k \square h_k)h_\ell = \mathrm{sgn}(h_k)h_\ell \quad (k, \ell \in \{0,1\})$. *Then there exists a (unique) homomorphism* Φ *of* $\big(T(\mathbb{R}), \{\ldots\}_S\big)$ *onto the subtriple* F *generated by* h_0, h_1 *such that* $\Phi(\chi^k) = h_k \quad (k = 0, 1)$ *where* S *is the linear extension of the map* $\chi^n \mapsto \mathrm{sgn}(h_0)[\mathrm{sgn}(h_0)\mathrm{sgn}(h_1)]^n \chi^n$.

Proof. We may assume without loss of generality that E is generated by $\{h_0, h_1\}$. Set $a_0 := \widetilde{h_0}$, $a_1 := \widetilde{h_1}$ and define recursively

$$a_{n+1} := \{a_n a_{n-1} a_n\} \quad (n > 1), \quad a_{n-1} := \{a_n a_{n+1} a_n\} \quad (n < 0)$$

in $\widetilde{E}$. By [14, I.4.4], $\{\widetilde{g}\widetilde{h}\widetilde{g}\}$ is a positive tripotent associated with both $\widetilde{g}$ and $\widetilde{h}$ whenever $\widetilde{g} \approx \widetilde{h}$ are positive tripotents in $\widetilde{E}$. Hence we see by induction on n , that a_n is a positive tripotent associated with a_{n-1} and a_{n+1} for every $n \in \mathbb{Z}$. That is, for all $n \in \mathbb{Z}$ we have $a_k \square a_k a_\ell = a_\ell$ if $n-1 \le k, \ell \le n+1$. Thus by the Jordan identity, also $a_{n+1} = \{a_n a_{n-1} a_n\}$ and $a_{n-1} = \{a_n a_{n+1} a_n\}$ for all $n \in \mathbb{Z}$. Therefore

$$\begin{aligned}(a_n \square a_n)a_{t\pm 1} &= 2\{\{a_n a_n a_t\}a_{t\mp 1} a_t\} - \{a_t\{a_n a_n a_{t\mp 1}\}a_t\} = \\ &= a_{t\pm 1} \qquad \text{if } (a_n \square a_n)a_t = a_t,\ (a_n \square a_n)a_{t\mp 1} = a_{t\mp 1}.\end{aligned}$$

Hence $(a_n \square a_n)a_{n+t} = a_{n+t} \quad (n, t \in \mathbb{Z})$ by induction on t . On the other hand

$$\begin{aligned}a_{n\pm 1} \square a_n &= \{a_n a_{n\mp 1} a_n\} \square)a_n = [a_n \square a_{n\mp 1}, a_n \square a_n] + a_n \square \{a_n a_n a_{n\mp 1}\} = \\ &= -[a_n \square a_n, a_{n\mp 1} \square a_n] = \\ &= -\{a_n a_n a_n\} \square a_{n\mp 1} + a_n \square \{a_{n\mp 1} a_n a_{n\mp 1}\} = a_n \square a_{n\mp 1} \ .\end{aligned}$$

Thus we can conclude

$$a_n \square a_{n-1} = a_1 \square a_0 \ , \quad a_{n-1} \square a_n = a_0 \square a_1 \ , \quad a_n \square a_n = a_0 \square a_0 \qquad (n \in \mathbb{Z}) \ .$$

It follows $\{a_k a_\ell a_m\} = a_{k-\ell+m}$ for any $k, \ell, m \in \mathbb{Z}$ with $|k - \ell| \le 1$. Hence we can prove at once

$$\{a_k a_\ell a_m\} = a_{k-\ell+m} \qquad (k, \ell, m \in \mathbb{Z}) \tag{3.9}$$

by induction on $|k - \ell|$. Suppose (3.9) holds whenever $|k - \ell| \le n$. If $k, \ell, m \in \mathbb{Z}$ and $\ell \le k \le \ell + n \le n$ then

$$\begin{aligned}\sigma\varepsilon^{k-1}\{a_{k+1} \quad a_\ell a_m\} &= \{\{a_k a_{k-1} a_k\}a_\ell a_m\} = \\ &= \{a_k a_{k-1}\{a_k a_\ell a_m\}\} + \{a_k\{a_{k-1} a_k a_\ell\}a_m\} - \{a_k a_\ell\{a_k a_{k-1} a_m\}\} = \\ &= \sigma\varepsilon^\ell\{a_k a_{k-1} a_{k-\ell+m}\} + \sigma\varepsilon^k\{a_k a_{\ell-1} a_m\} - \sigma\varepsilon^{k-1}\{a_k a_\ell a_{m+1}\} = \\ &= \varepsilon^{\ell+k-1} a_{k-\ell+m+1} + \sigma\varepsilon^k\{a_k a_{\ell-1} a_m\} - \varepsilon^{k-1+\ell} a_{k-\ell+m+1} = \\ &= \sigma\varepsilon^k\{a_k a_{\ell-1} a_m\} \ .\end{aligned}$$

Similarly $\{a_k a_{\ell+1} a_m\} = \varepsilon\{a_{k-1} a_\ell a_m\}$ $(k \le \ell \le k+n)$. Therefore, given any fixed $k, \ell, m \in \mathbb{Z}$ with $|k-\ell| \le n+1$, we have

$$\{a_{k+t} a_{\ell+t} a_m\} = \{a_k a_\ell a_m\} \qquad (t \in \mathbb{Z}) \ .$$

With $t := m - \ell$, we get

$$\begin{aligned}\{a_k a_\ell a_m\} = \{a_{k+\ell-m} a_m a_m\} = (a_m \,\square\, a_m) a_{k+\ell-m} = \\ = a_{k-\ell+m} \qquad (|k-\ell| = n+1)\end{aligned}$$

which completes the induction argument.

Define $h_n := P a_n$ $(0, 1 \ne n \in \mathbb{Z})$. Notice that also $h_0 = P a_0$ and $h_1 = P a_1$. The relation $a_{n+1} = \{a_n a_{n-1} a_n\}$ means

$$h_{n+1} \oplus [\mathrm{sgn}(h_{n+1}] = [\mathrm{sgn}(h_{n-1})\{h_n h_{n-1} h_n\}] \oplus \{h_n h_{n-1} h_n\}$$

whence $\mathrm{sgn}(h_{n-1}) = \mathrm{sgn}(h_{n+1})$ $(n \in \mathbb{Z})$. That is $\mathrm{sgn}(h_n) = \sigma \varepsilon^n$ $(n \in \mathbb{Z})$ with $\sigma := \mathrm{sgn}(h_0)$ and $\varepsilon := \mathrm{sgn}(h_1)\mathrm{sgn}(h_0)$. Substituting this into (3.9) we get

$$\{h_k h_\ell h_m\} = \sigma \varepsilon^\ell h_{k-\ell+m} \qquad (k, \ell, m \in \mathbb{Z}) \ .$$

The function $n \mapsto \sigma\varepsilon^n$ satisfies the sign condition (S) on the weight figure $W := \mathbb{Z}$ of the non-nil weighted grid $\{\chi^n : \ n \in \mathbb{Z}\}$ of $T(\mathbb{R})$. Hence the linear extension S of $\chi^n \mapsto \sigma\varepsilon^n \chi^n$ is an involution automorphism of $T(\mathbb{R})$ giving rise to the triple product $\{\ldots\}_S$. Clearly $\{\chi^k \chi^\ell \chi^m\}_S = \sigma\varepsilon^\ell \chi^{k-\ell+m}$ $(k, \ell, m \in \mathbb{Z})$ which completes the proof.

4. Basic configurations

We apply the theory of elementary COG configurations developed in [14, Ch.I] (and [15, 2.5] (with an abstract treatment) to the setting of weighted grids of non-nil tripotents. As in the previous section, E is a Jordan* triple, $\widetilde{E} := E \oplus E$ with the triple product (3.1) and for a tripotent $g \in E$ we write

$$\widetilde{g} := g \oplus [\mathrm{sgn}(g) g] \ .$$

Throughout the whole section $G := \{g_w : \ w \in W\}$ denotes a weighted grid in E . It is straightforward to see that two non-nil tripotents $g, h \in E$ are eigenvectors for both $g \square g$ and $h \square h$ iff the pair $\widetilde{g}, \widetilde{h}$ has the same property. with the modules of the respective eigenvalues. In particular $(g_u \square g_u) g_v \in \mathbb{R} g_v$ $(u, v \in W)$, and hence $\widetilde{G}$ consists of pairwise Peirce compatible [14, Ch. I] positive tripotents. Therefore there exists a matrix $\big(\pi_{uv}\big)_{u,v \in W}$ with entries in $\{0, 1, 2\}$ which we call the *Peirce matrix* of G such that

$$(g_u \square g_u) g_v = \frac{1}{2} \mathrm{sgn}(g_u) \pi_{uv} g_v \qquad (u, v \in W) \ .$$

On W we introduce the COG relations [15, 3.0] by means of those on $\widetilde{G}$:

$$u \top v \quad \text{if} \ \ \pi_{uv} = \pi_{vu} = 1 \ , \qquad u \perp v \quad \text{if} \ \ \pi_{uv} = \pi_{vu} = 0 \ ,$$
$$u \approx v \quad \text{if} \ \ \pi_{uv} = \pi_{vu} = 2 \ , \qquad u \vdash v \quad \text{if} \ \ \pi_{uv} = 2 \ \text{ and } \ \pi_{vu} = 1 \ .$$

By writing also $u \dashv v$ for $v \vdash u$, it is well-known that for any pair $u, v \in W$ we have uRv with exactly one of the relations $R := \top, \perp, \vdash, \dashv, \approx$.

Modifying slightly the notation of [15, 3.1], we call a tuple $(w_1, \ldots, w_4)$ a *triangle* if $w_1 = w_3 \vdash w_2 \dashv w_4 \perp w_1$, *quadrangle* if $w_1 \top w_2 \top w_3 \perp w_1 \top w_4 \top w_3 \top w_2 \perp w_4$, *diamond* if $w_1 \vdash w_2 \dashv w_3 \top w_1 \vdash w_4 \dashv w_3 \vdash w_2 \perp w_4$. Triangles, quadrangles and diamonds are referred as *elementary COG configurations.* A tuple (w_1, w_2, w_3) is an *incomplete* elementary configuration if (w_1, w_2, w_3, w_4) is a basic configuration for some $w_4 \in W$. In the sequel we shall write

$$u^* := \big[t(\in W) \mapsto \pi_{tu}\big] \qquad (u \in W)$$

for the column vectors of the Peirce matrix. By the Jordan identity (J) (applied with $a := b := g_t$, $x := g_{u_1}$, $y := g_{u_2}$, $z := g_{u_3}$) we have

$$\text{(J}'\text{)} \qquad u_1^* - u_2^* + u_3^* = (u_1 - u_2 + u_3)^* \qquad \text{if } u_1, u_2, u_3, u_1 - u_2 + u_3 \in W .$$

These vectors distinguish only non-associated points:

$$\text{(4.1)} \qquad u_n := u + n(v - u) \in W \text{ with } u_n^* = u^* \text{ for all } n \in \mathbb{Z} \qquad \text{iff} \qquad u \approx v .$$

Indeed, by Theorem 3.8 we have $u_n \in W$ for any $n \in \mathbb{Z}$ and, taking (J$'$) into account, the sequence $\{u_n^* : n \in \mathbb{Z}\}$ is arithmetic. Since $u_n^*(t) \in \{0, 1, 2\}$ $(n \in \mathbb{Z})$ for any fixed $t \in W$, this is possible only if all the terms u_n^* are the same. The converse implication is trivial since $\pi_{u_n u_n} = 2$ $(n \in \mathbb{Z})$.

Definition 4.2. A tuple $(u_1, u_2, u_3, u_4) \in W^4$ is a *basic configuration* for the non-nil weighted grid $\{g_w : w \in W\}$ if $\{u_1 u_2 u_3\} \neq 0$ and $u_4 = u_1 - u_2 + u_3$.

Remark 4.3. By the Jordan identity $u_4^* = u_1^* - u_2^* + u_3^*$ whenever $(u_1, \ldots, u_4)$ is a basic configuration. If we choose a representant from each equivalence class of the relation $\approx$ on W then, for the family W_0 of the chosen elements, the set $\{\widetilde{g_w} : w \in W_0\}$ is a cog of (positive) tripotents in the sense of [15, 3.1]. Moreover this cog of tripotents is closed in the sense of [15, 3.2], since $\{\widetilde{g_u}\widetilde{g_v}\widetilde{g_w}\} \neq 0$ and hence $u - v + w \in W$ with $u - v + w \approx t$ for some $t \in W_0$ whenever (u, v, w) is an incomplete elementary COG configuration (see [14, Ch.I]). According to [15, 2.1], if $u_1, \ldots, u_4 \in W_0$ are at least three distinct points such that $u_k \not\approx u_\ell$ for $u_k \neq u_\ell$ and $u_4^* = u_1^* - u_2^* + u_3^*$ then for some index permutation τ, $(u_{\tau(1)}, u_{\tau(2)}, u_{\tau(3)}, u_{\tau(4)})$ is an elementary COG configuration. Thus (taking into account arbitrariness in the choice of W_0), *given a basic configuration* $(u_1, \ldots, u_4)$ *in* W, *there exists an index permutation* τ *such that* $(u_{\tau(1)}, \ldots, u_{\tau(4)})$ *is an elementary COG configuration or* $u_{\tau(1)} \approx u_{\tau(2)}$, $u_{\tau(1)} - u_{\tau(2)} = u_{\tau(3)} - u_{\tau(4)}$. *In the latter case necessarily* $u_{\tau(k)} \mathrm{R} u_{\tau(\ell)}$ $(k = 1, 2;\ \ell = 3, 4)$ *for some or the relations* $\mathrm{R} \in \{\top, \perp, \vdash, \dashv, \approx\}$.
Here we can exclude the case $\mathrm{R} = \perp$: if $u_\tau(1) \approx u_{\tau(2)} \perp u_\tau(3) \approx u_{\tau(4)}$ then for any index permutation θ we have $u_{\theta(1)} \perp u_{\theta(2)}$ or $u_{\theta(2)} \perp u_{\theta(3)}$ and hence $\{\widetilde{g_{u_{\theta(1)}}}\widetilde{g_{u_{\theta(1)}}}\widetilde{g_{u_{\theta(1)}}}\} = 0$ entailing $\{g_{u_{\theta(1)}} g_{u_{\theta(1)}} g_{u_{\theta(1)}}\} = 0$.

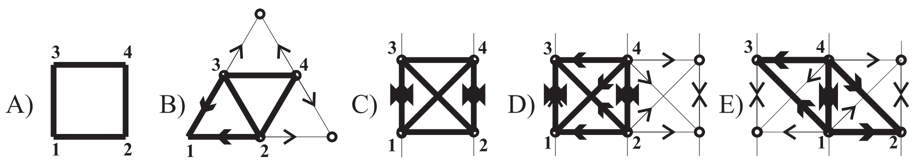
Consider the positive tripotents $h_k := \widetilde{g_{u_{\tau(k)}}}(= g_{u_{\tau(k)}} \oplus [\mathrm{sgn}(g_{u_{\tau(k)}}) g_{u_{\tau(k)}}])$ in $\widetilde{E}$. According to [14, Ch.I], the subtriple H of $\widetilde{E}$ generated by $\{h_1, \ldots, h_4\}$ is

$\mathrm{Span}_{k=1}^4 h_k$ if $(h_1,\dots,h_4)$ is a triangle or quadrangle while $H=\mathrm{Span}_{k=1}^6 h_k$ where $h_5 := \{h_1h_2h_1\}$ and $h_6 := \{h_3h_2h_3\}$ are positive tripotents if $(h_1,\dots,h_4)$ is a a diamond. On the other hand, as we have seen, if $h_1 \approx h_2$ and $h_3 \approx h_4$ then both the couples $\{h_1,h_2\}$ and $\{h_3,h_4\}$ generate strings of pairwise associated tripotents with the same Peirce vectors. Therefore any (possibly degenerate) parallelogram $(w_1,\dots,w_4)$ in W with $w_1-w_2=w_3-w_4$ and $w_1^*-w_2^*=w_3^*-w_4^*$ is embedded into a subset of W which is the affine image of one of the below forms

where we can read the COG relations between the vertices as follows: for $u,v(\in W)$ we draw g —— h if $g \top h$, g —>—– h if $g \vdash h$, g —><—– h if $g \approx h$, and g is not connected with h for $g \perp h$.

Proposition 4.4. *Let $(u_1,\dots,u_4)$ be a basic configuration for G such that $u_k \not\approx u_\ell$ for some k,ℓ. Then $\prod_{k=1}^4 \mathrm{sgn}(g_{u_k})=1$.*

Proof. By the previous remark we may assume that, with a suitable index permutation τ, the Peirce coefficients of the points $w_k := u_{\tau(k)}$ $(k=1,\dots,4)$ correspond to one of the following graphs



and we have $w_1-w_2=w_3-w_4$ along with $w_1^*-w_2^*=w_3^*-w_4^*$ (the case of triangles is covered by graph E with $w_4=w_1$). Observe that, in any case, we have $\pi_{w_3w_4}<2$. For short abbreviate $\mathbf{k} := g_{w_k}$, $\pi_{k\ell} := \pi_{w_k u_\ell}$. and let $\overline{\mathbf{4}} := \{\mathbf{213}\}$. Notice that $\overline{\mathbf{4}} \in \mathbb{C}g_{w_2-w_1+w_3} = \mathbb{C}g_{w_4}$. Hence $\{\mathbf{12}\overline{\mathbf{4}}\} \in \mathbb{C}g_{w_1-w_2+w_4} = \mathbb{C}g_{w_3}$. On the other hand $\{\mathbf{123}\}=0$ because otherwise we would have $v := w_1-w_2+w_3 \in W$ and $v^* = w_1^*-w_2^*+w_3^* = 2u_3^*-u_4^*$ with $\pi_{w_3v} = 2\pi_{\mathbf{34}}-\pi_{\mathbf{33}} = 4-\pi_{\mathbf{34}} > 2$. Therefore

$$(\overline{\mathbf{4}}\,\square\,\overline{\mathbf{4}})\mathbf{3} = \frac{1}{8}\prod_{k=1}^{3}\mathrm{sgn}(\mathbf{k})(2-\pi_{34})(\pi_{21}\pi_{13}-\pi_{12}\pi_{23})\mathbf{3}$$

because

$$\begin{aligned}
\{\overline{\mathbf{4}}\overline{\mathbf{4}}\mathbf{3}\} &= -\{\mathbf{12}\{\overline{\mathbf{4}}\mathbf{33}\}\} + \{\{\mathbf{12}\overline{\mathbf{4}}\}\mathbf{33}\} + \{\overline{\mathbf{4}}\mathbf{3}\{\mathbf{123}\}\} = \\
&= -\frac{1}{2}\mathrm{sgn}(\mathbf{3})\pi_{34}\{\mathbf{12}\overline{\mathbf{4}}\} + \frac{1}{2}\mathrm{sgn}(\mathbf{3})\pi_{33}\{\mathbf{12}\overline{\mathbf{4}}\} = \\
&= \frac{1}{2}\mathrm{sgn}(\mathbf{3})(2-\pi_{34})\{\mathbf{12}\overline{\mathbf{4}}\} \quad \text{where} \\
\{\mathbf{12}\overline{\mathbf{4}}\} &= \{\mathbf{12}\{\mathbf{213}\}\} = \{\{\mathbf{122}\}\mathbf{13}\} - \{\mathbf{2}\{\mathbf{221}\}\mathbf{3}\} + \{\mathbf{21}\{\mathbf{123}\}\} = \\
&= \frac{1}{2}\mathrm{sgn}(\mathbf{2})\pi_{21}\{\mathbf{113}\} - \frac{1}{2}\mathrm{sgn}(\mathbf{1})\pi_{12}\{\mathbf{223}\} = \\
&= \frac{1}{4}\mathrm{sgn}(\mathbf{2})\pi_{21}\mathrm{sgn}(\mathbf{1})\pi_{13}\mathbf{3} - \frac{1}{4}\mathrm{sgn}(\mathbf{1})\pi_{12}\mathrm{sgn}(\mathbf{2})\pi_{23}\mathbf{3}\ .
\end{aligned}$$

A case by case inspection shows that we have always $(2-\pi_{34})(\pi_{21}\pi_{13}-\pi_{12}\pi_{23}) = 2$. Hence, in any case $\overline{\mathbf{4}} \in \frac{1}{2}\mathbb{T}\,\mathbf{4}$ and $\mathrm{sgn}(\mathbf{4}) = \mathrm{sgn}(2\cdot\overline{\mathbf{4}}) = \prod_{k=1}^{3}\mathrm{sgn}(\mathbf{k})$.

5. The semisimple case

Definition 5.1. A weighted grid $G := \{g_w :\ w \in W\}$ is *semisimple* if it consists of pairwise non-associated non-nil tripotents (i.e. $\{g_u g_u g_v\} \notin \{\pm g_v\}$ or $\{g_v g_v g_u\} \notin \{\pm g_u\}$ for different $u, v \in W$).

Throughout this section E denotes a Jordan* triple spanned by a semisimple weighted grid $G := \{g_w :\ w \in W\}$. We shall use the notations π_{uv} resp. u^* $(u, v \in W)$ established in Section 3 for the entries and the column vectors of the Peirce matrix of G along with the COG relations $\top, \perp, \vdash, \dashv$. We write

$$S := \big[\ \text{linear extension of} \quad \{w \mapsto \mathrm{sgn}(g_w) :\ w \in W\}\ \big]\ .$$

Remark 5.2. By the semisimplicity of G, $u^* \neq v^*$ for $u \neq v$ in W. Conversely, by Lemma 3.8, the columns of the Peirce matrix distinguish the points of the weight figure only for semisimple weighted grids.

As a consequence of Proposition 4.4, $\prod_{k=1}^{4}\mathrm{sgn}(g_{u_k}) = 1$ for every basic configuration $(u_1, \ldots, u_4)$. Therefore, by Remark 3.4(3), the operator S is an involution automorphism of E giving rise to the sign transformed triple product

$$[xyz] := \{xyz\}_S = \{x(Sy)z\} \quad \text{with} \quad \mathrm{sgn}_{[\ldots]}(g_w) = 1 \quad (w \in W)\ .$$

Notice also that $\{\ldots\} = [\ldots]_S$. Furthermore the Peirce coefficients of G are the same for both the products $\{\ldots\}$ and $[\ldots]$.

According to the previous observation, with respect to the operation $\{\ldots\}_S$, G can be regarded as a multiplicatively closed set (i.e. $\{g_u g_v g_w\} \in \mathbb{C} g_t$ for some $t \in W$ for any $u, v, w \in W$) of pairwise Peirce compatible pairwise non-associated positive tripotents. Thus, by classical grid theory [14, Ch.II] and [15, 3.3-8],

G consists of multiples of a standard $\{\ldots\}_S$-grid $H := \{h_w :\ w \in W\}$.

In particular, W with the relations $\top, \perp, \vdash$ is a closed abstract COG in the sense of [15, 2.14]. Hence we can achieve a classification of semisimple weighted grids once we determine the affine shape of non-degenerate weight figures and the possible sign distributions in terms of the Peirce matrix whose structure is completely known.

Since $(W, \top, \perp, \vdash)$ is a closed COG, by [15, Theorem A] and its constructive proof, there exists an inner product $(\,.\,|\,.\,)$ on the real vector space W^* spanned by the functions

$$w^* : u \mapsto \pi_{uw} \qquad (w \in W\}$$

such that

$$(R, R_1) \quad \text{where} \quad R := R_1 \cup R_0 \cup R_{-1} \quad \text{with}$$
$$R_1 := \{w^* : \ w \in W\}, \quad R_0 := \{u^* - v^* : \ (u^*|v^*) \neq 0, \ u \neq v\}, \quad R_{-1} := -R_1$$

is a 3-graded root system and

$$\text{(5.3)} \qquad \pi_{uv} = \frac{2(u^*|v^*)}{(v^*|v^*)}\ , \quad (u^*|u^*) = \begin{cases} 2 & \text{if } \ \exists w \ \ u \vdash w \\ 4 & \text{else} \end{cases} \qquad (u, v \in W)\ .$$

Notice also that, by [15, 2.5], four (not necessarily distinct) points $u_1^*, \ldots, u_4^* \in R_1$ form a parallelogram ($u_{\tau(1)}^* - u_{\tau(2)}^* + u_{\tau(3)}^* = u_{\tau(4)}^*$ for some index permutation τ) if and only if $(u_{\theta(1)}, \ldots, u_{\theta(4)})$ is a basic configuration or trivially $u_{\theta(1)} = u_{\theta(2)}$ and $u_{\theta(3)} = u_{\theta(4)}$ for some index permutation θ. On the other hand, by the Jordan identity (J), $(u - v + w)^* = u^* - v^* - w^*$ whenever $u, v, w, u - v + w \in W$ and $\{g_u g_v g_w\} \neq 0$. According to (4.1), the correspondence $w \mapsto w^*$ is one-to-one. Hence, by Lemma 4.4 we see that

$$\text{(5.4)} \qquad u_1 - u_2 = u_3 - u_4 \iff u_1^* - u_2^* = u_3^* - u_4^* \qquad (u_1, \ldots, u_4 \in W)\ .$$

Recall that 3-graded root systems can be generated by so-called grid bases. Taking into account [15, 1.5(7)]), we can reformulate the definition as follows. A *grid base* B of (R, R_1) (with $R_1 = R_1^B$ in the terminology of [15, 1.5]) is a maximal linearly independent subset of R_1 such that for any $C \subset R_1$,

$$C = R_1 \quad \text{if} \quad B \subset C \ \text{ and } \ \alpha - \beta + \gamma \in C \ \text{ whenever } \ \alpha, \beta, \gamma \in C \ \text{ with } \ \alpha - \beta + \gamma \in R_1 .$$

For an explicit construction of grid bases see [15, 3.6]).

Lemma 5.5. *Let* $(V, +)$ *be a connected commutative real Lie group. Assume* $\psi_0 : R_1 \to V$ *is a mapping such that*

$$\textstyle\sum_{k=1}^4 (-1)^k \psi_0(\alpha_k) = 0 \quad \textit{whenever} \ \sum_{k=1}^4 (-1)^k \alpha_k = 0\ , \ \alpha_1, \ldots, \alpha_4 \in R_1\ .$$

Then ψ_0 *extends to a homomorphism* $\psi : \mathrm{Span}_{\mathbb{R}} R_1 \to V$ *of the form* $\psi = \exp \phi$ *where* ϕ *is a linear map of* $\mathrm{Span}_{\mathbb{R}} R_1$ *into the Lie algebra of* V.

Proof. Choose a grid base B in R_1. Since V is connected and commutative, the exponential map of the Lie algebra L of V is a surjective submersion [17]. Hence for any $\beta \in B$ there exists some $\lambda_\beta \in L$ such that $\exp(\lambda_\beta) = \psi_0(\beta)$ $(\beta \in B$. Since B is a vector space basis in $\mathrm{Span}_{\mathbb{R}} R$, there is a (unique) linear $\phi : \mathrm{Span}_{\mathbb{R}} R \to V$ extending the map $\beta \mapsto \lambda_\beta$. Let $C := \{\alpha \in R_1 : \exp(\phi(\alpha)) = \psi_0(\alpha)\}$. The hypothesis $\sum_{k=1}^4 (-1)^k \psi_0(\alpha_k) = 0$ for $\sum_{k=1}^4 (-1)^k \alpha_k = 0$ implies $\alpha_1 - \alpha_2 + \alpha_3 \in C$ for $\alpha_1, \alpha_2, \alpha_3 \in C$. By the definition of ϕ, $B \subset C$ whence $C = R_1$.

Corollary 5.6. *For a linear map* $L \colon E \to E$ *commuting with* $\{g_w \square g_w : w \in W\}$,

(i) $L \in \mathrm{Der}_*(E)$ *if and only if* $Lg_w = \phi(w^*) g_w$ $(w \in W)$ *for some linear* $\phi : W^* \to \mathbb{R}$,
(ii) $L \in \mathrm{Aut}(E)$ *if and only if* $Lg_w = e^{(i\phi(w^*))}$ $(w \in W)$ *for some linear* $\phi : W^* \to \mathbb{R}$.

Proof. Observe that each g_w is an eigenvector of L. Indeed, if $w \in W$ and $Lg_w = \sum_{v \in W} \mu_v g_v$ then $0 = (g_u \square g_u) L g_w - L (g_u \square g_u) g_w == 2 \sum_{v \in W} \mu_v (\pi_{uv} - \pi_{uw}) g_v$ for any $u \in W$. Hence $\mu_v (v^* - w^*) = 0$ $(v \in W)$. However, by (S) we have $v^* \neq w^*$ for $v \neq w$. Thus in any case we can write $Lg_w = \lambda_w g_w$ with suitable constants $\lambda_w \in \mathbb{C}$ $(w \in W)$.

(i) According to Lemma 2.2, we have $L \in \mathrm{Der}_*(E)$ if and only if $\lambda_{u-v+w} = \lambda_u - \lambda_v + \lambda_w$ whenever $u, v, w, u - v + w \in W$. Since the map $w^* \mapsto w$ is parallelogram preserving (5.4), the statement is immediate from 5.5 applied with $V := \mathbb{R}$ and $\psi_0 : w^* \mapsto \lambda_w$.

(ii) Notice that we have $L \in \mathrm{Aut}(F)$ if and only if $|\lambda_w| = 1$ $(w \in W)$ and $\lambda_{u-v+w} = \lambda_u \overline{\lambda_v} \lambda_w$ $(u, v, w, u - v + w \in W)$. To complete the proof we apply 5.5 to the multiplicative group $\mathbb{T}$ $(:= \{\zeta \in \mathbb{C} : |\zeta| = 1\})$ and the map $\psi_0 : w^* \mapsto \lambda_w$.

Remark 5.7. As an immediate consequence of (i), R_1 is a non-degenerate weight figure for G and $\mathrm{Span}_{\mathbb{R}}\{g_w \square g_w : w \in W\}$ is an M-family in E.

We summarize our considerations in the framework of classical grid theory as follows.

Theorem 5.8. *Let* $G := \{g_w : w \in W\}$ *be a weighted grid of non-nil tripotents. In terms of the family* $R_1 := \{w^* : w \in W\}$ *of the Peirce vectors* $w^* := [u \mapsto \pi_{uw}]$ *where* $\pi_{uv} := [\lambda \in \{0,1,2\} : (g_u \square g_u) g_v = \frac{1}{2} \mathrm{sgn}(g_u) \pi_{uv} g_v]$, *the following statements are equivalent.*

(i) G *is semisimple.*
(ii) W *does not contain any non-degenerate affine copy* $\{w_n : n \in \mathbb{Z}\}$ *of* $\mathbb{Z}$ *with* $w_n^* = w_m^*$ $(m, n \in \mathbb{Z})$.
(iii) *For some linear mapping* L *of the space* $W^* := \mathrm{Span}_{\mathbb{R}} R_1$ *onto* $\mathrm{Span}_{\mathbb{R}} W$ *we have* $w = Lw^*$ $(w \in W)$. *There exists an inner product* $(\,.\,|\,.\,)$ *on* W^* *satisfying* (5.3) *with respect to which* R_1 *is the* 1*-component of a* 3*-graded root system. Furthermore*

$$\mathrm{sgn}(g_w) = (-1)^{\Psi(w^*)} \qquad (w \in W)$$

for some linear functional $\Psi : W^* \to \mathbb{R}$ *assuming integral values on* R_1 .

Proof. To complete the proof we only have to establish the sign formula in (iii). Assume (i). By Lemma 4.4 we have $\prod_{k=1}^4 \mathrm{sgn}(g_{u_k}) = 1$ whenever $\sum_{k=1}^4 u_k^* = 0$ $(u_1, \ldots, u_4 \in W)$. An application of Lemma 5.5 with $V := (\mathbb{T}, \cdot)$ yields the existence of a linear functional $\psi : W^* \to \mathbb{R}$ such that $\mathrm{sgn}(g_w) = e^{i\psi(w^*)}$ $(w \in W)$. Thus the choice $\Psi := \pi^{-1}\psi$ suits our requirements.

6. Grid triples of $\mathbb{Z}^2$ type

Throughout this section, F denotes a Jordan* triple spanned by a non-nil weighted grid $G := \{g_w : w \in W\}$ with non-degenerate weight figure W. For short we write $\lambda_{ef} := [\lambda \in \{0, \pm\frac{1}{2}, \pm 1\} : (e \square e)f = \lambda f]$ for $e, f \in G$. We use also the direct notation $e\mathrm{R}f$ $(e, f \in G,\ \mathrm{R} = \top, \perp, \vdash, \dashv, \approx)$ for the COG relations of the elements of G (defined in terms of (3.1)).

Proposition 6.1. *Let* $(w_k : k \in \mathbb{Z})$ *be an arithmetic sequence in* W *and suppose* $u, v \in W$ *with* $u - v = N(w_1 - w_0)$. *Then with the notations* $e := g_u$, $f := g_v$, $a_n := g_{w_n}$ $(n \in \mathbb{Z})$ *and* $F_0 := \mathrm{Span}_{n \in \mathbb{Z}}\, a_n$, *for some* $\xi \in \mathbb{C}$ *we have*

$$e \square f | F_0 = \xi a_N \square a_0 | F_0 , \quad f \square e | F_0 = \overline{\xi} a_0 \square a_N | F_0 \qquad (n \in \mathbb{Z}) .$$

Proof. By setting $\sigma := \mathrm{sgn}(a_0)$ and $\varepsilon := \mathrm{sgn}(a_0)\mathrm{sgn}(a_1)$, we may assume

$$a_{k-\ell+m} = \sigma\varepsilon^\ell \{a_k a_\ell a_m\} \qquad (k, \ell, m \in \mathbb{Z}) ,$$
$$e \square f : a_n \mapsto \xi_n a_{n+N} , \quad f \square e : a_n \mapsto \eta_n a_{n-N}$$

for suitable coefficients $\xi_n, \eta_n \in \mathbb{C}$ $(n \in \mathbb{Z})$. Thus

$$\begin{aligned} \{ef\{a_k a_\ell a_m\}\} &= \{\{efa_k\}a_\ell a_m\} - \{a_k\{fea_\ell\}a_m\} + \{a_k a_\ell\{efa_m\}\} \\ \sigma\varepsilon^\ell\{efa_{k-\ell+m}\} &= \xi_k\{a_{k+N}a_\ell a_m\} - \overline{\eta_\ell}\{a_k a_{\ell-N} a_m\} + \xi_m\{a_k a_\ell a_{m+N}\} \\ \xi_{k-\ell+m}\sigma\varepsilon^\ell a_{k-\ell+m+N} &= [\xi_k\sigma\varepsilon^\ell - \overline{\eta_\ell}\sigma\varepsilon^{\ell-N} + \xi_m\sigma\varepsilon^\ell]a_{k-\ell+m+N} \\ \xi_k + \xi_m - \xi_{k-\ell+m} &= \overline{\eta_\ell}\varepsilon^N \qquad (k, \ell, m \in \mathbb{Z}) . \end{aligned}$$

In particular (with $\ell := 0$) we have $\xi_k + \xi_m - \xi_{k+m} = \overline{\eta_0}\varepsilon^N$ $(k, m \in \mathbb{Z})$. That is $\xi'_k + \xi'_m = \xi'_{k+m}$ $(k, m \in \mathbb{Z})$ for the values $\xi'_n := \xi_n - \overline{\eta_0}\varepsilon^N$. It follows by induction that $\xi'_n = n\xi'_1$ $(n \in \mathbb{Z})$ and hence the sequence $(\xi_n : n \in \mathbb{Z})$ is arithmetic. On the other hand (with $k = \ell = m =: n$) also $\xi_n = \overline{\eta_n}\varepsilon^N$ $(n \in \mathbb{Z})$. Thus for some $\alpha, \beta \in \mathbb{C}$,

$$\xi_n = n\alpha + \beta , \quad \eta_n = \varepsilon^N(n\overline{\alpha} + \overline{\beta}) \qquad (n \in \mathbb{Z}) .$$

Since $u - v = N(w_1 - w_0)$, we have $\lambda_{ge} - \lambda_{gf} = N(\lambda_{ga_1} - \lambda_g a_0) = 0$ $(g \in G)$. Thus $e \approx f$ and therefore $\lambda_{ee} e \square e = \lambda_{ff} f \square f$ [on the base space if we consider partial Jordan* triples]. It follows

$$[e \square f, f \square e] = \{eff\} \square e - f \square \{eef\} = \lambda_{fe} e \square e - \lambda_{ef} f \square f = \lambda_{ff} e \square e - \lambda_{ee} f \square f = 0 .$$

This means that $\{ef\{fea_n\}\} = \{fe\{efa_n\}\}$ i.e. $\eta_n\{efa_{n-N}\} = \xi_n\{fea_{n+N}\}$ or $\eta_n\xi_{n-N}a_n = \xi_n\eta_{n+N}a_n \quad (n \in \mathbb{Z})$. Thus

$$\begin{aligned} &\eta_n\xi_{n-N} = \xi_n\eta_{n+N} \qquad (n \in \mathbb{Z}) \\ &\varepsilon^N(n\overline{\alpha} + \overline{\beta})[(n-N)\alpha + \beta] = (n\alpha + \beta)\varepsilon^N[(n+N)\overline{\alpha} + \overline{\beta}] \\ &n^2|\alpha|^2 + n[-N|\alpha|^2 + 2\mathrm{Re}\alpha\overline{\beta}] + [-N\alpha\overline{\beta} + |\beta|^2] = \\ = \; &n^2|\alpha|^2 + n[N|\alpha|^2 + 2\mathrm{Re}\alpha\overline{\beta}] + [-N\beta\overline{\alpha} + |\beta|^2] \end{aligned}$$

which is possible only if $\alpha = 0$. Therefore $\xi_n = \xi_0$ and $\eta_n = \eta_0 = \varepsilon^N\overline{\xi_0}$ for every index $n \in \mathbb{Z}$.

Definition 6.2. Henceforth, throughout this section, we use the notation

$$\left[\frac{a\,\square\, b}{c\,\square\, d}\right] := \left[\xi \in \mathbb{C} : \; (a\,\square\, b)d = \xi(c\,\square\, d)d = \xi\mathrm{sgn}(d)c\right] \qquad (a,b,c,d \in G \;,\; c \approx d) \;.$$

Lemma 6.3. *Suppose $u,v,w,z \in W$ with $u - v = w - z$ and $g_u \approx g_v \approx g_w \approx g_z$. Then for the tripotents $a := g_u$, $b := g_v$, $c := g_w$, $d := g_z$ we have*

$$\left[\frac{a\,\square\, b}{c\,\square\, d}\right]\mathrm{sgn}(d)\mathrm{sgn}(a) = \left[\frac{d\,\square\, b}{c\,\square\, a}\right] \;, \quad \left[\frac{a\,\square\, b}{c\,\square\, d}\right] = \left[\frac{b\,\square\, a}{d\,\square\, c}\right]^{-} \;.$$

Proof. We have

$$\begin{aligned} \{abd\} &= (a\,\square\, b)d = \left[\frac{a\,\square\, b}{c\,\square\, d}\right](c\,\square\, d)d = \left[\frac{a\,\square\, b}{c\,\square\, d}\right]\mathrm{sgn}(d)c \;, \\ \{dba\} &= (d\,\square\, b)a = \left[\frac{d\,\square\, b}{c\,\square\, a}\right](c\,\square\, a)a = \left[\frac{d\,\square\, b}{c\,\square\, a}\right]\mathrm{sgn}(a)c \;. \end{aligned}$$

Since $\{abd\} = \{dba\}$, this proves the first equality. The second one is immediate from the previous lemma.

Corollary 6.4. *In particular, if $a \approx b \in G$ then $(Q_b a)\,\square\, b = \mathrm{sgn}(a)\mathrm{sgn}(b) b\,\square\, a$ and $b\,\square\,(Q_b a) = \mathrm{sgn}(a)\mathrm{sgn}(b) a\,\square\, b$ for $Q_b a := \{bab\}$.*

Proposition 6.5. *Suppose $\{u_k : \; k \in \mathbb{Z}\}$ and $\{v_k : \; k \in \mathbb{Z}\}$ are two strings in W such that $u_k - u_{k-1} = v_\ell - v_{\ell-1} \quad (k,\ell \in \mathbb{Z})$. Set*

$$a_k := g_{u_k} \;, \quad b_k := g_{v_k} \;, \quad c_k := \mathrm{sgn}(a_k) Q_{b_k} a_k \;, \quad \xi_k := \left[\frac{b_k\,\square\, b_0}{a_k\,\square\, a_0}\right]$$

and assume $\{a_k a_\ell a_m\} = \alpha\sigma^\ell a_{k-\ell+m}$, $\{b_k b_\ell b_m\} = \beta\tau^\ell b_{k-\ell+m}$, $a_k \approx b_\ell \quad (k,\ell,m \in \mathbb{Z})$. Then

$$\xi_0 = \alpha\beta \;, \quad \xi_{-k} = (\sigma\tau)^k\overline{\xi_k} \qquad (k \in \mathbb{Z})$$

and there exists a sequence $(\lambda_t : \; t \in \mathbb{Z})$ in $\mathbb{T}$ such that

$$\begin{aligned} &\xi_{j+t} - \alpha\beta(1 + \sigma^t\tau^t)\xi_t\xi_j + \lambda_t\xi_{j-t} = 0 \;, \\ &Q_{b_j} a_{j+t} = \lambda_t \mathrm{sgn}(a_{j+t}) c_{j-t} \qquad (j,t \in \mathbb{Z}) \;. \end{aligned}$$

Proof. By definition, $\xi_0 = \left[\frac{b_0 \square b_0}{a_0 \square a_0}\right] = \mathrm{sgn} b_0 / \mathrm{sgn}(a_0) = \alpha/\beta = \alpha\beta$. Furthermore

$$\xi_{-k} = \left[\frac{b_{-k} \square b_0}{a_{-k} \square a_0}\right] = \left[\frac{b_0 \square b_{-k}}{a_0 \square a_{-k}}\right]^{-} = \frac{\tau^k}{\sigma^k}\left[\frac{b_k \square b_0}{a_k \square a_0}\right]^{-} = (\sigma\tau)^k \overline{\xi_k} \ .$$

For every $k, \ell, m \in \mathbb{Z}$ we have

$$\begin{aligned}
\left[\frac{b_k \square b_\ell}{a_k \square a_\ell}\right] &= \frac{\tau^\ell}{\sigma^\ell}\left[\frac{b_{k-\ell} \square b_0}{a_{k-\ell} \square a_0}\right] = (\sigma\tau)^\ell \xi_{k-\ell} \ , \\
\left[\frac{a_k \square a_\ell}{b_k \square b_\ell}\right] &= \alpha\sigma^k\beta\tau^\ell\left[\frac{b_\ell \square a_\ell}{b_k \square a_k}\right] = \alpha\sigma^k\beta\tau^\ell\left[\frac{a_\ell \square b_\ell}{a_k \square b_k}\right]^{-} = \\
&= \alpha\sigma^k\beta\tau^\ell\alpha\sigma^\ell\beta\tau^k\left[\frac{b_k \square b_\ell}{a_k \square a_\ell}\right]^{-} = (\sigma\tau)^{k+\ell}\overline{\xi_{k-\ell}} = \xi_{\ell-k} \ , \\
\{a_k a_\ell b_m\} &= \left[\frac{a_k \square a_\ell}{b_k \square b_\ell}\right]\{b_k b_\ell b_m\} = \xi_{\ell-k}\beta\tau^\ell b_{k-\ell+m} \ , \\
\{b_k b_\ell a_m\} &= \left[\frac{b_k \square b_\ell}{a_k \square a_\ell}\right]\{a_k a_\ell a_m\} = (\sigma\tau)^\ell \xi_{k-\ell}\alpha\sigma^\ell a_{k-\ell+m} = \\
&= \alpha\tau^\ell \xi_{k-\ell} a_{k-\ell+m} \ .
\end{aligned}$$

Using these relations, we evaluate the identity

$$\begin{aligned}
\{b_k a_{k+t}\{b_k b_{k-t} a_i\}\} = \{\{b_k a_{k+t} b_k\} b_{k-t} a_i\} - \\
-\{b_k\{a_{k+t} b_k b_{k-t}\} a_i\} + \{b_k b_{k-t}\{b_k a_{k+t} a_i\}\} \ .
\end{aligned}$$

It follows

$$\begin{aligned}
&\{(Q_{b_k} \quad a_{k+t}) b_{k-t} a_i\} = \\
&\quad = \alpha\tau^{k-t}\xi_t\{b_k a_{k+t} a_{i+t}\} - \beta\tau^{k+t}\xi_{k+t-i}\{b_k b_{k-t} b_{i-t}\} + \alpha\tau^k\overline{\xi_{-t}}\{b_k a_k a_i\} = \\
&\quad = \alpha\tau^{k-t}\xi_t\beta\tau^{k+t}\xi_{k-i} b_i - \beta\sigma^{k-t}\xi_{k+t-i}\beta\tau^{k-t} b_i + \alpha\tau^k(\sigma^t\tau^t\xi_t)\beta\tau^k\xi_{k-i} b_i = \\
&\quad = \big[\alpha\beta(1+\sigma^t\tau^t)\xi_t\xi_{k-i} - \xi_{k-i+t}\big] b_i \ .
\end{aligned}$$

Since $b_k = g_{v_k} \approx g_{u_{k+t}} = a_{k+t}$, we have $Q_{b_k} a_{k+t} \in \mathbb{T}\, g_{2v_k - u_{k+t}}$. Since the sequences (u_n) , (v_n) are arithmetic with the same difference, $2v_k - u_{k+t} = 2v_{k-t} - u_{k-t}$. Thus $c_{k-t} = Q_{b_{k-t}} a_{k-t} \in \mathbb{T}\, g_{2v_{k-t} - u_{k-t}}$, and we have

$$Q_{b_k} a_{k+t} = \Lambda_{kt} c_{k-t} \qquad \text{for some} \ \ \Lambda_{kt} \in \mathbb{T} \ .$$

By Corollary 6.4,

$$\begin{aligned}
&(Q_{b_k} a_{k+t}) \square b_{k-t} = \Lambda_{kt} c_{k-t} \square b_{k-t} = \Lambda_{kt}\mathrm{sgn}(b_{k-t})\mathrm{sgn}(a_{k-t}) b_{k-t} \square a_{k-t} \ , \\
&\{(Q_{b_k} a_{k+t}) b_{k-t} a_i\} = \Lambda_{kt}\alpha\sigma^{k-t}\beta\tau^{k-t}\{b_{k-t} a_{k-t} a_i\} = \Lambda_{kt}\alpha\sigma^{k-t}\xi_{k-t-i} b_i \ , \\
&\Lambda_{kt}\alpha\sigma^{k-t}\xi_{k-t-i} = \alpha\beta(1+\sigma^t\tau^t)\xi_t\xi_{k-i} - \xi_{k-i+t} \ .
\end{aligned}$$

Since $\xi_0 = \alpha\beta \in \{\pm 1\}$, by substituting $i := k - t$, we see that the coefficient

$$\alpha\sigma^{t-k}\Lambda_{kt} = \xi_0[\alpha\beta(1 - \sigma^t\tau^t)\xi_t^2 - \xi_{2t}] =: \lambda_t$$

is independent of the index k and has absolute value 1 . Thus

$$\lambda_t \xi_{(k-i)-t} = \alpha\beta(1+\sigma^t\tau^t)\xi_t\xi_{k-i} - \xi_{(k-i)+t} \ , \quad Q_{b_k} a_{k+t} = \Lambda_{kt} c_{k-t} = \alpha\sigma^{k+t}\lambda_t c_{k-t}$$

for any $i,k,t \in \mathbb{Z}$. We complete the proof by substituting $j := k-i$.

Remark 6.6. It is well-known from elementary linear algebra that the bilateral shift

$$T : (z_n : \ n \in \mathbb{Z}\,) \mapsto (z_{n+1} : \ n \in \mathbb{Z}\,)$$

on the sequence space $\mathcal{S} := \big\{ (z_n : \ n \in \mathbb{Z}\,) : \ z_0, z_{-1}, z_1, \ldots \in \mathbb{C} \big\}$ has the following spectral property:

$$\big\{ (n^k \omega^n : \ n \in \mathbb{Z}\,) : \ \omega \in \mathbb{C} \setminus \{0\}, \ k = 0,1,\ldots \big\} \text{ is a basis in } \ \mathcal{S}_0$$

where $\mathcal{S}_0 := \big\{ z \in \mathcal{S} : \ \exists \ p \text{ polynomial} \quad p(T)z = 0 \big\}$. Moreover, each sequence $(n^k \omega^k : \ n \in \mathbb{Z}\,)$ is an *eigenvector of order* k (with eigenvalue $\omega \neq 0$).

Corollary 6.7. *There exist* $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ *and* $\delta \in \mathbb{R}$ *such that*

$$\xi_n = \frac{\alpha\beta}{2}(1+i\delta)\omega_1^n + \frac{\alpha\beta}{2}(1-i\delta)\omega_2^n \ , \quad \lambda_n = (\omega_1\omega_2)^n \qquad (n \in \mathbb{Z}\,) \ .$$

The following alternatives hold

i) $\sigma\tau = 1$, $\delta = 0$ *and* $|\omega_1| = |\omega_2| = 1$,

ii) $\sigma\tau = 1$, $\delta = 0$ *and* $\omega_2 = \overline{\omega_1}^{-1}$,

iii) $\sigma\tau = -1$, $\delta \in \mathbb{R}$ *arbitrary and* $\omega_1 = -\omega_2$, $|\omega_1| = 1$.

Proof. By Proposition 6.5, $\xi_{j+2t} - \alpha\beta(1+\sigma^t\tau^t)\xi_t\xi_{j+t} + \lambda_t\xi_j \equiv 0$. Thus with the notation of the previous Remark,

$$\big[T^{2t} - \alpha\beta(1+\sigma^t\tau^t)\xi_t T^t + \lambda_t \big] (\xi_n : \ n \in \mathbb{Z}\,) = 0 \qquad (t \in \mathbb{Z}\,) \ .$$

Let ω_1, ω_2 denote the roots of the polynomial $z^2 - \alpha\beta(1+\sigma\tau)\xi_1 z + \lambda_1$. Therefore we have only the possibilities

$$\begin{aligned} &1)\ \sigma\tau = 1 \ , \quad \xi_n = (A+Bn)\omega^n && \text{with } \ \omega \in \{\omega_1,\omega_2\} \ , \\ &2)\ \sigma\tau = 1 \ , \quad \xi_n = A\omega_1^n + B\omega_2^n && \text{with } \ \omega_1 \neq \omega_2 \ , \quad A,B \neq 0 \ , \\ &3)\ \sigma\tau = -1, \quad \xi_n = A\rho^n + B(-\rho)^n && \text{with } \ \rho \in \{\omega : \ \omega^2 = \lambda_1\} \end{aligned}$$

for some $A,B \in \mathbb{C}$.

Case 1. We show that necessarily $A = \alpha\beta$, $B = 0$, and $\lambda_t = \omega^{2t}$.

The condition $\xi_0 = \alpha\beta$ implies $A = \alpha\beta$. Hence the relation $\xi_{-n} = (\sigma\tau)^n \overline{\xi_n} = \overline{\xi_n}$ means

$$(\alpha\beta - nB)\omega^{-n} = (\alpha\beta + n\overline{B})\overline{\omega}^n \qquad (n \in \mathbb{Z}\,) \ .$$

Thus we must have $\omega^{-1} = \overline{\omega}$ and $B = -\overline{B}$. That is $|\omega|^2 = 1$ and $B \in i\mathbb{R}$. On the other hand

$$\begin{aligned} 0 &= \xi_{n+2t} - 2\alpha\beta\xi_t\xi_{n+t} + \lambda_t\xi_n = \\ &= [\alpha\beta+(n+2t)B]\omega^{n+2t} - 2\alpha\beta(\alpha\beta+tB)[\alpha\beta+(n+t)B]\omega^{n+2t} + \lambda_t(\alpha\beta+nB)\omega^n = \\ &= \big[-\alpha\beta\omega^{2t} - 2B(t\omega^{2t}) - 2\alpha\beta B^2(t^2\omega^{2t}) + \alpha\beta\lambda_t\big]\omega^n + \\ &\quad + B\big[-\omega^{2t} - 2\alpha\beta B(t\omega^{2t}) + \lambda_t\big]n\omega^n \ . \end{aligned}$$

For fixed $t \in \mathbb{Z}$, both the coefficients of ω^n and $n\omega^n$ should vanish in the last expression. Hence $B = 0$ implies $\lambda_t = \omega^{2t}$. From the assumption $B \neq 0$, we get the contradiction $\lambda_t = \omega^{2t} + 2\alpha\beta B(t\omega^{2t}) - 2\alpha\beta B^2(t^2\omega^{2t}) \equiv \omega^{2t} + 2\alpha\beta B(t\omega^{2t})$.

Case 2. For any fixed $t \in \mathbb{Z}$,

$$\begin{aligned} 0 &= \xi_{n+2t} - 2\alpha\beta\xi_t\xi_{n+t} + \lambda_t\xi_n = \\ &= \omega_1^n A\big[\big(1 - 2\alpha\beta A\big)\omega_1^{2t} - 2\alpha\beta B(\omega_1\omega_2)^t + \lambda_t\big] + \\ &\quad + \omega_2^n B\big[\big(1 - 2\alpha\beta B\big)\omega_2^{2t} - 2\alpha\beta A(\omega_1\omega_2)^t + \lambda_t\big] \qquad (n \in \mathbb{Z}). \end{aligned}$$

By assumption $A, B \neq 0$. Therefore, since the coefficients of ω_1, ω_2 must vanish,

$$\begin{aligned} \lambda_t &= 2\alpha\beta B(\omega_1\omega_2)^t + (2\alpha\beta A - 1)(\omega_1^2)^t = \\ &= 2\alpha\beta A(\omega_1\omega_2)^t + (2\alpha\beta B - 1)(\omega_2^2)^t \end{aligned}$$

for any $t \in \mathbb{Z}$. By assumption $\omega_1 \neq \omega_2$ and hence $\omega_1\omega_2 \neq \omega_1^2$, $\omega_1\omega_2 \neq \omega_2^2$ in this case. Thus the coefficients of the terms $(\omega_1\omega_2)$ should be the same i.e. $A = B$. Since $A + B = \xi_0 = \alpha\beta$, necessarily

$$A = B = \alpha\beta/2 \ , \quad \lambda_t = (\omega_1\omega_2)^t \qquad (t \in \mathbb{Z}) \ .$$

Since $|\lambda_t| = 1$, also $|\omega_1\omega_2| = 1$. On the other hand $\xi_{-n} = \overline{\xi_n}$ $(n \in \mathbb{Z})$. This means

$$\omega_1^{-n} + \omega_2^{-n} = \overline{\omega_1}^n + \overline{\omega_2}^n \qquad (n \in \mathbb{Z}) \ .$$

Since the sequences $\big(\zeta^n\big)$ $(\zeta \in \mathbb{C}\backslash\{0\})$ form a linearly independent family, it follows $\omega_1^{-1} \in \{\omega_2^{-1}, \overline{\omega_1}, \overline{\omega_2}\}$. However, $\omega_1^{-1} \neq \omega_1^{-1}$ whence $\omega_1^{-1} \in \{\overline{\omega_1}, \overline{\omega_2}\}$. Similarly $\omega_2^{-1} \in \{\omega_1^{-1}, \overline{\omega_1}\}$. Since $\omega_1 \neq \omega_2$, we have the subcases

2.1) $\omega_1^{-1} = \overline{\omega_1}$ and $\omega_2^{-1} = \overline{\omega_2}$ with $|\omega_1|^2 = |\omega_2|^2 = 1$

2.2) $\omega_1^{-1} = \overline{\omega_2}$ i.e. $\omega_1 = \omega, \ \omega_2 = \overline{\omega}^{-1}$.

Case 3. Since $|\rho| = \sqrt{|\lambda_1|} = 1$, the relations $\xi_0 = \alpha\beta$, $\xi_{-n} = (\sigma\tau)^n\overline{\xi_n} = (-1)^n\overline{\xi_n}$ imply $A + B = \alpha\beta$ and

$$0 = [A\rho^{-n} + B(-\rho)^{-n}] - (-1)^n\overline{[A\rho^n + B(-\rho)^n]} = (A - \overline{B})\overline{\rho}^n + (B - \overline{A})(-\overline{\rho})^n$$

for all $n \in \mathbb{Z}$. This is possible if and only if $A = (1 + i\delta)\alpha\beta/2$, $B = (1 - i\delta)\alpha\beta/2$ for some constant $\delta \in \mathbb{R}$. In this case

$$\begin{aligned} 0 &= \xi_{n+2t} - [1 + (-1)^t]\alpha\beta\xi_t\xi_{n+t} + \lambda_t\xi_n = \\ &= \frac{\alpha\beta}{2}\big((1 + i\delta)\rho^n + (1 - i\delta)(-\rho)^n\big)\left[\lambda_t - (-\rho^2)^t\right] \qquad (n \in \mathbb{Z}) \ , \end{aligned}$$

and hence $\lambda_t = (-\rho^2)^t$ for any fixed $t \in \mathbb{Z}$.

Remark 6.8. Henceforth we assume $W := \mathbb{Z}^2 \oplus 1$ and we use the abbreviation g_{pq} for the term $g_{(p,q,1)}$.

Lemma 6.9. Assume

$$\begin{aligned} \{g_{pi} g_{pj} g_{p_k}\} &= \mathrm{sgn}(g_{pj}) g_{p,i-j+k} \qquad (i,j,k \in \mathbb{Z},\ p = 0,1) ,\\ \{g_{iq} g_{jq} g_{kq}\} &= \mathrm{sgn}(g_{jq}) g_{i-j+k,q} \qquad (i,j,k,q \in \mathbb{Z}) . \end{aligned}$$

Then also

$$\{g_{pi} g_{pj} g_{pk}\} = \mathrm{sgn}(g_{pj}) g_{p,i-j+k} \qquad (i,j,k,p \in \mathbb{Z}) .$$

Proof. Taking into account 3.8, it suffices to verify

$$\mathrm{sgn}(g_{pk}) g_{p,k\pm 2} = \{a_{p,k\pm 1} g_{pk} g_{p,k\pm 1}\} = Q_{g_{p,k\pm 1}}(g_{pk})$$

for all $p,k \in \mathbb{Z}$. We prove this statement by induction. By assumption

$$\begin{aligned} &\mathrm{sgn}(g_{pk}) g_{p,k+2\zeta} = Q_{g_{p,k+\zeta}}(g_{pk}) \quad \text{for } p = 0,1 \text{ with } k=0,\ \zeta=1 \text{ or } k=1,\ \zeta=-1;\\ &\mathrm{sgn}(g_{jq}) g_{j+2\varepsilon,q} = Q_{g_{j+\varepsilon,q}}(g_{jq}) \quad \text{for any } q,j \in \mathbb{Z} \text{ and } \varepsilon = \pm 1 . \end{aligned}$$

Thus we can apply the following induction step:
For any $j,k \in \mathbb{Z}$, $\varepsilon,\zeta \in \{\pm 1\}$

$$\left.\begin{aligned} \mathrm{sgn}(g_{pk}) g_{p,k+2\zeta} = Q_{g_{p,k+\zeta}}(g_{pk}) \quad (p=j,j+\varepsilon,j+2\varepsilon)\\ \mathrm{sgn}(g_{jq}) g_{j+2\varepsilon,q} = Q_{g_{j+\varepsilon,q}}(g_{jq}) \quad (q=k,k+\zeta) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &\mathrm{sgn}(g_{jk}) g_{j+2\varepsilon,k+2\zeta} =\\ &= Q_{g_{j+2\varepsilon,k+\zeta}}(g_{j+2\varepsilon,k}). \end{aligned}$$

By setting

$$a_m := g_{j+m\varepsilon,k} \ , \quad b_m := g_{j+m\varepsilon,k+\zeta} \ , \quad c_m := g_{j+m\varepsilon,k+2\zeta} = Q_{b_m} a_m \qquad (m = 0,1,2),$$

we have to establish the relation $Q_{c_1} c_0 = \mathrm{sgn}(c_0) c_2$.

Since, for any p,q the subspaces $\mathrm{Span}_j \in \mathbb{Z}\, g_{pj}$, $\mathrm{Span}_i \in \mathbb{Z}\, g_{iq}$ are string triples, we have

$$\{a_k a_\ell a_m\} = \alpha \sigma^\ell \ , \quad \{b_k b_\ell b_m\} = \beta \tau^\ell \qquad (k,\ell,m \in \mathbb{Z}) .$$

with some $\alpha,\beta,\sigma,\tau \in \{\pm 1\}$. Thus we can apply Proposition 6.5 and its Corollary to the strings (a_n) , (b_n) , (c_n) . It follows in particular

$$Q_{b_1} a_2 = \lambda_1 c_0 \ , \quad Q_{b_1} a_0 = \lambda_{-1} c_2 = \lambda_1^{-1} = \overline{\lambda_1} c_2$$

for some $\lambda_1 \in \mathbb{T}$. Notice that for any $g \in G (= \{g_{ij} :\ i,j \in \mathbb{Z}\})$, $Q_g^2 = \mathrm{id}$ because $g \,\square\, g = \pm \mathrm{id}$. Thus we complete the proof by the argument

$$\begin{aligned} Q_{c_1} c_0 &= Q_{c_1}(\lambda_1 Q_{b_1} a_2) = \overline{\lambda_1} Q_{Q_{b_1} a_1} Q_{b_1} a_2 =\\ &= \overline{\lambda_1} (Q_{b_1} Q_{a_1} Q_{b_1}) Q_{b_1} a_2 = \overline{\lambda_1} Q_{b_1} Q_{a_1} a_2 =\\ &= \overline{\lambda_1} \mathrm{sgn}(a_2) Q_{b_1} a_0 = \mathrm{sgn}(c_0) c_2 \end{aligned}$$

since $\mathrm{sgn}(a_2)=\mathrm{sgn}(g_{j+2\varepsilon,k})=\mathrm{sgn}(g_{jk})=\mathrm{sgn}(g_{j,k+2\eta})=\mathrm{sgn}(c_0)$.

Remark 6.10. $G_p:=\{g_{pk}:\ k\in\mathbb{Z}\}$ and $G'_q:=\{g_{jq}:\ j\in\mathbb{Z}\}$ in 6.9 are weighted grids of with weight figure $\mathbb{Z}$. Therefore (cf. 3.8) we can define an equivalent non-nil weighted grid $G':=\{g'_{pq}:\ p,q\in\mathbb{Z}\}$ such that

$$\{g'_{pi}g'_{pj}g'_{pk}\}=\mathrm{sgn}(g'_{pj})g_{p,i-j+k}\ ,$$
$$\{g'_{iq}g'_{jq}g'_{kq}\}=\mathrm{sgn}(g'_{jq})g_{i-j+k,q}\quad (p,q,\ i,j,k\in\mathbb{Z})$$

by means of the following double recursion:

$$g'_{pq}=g_{pq}\qquad (p,q=0,1)\ ,$$
$$g'_{p,k+1}:=Q_{g'_{pk}}g'_{p,k-1}\quad (p=0,1;\ k>1),\quad g'_{p,k-1}:=Q_{g'_{pk}}g'_{p,k+1}\quad (p=0,1;\ k<0),$$
$$g'_{\ell+1,q}:=Q_{g'_{\ell,q}}g'_{\ell-1,q}\quad (q\in\mathbb{Z};\ \ell>1),\quad g'_{\ell-1,q}:=Q_{g'_{\ell,q}}g'_{\ell+1,q}\quad (q\in\mathbb{Z};\ \ell<0).$$

Corollary 6.11. *There exists an equivalent non-nil weighted grid* $\{g'_{k\ell}:\ k,\ell\in\mathbb{Z}\}$ *such that*

$$\{g'_ug'_vg'_w\}=\mathrm{sgn}(g'_{xy})g'_{u-v+w}\qquad (u,v,w\in\mathbb{Z}^2\ \textit{lie in one straight line})\ .$$

Proof. As we have seen, a non-nil weighted grid G' has the required property whenever $Q_ug_v=\mathrm{sgn}(g_v)g_{2u-v}$ for all $u,v\in\mathbb{Z}^2$. By Lemma 6.9, we may assume without loss of generality that

$$\{g_{pi}g_{pj}g_{pk}\}=\mathrm{sgn}(g_{pj})g_{p,i-j+k}atop\{g_{iq}g_{jq}g_{kq}\}=\mathrm{sgn}(g_{jq})g_{i-j+k,q}\qquad (p,q,i,j,k\in\mathbb{Z})\ .$$

Since, in any case $\mathrm{sgn}(g_{pq})=\mathrm{sgn}(g_{p+2,q})=\mathrm{sgn}(g_{p,q+2})$ $(p,q\in\mathbb{Z})$, we can write

$$\mathrm{sgn}(g_{pq})=\alpha\mu^p\nu^q\kappa^{p\cdot q}\qquad (p,q\in\mathbb{Z})$$

where $\alpha:=\mathrm{sgn}(g_{00})$, $\mu:=\mathrm{sgn}(g_{10})\mathrm{sgn}(g_{00})$, $\nu:=\mathrm{sgn}(g_{01})\mathrm{sgn}(g_{00})$ and $\kappa:=\prod_{k,\ell=0}^1\mathrm{sgn}(g_{k\ell})$.

Given any $x,y,q\in\mathbb{Z}$, we can apply Proposition 6.5 and its Corollary to the strings $a_p:=g_{p,y}$, $b_p:=g_{p,y+q}$, $c_p:=g_{p,y+2q}$ $(p\in\mathbb{Z})$. Since

$$\{a_ka_\ell a_m\}=\alpha_{yq}\sigma_{yq}^\ell a_{k-\ell+m},\quad \{b_kb_\ell b_m\}=\beta_{yq}\tau_{yq}^\ell\qquad (k,\ell,m\in\mathbb{Z})$$

where $\alpha_{yq}:=\mathrm{sgn}(a_0)$, $\beta_{yq}:=\mathrm{sgn}(b_0)$, $\sigma_{yq}:=\mathrm{sgn}(a_0)\mathrm{sgn}(a_1)$ and $\tau_{yq}:=\mathrm{sgn}(b_0)\mathrm{sgn}(b_1)$, it follows in particular

$$Q_{g_{x+p,y+q}}g_{xy}=Q_{b_{x+p}}a_x=\mathrm{sgn}(a_x)\lambda_{-p,yq}c_{x+2p}=\mathrm{sgn}(g_{x,y})\omega_{1,yq}^{-p}\omega_{2,yq}^{-p}g_{x+2p,y+2q}=$$
$$=\mathrm{sgn}(g_{xy})\Omega_{y,q}^pg_{x+2p,y+2q}\qquad (p\in\mathbb{Z})$$

with some constants $\Omega_{y,q}\in\mathbb{T}$ for fixed $y,q\in\mathbb{Z}$. Analogously, by arguing with the strings $a'_q:=g_{xq}$, $b'_q:=g_{x+p,q}$, $c'_q:=g_{x+2p,q}$ $(q\in\mathbb{Z})$, we get

$$Q_{g_{x+p,y+q}}g_{xy}=\mathrm{sgn}(g_{xy})(\Omega'_{x,p})^{-q}g_{x+2p,y+2q}\qquad (q\in\mathbb{Z})$$

with some $\Omega'_{x,p} \in \mathbb{T}$ for fixed x, p. Necessarily

$$(\Omega_{y,q})^p = (\Omega'_{x,p})^q \qquad (x, y, p, q \in \mathbb{Z}) .$$

Here $\Omega_{y,q} = (\Omega_{y,q})^1 = (\Omega'_{0,1})^q \quad (y, q \in \mathbb{Z})$. Similarly $\Omega'_{x,p} = (\Omega_{0,1})^p \quad (x, p \in \mathbb{Z})$. Hence $\Omega_{0,1} = \Omega'_{0,1}$ and, by denoting this common value by Ω,

$$Q_{x+p,y+q} g_{xy} = \mathrm{sgn}(g_{xy}) \Omega^{pq} g_{x+2p,y+2q} \qquad (x, y, p, q \in \mathbb{Z}) .$$

For $\zeta \in \mathbb{T}$, consider the non-nil weighted grid

$$G_\zeta := \{\zeta^{pq} g_{pq} : \ p, q \in \mathbb{Z}\} \qquad (\zeta \in \mathbb{T}) .$$

Since

$$\begin{aligned}
&\{(\zeta^{(x+p)(y+q)} g_{x+p,y+q})(\zeta^{xy} g_{xy})(\zeta^{(x+p)(y+q)} g_{x+p,y+q})\} = \\
&\quad = \zeta^{(x+2p)(y+2q)-2pq} \{g_{x+p,y+q} g_{xy} g_{x+p,y+q}\} = \\
&\quad = (\zeta^{-2}\Omega)^{pq} \zeta^{(x+2p)(y+2q)} \mathrm{sgn}(g_{xy}) g_{x+2p,y+2q} \qquad (x, y, p, q \in \mathbb{Z}),
\end{aligned}$$

by taking a square root $\zeta \in \{\omega : \ \omega^2 = \Omega\}$, the non-nil weighted grid $G' := G_\zeta$ suits our requirements.

Definition 6.12. Henceforth we shall use the notation

$$u \wedge v := \det \begin{pmatrix} u_1 u_2 \\ v_1 v_2 \end{pmatrix} = u_1 v_2 - u_2 v_1 \qquad \text{for } u =: (u_1, u_2), \ v := (v_1, v_2) \in \mathbb{Z}^2 .$$

Lemma 6.13. *Suppose* $\{g_u g_v g_w\} = \mathrm{sgn}(g_v) g_{u-v+w}$ *whenever the vectors* $u, v, w (\in \mathbb{Z}^2)$ *lie on one straight line. Then*

$$\begin{aligned}
&\{g_{z+u} g_z g_{z+w}\} = \mathrm{sgn}(g_z) \xi_{u \wedge v} g_{z+u+v} = \qquad (u, v, z \in \mathbb{Z}^2) \\
&\text{where} \quad \xi_n := \mathrm{sgn}(g_{01}) \mathrm{sgn}(g_{00}) \left[\frac{g_{n1} \,\square\, g_{01}}{g_{n0} \,\square\, g_{00}} \right] \qquad (n \in \mathbb{Z}) .
\end{aligned}$$

Proof. Observe that

$$\begin{aligned}
\{g_{z+u} g_z g_{z+v}\} \qquad &= \qquad \left[\frac{g_{z+u} \,\square\, g_z}{g_{z+u+v} \,\square\, g_{z+v}} \right] \{g_{z+u+v} g_{z+v} g_{z+v}\} = \\
&= S_z(u, v) g_{z+u+v}
\end{aligned}$$

for any $z, u, v \in \mathbb{Z}^2$ where

$$S_z(u, v) := \mathrm{sgn}(g_{z+v}) \left[\frac{g_{z+u} \,\square\, g_z}{g_{z+u+v} \,\square\, g_{z+v}} \right] .$$

Since the triple product is symmetric in the outer variables,

$$S_z(u, v) = S_z(v, u) \qquad (z, u, v \in \mathbb{Z}^2) .$$

If $v \in \mathbb{Z}^2$ and the constant $\theta \in \mathbb{R}$ is such that $\theta v \in \mathbb{Z}$, then for some integers k, n we have $\theta = k/n$, $n > 0$ and $w := (1/n)v \in \mathbb{Z}^2$. Thus

$$\begin{aligned}
S_z(u+\theta v, v) &= \mathrm{sgn}(g_{z+nw})\Big[\frac{g_{z+u+kw} \,\square\, g_z}{g_{z+u+(k+n)w} \,\square\, g_{z+nw}}\Big] = \\
&= \mathrm{sgn}(g_{z+u+kw})\Big[\frac{g_{z+nw} \,\square\, g_z}{g_{z+u+(k+n)w} \,\square\, g_{z+u+kw}}\Big] = \\
&= \frac{\mathrm{sgn}(g_{z+u+kw})}{[\mathrm{sgn}(g_{z+u+w})\mathrm{sgn}(g_{z+u})]^k}\Big[\frac{g_{z+nw} \,\square\, g_z}{g_{z+u+nw} \,\square\, g_{z+u}}\Big] = \\
&= \mathrm{sgn}(g_{z+u})\Big[\frac{g_{z+v} \,\square\, g_z}{g_{z+u+v} \,\square\, g_{z+u}}\Big] = S_z(v,u) = S_z(u,v)
\end{aligned}$$

whenever $z, u, v, \theta v \in \mathbb{Z}$. Hence, as it is well-known from elementary linear algebra of determinants, the functionals S_z satisfies

$$S_z(u,v) = S_z\big((u \wedge v, 0), (0,1)\big) \qquad (z, u, v \in \mathbb{Z}^2) \ .$$

In order to calculate S_z in terms of the coefficients ξ_n , let $z = (p,q), u, v \in \mathbb{Z}^2$ be arbitrarily fixed and write $d := u \wedge v$. For suitable $\alpha, \mu, \nu, \kappa \in \{\pm 1\}$ we have

$$\mathrm{sgn}(g_{xy}) = \alpha \mu^x \nu^y \kappa^{xy} \qquad (x, y \in \mathbb{Z}) \ .$$

Then

$$\begin{aligned}
S_z(u,v) &= S_z\big((d,0),(0,1)\big) = \mathrm{sgn}(g_{p,q+1})\Big[\frac{g_{p+d,q} \,\square\, g_{pq}}{g_{p+d,q+1} \,\square\, g_{p,q+1}}\Big] = \\
&= \mathrm{sgn}(g_{p,q+1})\frac{[\mathrm{sgn}(g_{1q})\mathrm{sgn}(g_{0q})]^p}{[\mathrm{sgn}(g_{1,q+1})\mathrm{sgn}(g_{0,q+1})]^p}\Big[\frac{g_{dq} \,\square\, g_{0q}}{g_{d,q+1} \,\square\, g_{0,q+1}}\Big] = \\
&= (\alpha\mu^p\nu^{q+1}\kappa^{pq+p})\kappa^p \mathrm{sgn}(g_{dq})\mathrm{sgn}(g_{0,q+1})\Big[\frac{g_{0,q+1} \,\square\, g_{0q}}{g_{d,q+1} \,\square\, g_{dq}}\Big] = \\
&= \alpha\mu^{p+d}\nu^q\kappa^{(p+d)q}\frac{[\mathrm{sgn}(g_{01})\mathrm{sgn}(g_{00})]^q}{[\mathrm{sgn}(g_{d1})\mathrm{sgn}(g_{d0})}\Big[\frac{g_{01} \,\square\, g_{00}}{g_{d1} \,\square\, g_{d0}}\Big] \\
&= \alpha\mu^{p+d}\nu^q\kappa^{pq}\mathrm{sgn}(g_{d1})\mathrm{sgn}(g_{00})\Big[\frac{g_{01} \,\square\, g_{d1}}{g_{00} \,\square\, g_{d0}}\Big] = \\
&= \alpha\mu^p\nu^{q+1}\kappa^{pq+d}\Big[\frac{g_{d1} \,\square\, g_{01}}{g_{d0} \,\square\, g_{00}}\Big]^{-} = \mathrm{sgn}(g_{pq})\kappa^d\overline{\xi_d} \ .
\end{aligned}$$

By Proposition 6.5, we have $\xi_{-d} = \Big(\prod_{k,\ell=0}^1 \mathrm{sgn}(g_{k\ell})\Big)^d \overline{\xi_d} = \kappa^d \overline{\xi_d}$. Therefore

$$S_z(u,v) = \mathrm{sgn}(g_z)\xi_{-u\wedge v} \qquad (z, u, v \in \mathbb{Z}^2) \ .$$

Since $S_z(v,u) = S_z(u,v)$, necessarily $\xi_d = \xi_{-d} \qquad (d \in \mathbb{Z})$ which completes the proof.

Theorem 6.14. *Let E be a Jordan* triple spanned by a non-nil weighted grid $G := \{g_{pq} :\ p, q \in \mathbb{Z}\}$ over the non-degenerate weight figure $\mathbb{Z}^2 \oplus 1$ (notation see 6.8) . If $\prod_{k=1}^4 g_{u^{(k)}} = 1$ whenever the vectors $u^{(1)}, \ldots, u^{(4)}$ form a parallelogram* *then there exists an equivalent non-nil weighted grid* $G' := \{g'_{pq} :\ p,q \in \mathbb{Z}\}$ *of* E *along with a constant* $\omega \in \mathbb{T} \cup \mathbb{R} \setminus \{0\}$ *such that*

$$\{g'_{z+u} g'_z g'_{z+v}\} = \big[\frac{1}{2}\omega^{u\wedge v} + \frac{1}{2}\omega^{-u\wedge v}\big]\operatorname{sgn}(g'_z) g'_{z+u+v} \qquad (z,u,v \in \mathbb{Z}^2).$$

Otherwise there exists an equivalent non-nil weighted grid $G' := \{g'_{pq} :\ p,q \in \mathbb{Z}\}$ *along with a constant* $\delta \in \mathbb{R}$ *such that*

$$\{g'_{z+u} g'_z g'_{z+v}\} = \operatorname{Re}\big((1+i\delta) i^{u\wedge v}\big)\operatorname{sgn}(g'_z) g'_{z+u+v} \qquad (z,u,v \in \mathbb{Z}^2).$$

The above operations determine a Jordan triple structure for each value* $\omega \in \mathbb{T} \cup \mathbb{R} \setminus \{0\}$ *and* $\delta \in \mathbb{R}$*, respectively.*

Proof. Let G' be a non-nil weighted grid with the properties described in the previous Lemma. For the sake of simplicity, we may assume $G = G'$ without danger of confusion. We know already that we can parameterize the signs of the grid elements as

$$\operatorname{sgn}(g_{pq}) = \alpha \mu^p \nu^q \kappa^{pq} \qquad (p,q \in \mathbb{Z})$$

and the triple product has the form

$$\{g_{z+u} g_z g_{z+v}\} = \frac{1}{2}\operatorname{sgn}(g_z)\left[(1+i\delta)\omega_1^{u\wedge v} + (1-i\delta)\omega_1^{-u\wedge v}\right] g_{z+u+v}$$

on the grid G with suitable constants $\delta \in \mathbb{R}$, $\omega_1, \omega_2 \in \mathbb{C}\setminus\{0\}$. It is straightforward to verify that

$$\prod_{k=1}^{4} \operatorname{sgn}(g_{v_k}) = \kappa^{(v_1-v_2)\wedge(v_3-v_2)} \qquad \text{whenever} \quad v_4 = v_1 - v_2 + v_3 \ .$$

Furthermore we have established that only the following three cases can occur

$$1)\ \kappa{=}1{=}|\omega_1|{=}|\omega_2|,\ \delta = 0; \quad 2)\ \kappa{=}1{=}\omega_1\overline{\omega_2},\ \delta = 0; \quad 3)\ \kappa{=}{-1},\ \omega_1{=}{-\omega_2}{\in}\mathbb{T}\,.$$

Moreover, since $Q_{g_{x+p,y+q}} g_{xy} = \operatorname{sgn}(g_{xy}) g_{x+2p,y+2q} = \operatorname{sgn}(g_{xy})(\omega_1\omega_2)^{-pq} g_{x+2p,y+2q}$ $(p,q \in \mathbb{Z})$, we have

$$\omega_1\omega_2 = 1 \ .$$

Thus, in Case 1) $\omega_2 = \overline{\omega} \in \mathbb{T}$; in Case 2) $\omega_1, \omega_2 \in \mathbb{R}$ with $\omega_2 = \omega_1^{-1}$; in Case 3) $\omega_2 = -\omega_1 = \pm i$. Therefore we have actually the following two cases

(i) $\kappa = 1\ ,\quad \omega_1 = \omega\ ,\quad \omega_2 = \omega^{-1}\ ,\quad \delta = 0$ for some $\omega \in \mathbb{T} \cup \mathbb{R} \setminus \{0\}$,

(ii) $\kappa = -1\ ,\quad \omega_1 = i\ ,\quad \omega_2 = -i\ .$

To complete the proof, it suffices to check that the sesqui-trilinear extensions of the operations

$$\{g_{z+u} g_z g_{z+v}\}^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa} = \frac{1}{2}\alpha\mu^{z_1}\nu^{z_2}\kappa^{z_1 z_2}\left[(1+i\delta)\omega_1^{u\wedge v} + (1-i\delta)\omega_1^{-u\wedge v}\right] g_{z+u+v}$$

satisfy the Jordan identity whenever the parameters $\alpha, \mu, \nu, \kappa \in \{\pm 1\}$, $\omega_1, \omega_2 \in \mathbb{C}$, $\delta \in \mathbb{R}$ satisfy the relations described in Cases (i),(ii).

Moreover it is enough to check the Jordan identity only for the grid elements. Since for the triple product $\{\ldots\} := \{\ldots\}^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa}$ we have $\{g_a g_b\{g_x g_y g_z\}\}, \{\{g_a g_b g_x\}g_y g_z\}, \{g_x\{g_b g_a g_y\}g_z\}, \{g_x g_y\{g_a g_b g_z\}\} \in \mathbb{C} g_{a-b+x-y+z}$, we have to prove that

$$D^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa}(a,b,x,y,z) = 0 \qquad (a,b,x,y,z \in \mathbb{Z}^2)$$

where the function $D^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa}$ is defined by the relation

$$\frac{1}{4}(\alpha\mu^{b_1}\nu_{b_2}\kappa^{b_1 b_2})(\alpha\mu^{y_1}\nu_{y_2}\kappa^{y_1 y_2})D^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa}(a,b,x,y,z)g_{a-b+x-y+z} :=$$
$$:= \{g_u g_v\{g_x g_y g_z\}\} - \{\{g_u g_v g_x\}g_y g_z\} + \{g_x\{g_v g_u g_y\}g_z\} - \{g_x g_y\{g_u g_v g_z\}\}$$

for $\{\ \} := \{\ \ \}^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa}$. Let $a,b,x,y,z \in \mathbb{Z}^2$ be arbitrarily fixed.

Case (i). We can write

$$\{g_u g_v g_z\}^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa} = \big[\frac{1}{2}\omega^{(u-v)\wedge(w-v)} + \frac{1}{2}\omega^{-(u-v)\wedge(w-v)}\big]\alpha\mu^{v_1}\nu^{v_2}g_{u-v+w}$$

with suitable $0 \neq \omega \in \mathbb{T} \cup \mathbb{R}$. Therefore, in this case

$$\begin{aligned}
&D^{\omega,1/\omega,0}_{\alpha,\mu,nu,1}(a,b,x,y,z) = \\
&= \big(\omega^{(x-y)\wedge(z-y)} + \omega^{-(x-y)\wedge(z-y)}\big)\big(\omega^{(a-b)\wedge(x-y+z-b)} + \omega^{-(a-b)\wedge(x-y+z-b)}\big) - \\
&- \big(\omega^{(a-b)\wedge(x-b)} + \omega^{-(a-b)\wedge(x-b)}\big)\big(\omega^{a-b+x-y)\wedge(z-y)} + \omega^{-(a-b+x-y)\wedge(z-y)}\big) + \\
&+ \big(\overline{\omega}^{(b-a)\wedge y-a)} + \overline{\omega}^{-(b-a)\wedge(y-a)}\big)\big(\overline{\omega}^{(x-b+a-y)\wedge z-b+a-y)} + \overline{\omega}^{-(x-b_a-y)\wedge(z-b+a-y)}\big) - \\
&- \big(\omega^{(a-b)\wedge(z-b)} + \omega^{-(a-b)\wedge(z-b)}\big)\big(\omega^{(x-y)\wedge(a-b+z-y)} + \omega^{-(x-y)\wedge(a-b+z-y)}\big)
\end{aligned}$$

for $\omega \in \mathbb{T} \cup \mathbb{R} \setminus \{0\}$ and $a,b,x,y,z \in \mathbb{Z}^2$. Since

$$\overline{\omega}^d + \overline{\omega}^{-d} = \omega^d + \omega^{-d} \qquad (\omega \in \mathbb{T} \cup \mathbb{R} \setminus \{0\}\ ,\quad d \in \mathbb{R})\ ,$$

the identity $D^{\omega,1/\omega,0}_{\alpha,\mu,nu,1}(a,b,x,y,z) = 0$ holds. Namely, by setting

$$\begin{aligned}
A &:= (x-y)\wedge(z-y) = x\wedge z - y\wedge z - x\wedge y\ , \\
B &:= (a-b)\wedge(x-b) = a\wedge x - b\wedge x - a\wedge b\ , \\
C &:= (b-a)\wedge(y-a) = b\wedge y - a\wedge y + a\wedge b\ , \\
D &:= (a-b)\wedge(z-b) = a\wedge z - b\wedge z - a\wedge b\ ,
\end{aligned}$$

we have

$$\begin{aligned}
(a-b)\wedge(x-y+z-b) &= B + C + D\ , \\
(z-y)\wedge(a-b+x-y) &= -A - C - D\ , \\
(x-b+a-y)\wedge(z-b+a-y) &= A - B + D\ , \\
(x-y)\wedge(a-b+z-y) &= A - B - C\ .
\end{aligned}$$

Hence indeed

$$\begin{aligned}
&\Big(\omega^{(x-y)\wedge(z-y)}+\omega^{-(x-y)\wedge(z-y)}\Big)\Big(\omega^{(a-b)\wedge(x-y+z-b)}+\omega^{-(a-b)\wedge(x-y+z-b)}\Big)-\\
&-\Big(\omega^{(a-b)\wedge(x-b)}+\omega^{-(a-b)\wedge(x-b)}\Big)\Big(\omega^{(z-y)\wedge(a-b+x-y)}+\omega^{-(z-y)\wedge(a-b+x-y)}\Big)+\\
&+\Big(\overline{\omega}^{(b-a)\wedge(y-a)}+\overline{\omega}^{-(b-a)\wedge(y-a)}\Big)\Big(\overline{\omega}^{(x-b+a-y)\wedge(z-b+a-y)}+\overline{\omega}^{-(x-b_a-y)\wedge(z-b+a-y)}\Big)-\\
&-\Big(\omega^{(a-b)\wedge(z-b)}+\omega^{-(a-b)\wedge(z-b)}\Big)\Big(\omega^{(x-y)\wedge(a-b+z-y)}+\omega^{-(x-y)\wedge(a-b+z-y)}\Big)=\\
&=(\omega^{A}+\omega^{-A})(\omega^{B+C+D}+\omega^{-B-C-D})-(\omega^{B}+\omega^{-B})(\omega^{-A-C-D}+\omega^{A+C+D})+\\
&+(\omega^{C}+\omega^{-C})(\omega^{A-B+D}+\omega^{-A+B-D})-(\omega^{D}+\omega^{-D})(\omega^{A-B-C}+\omega^{-A+B+C})=0.
\end{aligned}$$

Case (ii). With some $\delta\in\mathbb{R}$ we can write

$$\{g_u g_v g_w\}^{\omega_1,\omega_2,\delta}_{\alpha,\mu,\nu,\kappa}=\mathrm{Re}\big[(1+i\delta)i^{(u-v)\wedge(w-v)}\big]\alpha\mu^{v_1}\nu^{v_2}(-1)^{v_1v_2}g_{u-v+w}\ .$$

Therefore, by setting $\gamma:=(1+i\delta)/2$, with the same terms A,B,C,D as above

$$\begin{aligned}
D^{i,-i,\delta}_{\alpha,\mu,\nu,-1}&=4\mathrm{Re}(\gamma i^{(x-y)\wedge z-y)})\mathrm{Re}(\gamma i^{(a-b)\wedge(x-y+z-b)})-\\
&\quad-4\mathrm{Re}(\gamma i^{(a-b)\wedge(x-b)})\mathrm{Re}(\gamma i^{(a-b+x-y)\wedge(z-y)})+\\
&\quad+(-1)^{(b-a)\wedge(y-a)}4\mathrm{Re}(\gamma i^{(b-a)\wedge(y-a)})\mathrm{Re}(\gamma i^{-(x-b+a-y)\wedge(z-b+a-y)})-\\
&\quad-4\mathrm{Re}(\gamma i^{(a-b)\wedge(z-b)})\mathrm{Re}(\gamma i^{(x-y)\wedge(a-b+z-y)})=\\
&=2\mathrm{Re}\big[\gamma i^{A}(\gamma i^{B+C+D}+\overline{\gamma}i^{-B-C-D})-\gamma i^{B}(\gamma i^{A+C+D}+\overline{\gamma}i^{-A-C-D})+\\
&\qquad+\gamma i^{-C}(\gamma i^{A-B+D}+\overline{\gamma}i^{-A+B-D})-\gamma i^{D}(\gamma i^{A-B-C}+\overline{\gamma}i^{-A+B+C})\big]=\\
&=2\mathrm{Re}\big[\gamma^2(i^{A+B+C+D}-i^{A+B+C+D}+i^{A-B-C+D}-i^{A-B-C+D})+\\
&\quad+|\gamma|^2(i^{A-B-C-D}-i^{-A+B-C-D}+i^{-A+B-C-D}-i^{-A+B+C+D})\big]=\\
&=2|\gamma|^2\mathrm{Re}(i^{A-B-C-D}-i^{-A+B+C+D})=0\ .
\end{aligned}$$

Remark 6.15. The grid triples $F^{\omega,1/\omega,0}_{\alpha,\mu,\nu,1}$ with the triple products $\{\quad\}^{\omega,1/\omega,0}_{\alpha,\mu,\nu,1}$ are pairwise non-isomorphic for different parameters $\omega\in\mathbb{T}_+\cup(0,1]$ where $\mathbb{T}_+:=\{\zeta\in\mathbb{T}:\ \mathrm{Re}(\zeta),\mathrm{Im}(\zeta)\geq 0\}$. On the other hand, $F^{\omega,1/\omega,0}_{\alpha,\mu,\nu,1}$, $F^{1/\omega,\omega,0}_{\alpha,\mu,\nu,1}$ and $F^{-\omega,-1/\omega,0}_{\alpha,\mu,\nu,1}$ are isomorphic to each other for any $\alpha,\mu,\nu\in\{\pm1\}$ and $\omega\in\mathbb{T}\cup\mathbb{R}\setminus\{0\}$.

The grid triples $F^{i,-i,\delta}_{\alpha,\mu,\nu,-1}$ with the triple products $\{\quad\}^{i,-i,\delta}_{\alpha,\mu,\nu,-1}$ are pairwise non-isomorphic for different parameters $\delta\in[0,\infty)$. On the other hand, $F^{i,-i,\delta}_{\alpha,\mu,\nu,-1}$, $F^{-i,i,\delta}_{\alpha,\mu,\nu,-1}$ and $F^{i,-i,-\delta}_{\alpha,\mu,\nu,-1}$ are isomorphic to each other for any $\alpha,\mu,\nu\in\{\pm1\}$ and $\delta\in\mathbb{R}$.

Remark 6.16. Given $\delta\in[0,\infty)$, by setting $\theta:=\mathrm{arcotg}\,\delta$, the triple product $\{\quad\}^{i,-i,\delta}_{1,1,1,-1}$ of the grid triple $E_\theta:=F^{i,-i,\mathrm{cotg}\,\theta}_{1,1,1,-1}$ has the form

$$\{g_u g_v g_w\}^{i,-i,\mathrm{cotg}\,\theta}_{1,1,1,-1}=(-1)^{v_1v_2}\frac{\sin[\theta-(u-v)\wedge(w-v)\pi/2]}{\sin\theta}g_{u-v+w}\ .$$

Therefore the scaled operation

$$\{g_u g_v g_w\}_\theta:=(-1)^{v_1v_2}\sin[(u-v)\wedge(w-v)\pi/2-\theta]g_{u-v+w}$$

is a triple product on E_θ for any $\theta \in (0, \pi/2]$. Thus, by passing to the limit $\theta \downarrow 0$, the operation $\{\ \ \}_0$ is also a well-defined non-trivial Jordan* triple product on the vector space $\mathrm{Span}_{w\in\mathbb{Z}^2} g_w$ such that

$$(g_w \square g_w)_0 := \{g_w g_w \cdot\}_0 = 0 \qquad (w \in \mathbb{Z}^2) \ .$$

6.1. *Sub-headings*

Sub-headings should be typeset in boldface italic and capitalize the first letter of the first word only. Section number to be in boldface Roman.

6.1.1. *Sub-subheadings*

Typeset sub-subheadings in medium face italic and capitalize the first letter of the first word only. Section number to be in Roman.

6.2. *Numbering*

Sections, sub-sections and sub-subsections are to be numbered in Arabic. Sections and sub-sections in boldface while sub-subsections in Roman.

6.3. *Lists of items*

Lists may be laid out with each item marked by a dot:

- item one,
- item two.

Items may also be numbered in lowercase Roman numerals:

(i) item one
(ii) item two
  (a) Lists within lists can be numbered with lowercase Roman letters,
  (b) second item.

7. Equations

Displayed equations should be numbered consecutively in each section, with the number set flush right and enclosed in parentheses.

$$\mu(n,t) = \frac{\sum_{i=1}^{\infty} 1(d_i < t, N(d_i) = n)}{\int_{\sigma=0}^{t} 1(N(\sigma) = n)d\sigma} \ . \tag{7.1}$$

Equations should be referred to in abbreviated form, e.g. "Eq. (7.1)" or "(4.1)". In multiple-line equations, the number should be given on the last line.

Centralize displayed equations with the page width. Standard English letters like x are to appear as x (italicized) in the text if they are used as mathematical symbols. Punctuation marks are used at the end of equations as if they appeared directly in the text.

8. Theorem Environments

Theorem 8.1. *Theorems , lemmas , propositions , corollaries are to be numbered consecutively in the paper or in each section. Use italic for the body and upper and lower case boldface for the declaration.*

Remark 8.1. Remarks, examples, definitions are to be numbered consecutively in the paper or in each section. Use Roman for the body and upper and lower case boldface , for the declaration.

Proof. The word 'Proof' should be type in boldface. Proofs should end with a box. □

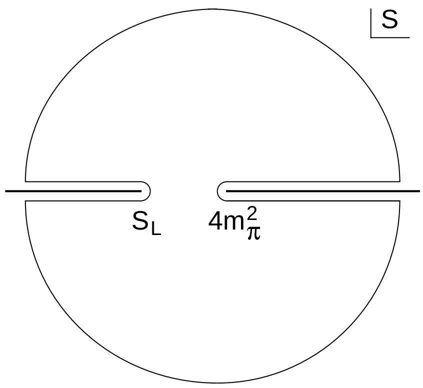


Fig. 1. A schematic illustration of dissociative recombination. The direct mechanism, $4m_\pi^2$ is initiated when the molecular ion S_L captures an electron with kinetic energy.

9. Illustrations and Photographs

Figures are to be inserted in the text nearest their first reference. eps files or postscript files are preferred. If photographs are to be used, only black and white ones are acceptable.

Figures are to be sequentially numbered in Arabic numerals. Centralize the caption and place it below the figure. Typeset in 8 pt Times Roman with baselineskip of 10 pt. Use double spacing between a caption and the text that follows immediately.

Previously published material must be accompanied by written permission from the author and publisher.

10. Tables

Tables should be inserted in the text as close to the point of reference as possible. Some space should be left above and below the table.

Table 1. Comparison of acoustic for frequencies for piston-cylinder problem.

Piston mass	Analytical frequency (Rad/s)[a]	TRIA6-S_1 model (Rad/s)[b]	% Error
1.0	281.0	280.81	0.07
0.1	876.0	875.74	0.03
0.01	2441.0	2441.0	0.0
0.001	4130.0	4129.3	0.16

Note: Table notes.
[a]Table footnote A.
[b]Table footnote B.

Tables should be numbered sequentially in the text in Arabic numerals. Captions are to be centralized above the tables. Typeset tables and captions in 8 pt Times Roman with baselineskip of 10 pt.

If tables need to extend over to a second page, the continuation of the table should be preceded by a caption, e.g. "Table 2. (*Continued*)"

11. Footnotes

Footnotes should be numbered sequentially in superscript letters. [b]

Appendix A. Appendices

Appendices should be used only when absolutely necessary. They should come before the Acknowledgment. If there is more than one appendix, number them alphabetically. Number displayed equations in the way, e.g. (A.1), (A.2), etc.

[b] Footnotes should be typeset in 8 pt Times Roman at the bottom of the page.

$$f(j\delta, i\delta) \cong \frac{\pi}{M} \sum_{n=1}^{M} Q_{\theta_n}(j\cos\theta_n + i\sin\theta_n)\,. \qquad \text{(A.1)}$$

Note Added

Should be placed before Acknowledgment.

Acknowledgment

This section should come before the References and should be unnumbered. Funding information may also be included here.

References

References are to be listed in alphabetical order of the author's name and cited in the text in Arabic numerals within square brackets. They can be referred to indirectly, e.g. "… in the statement [2]." or used directly, e.g. "… see [2] for examples." List references using the style shown in the following examples. For journal names, use the standard abbreviations. Typeset references in 9 pt Roman with baselineskip of 11 pt.

References

1. N. Bellomo and L. M. de Socio, Initial boundary value problems for the semidiscrete Boltzmann equation, *J. Math. Anal. Appl.* **128** (1987) 112–124.
2. H. Cabanes, Global solution of the discrete Boltzmann equation, in *Mathematical Problems in the Kinetic Theory of Gases*, eds. H. Neunzert and D. Pack (Lang Verlag, 1979), pp. 22–45.
3. R. Gatignol, *Theorie Cinetique des Gaz a Repartition Discrete des Vitesses*, Springer Lect. Notes in Phys., Vol. 36 (Springer, 1975).
4. E. Longo and N. Bellomo, On the semidiscrete Boltzmann equation with multiple collisions, in *Symposium: Progress en Mecanique*, Springer Lect. Notes in Phys., eds. R. Gatignol and Soubbaramayer, to appear.
5. P. Resibois and M. De Leener, *The Mathematical Theory of the Kinetic Theory of Fluids* (Wiley, 1977).
6. C. J. Van der Hoek, Contamination of a well in a uniform background flow, Report Dept. of Math., Agricult. Univ. Wageningen Technical Note 90–03 (1990).