Continuous Reinhardt domains from a Jordan view point

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Abstract. As a natural extension of bounded complete Reinhardt domains in $\mathbb{C}^N$ to spaces of continuous functions, continuous Reinhardt domains (CRD) are bounded open connected solid sets in commutative C*-algebras with respect to the natural ordering. We give a complete parametric description for the structure of holomorphic isomorphisms between CRDs and characterize the partial Jordan triple structures which can be associated with some CRD. On the basis of these results, we test two conjectures concerning the Jordan structure of bounded circular domains. It turns out that both the problems of the bidualization and the unique extension of inner derivations have positive solution in the setting of CRDs.

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1. Introduction.

A classical complete Reinhardt domain is an open connected subset in the space $\mathbb{C}^n$ of all complex n-tuples, being invariant under all coordinate multiplications $(z_1,\ldots,z_n)\mapsto(\lambda_1 z_1,\ldots,\lambda_n z_n)$ with $|\lambda_1|,\ldots,|\lambda_n|\le 1$. Regarding $\mathbb{C}^n$ as a the complex ordered space of the functions $z:\{1,\ldots,n\}\to\mathbb{C}$, this property can be stated as

$$\text{(CR)} \qquad f\in D \quad \text{and} \quad |g|\le|f| \Rightarrow \; g\in D\,.$$

Postulating (CR) in terms of the order absolute value, we can speak of bounded complete Reinhardt domains in complex Banach lattices in a natural manner.

In 1974 Sunada [18] has achieved a rather thorough description of classical bounded Reinhardt domains containing the origin from the viewpoint of holomorphic equivalence. Later on several authors investigated holomorphic equivalence of generalized Reinhardt domains in atomic Banach lattices [2,3,12]. Motivated by an interesting work of Vigué [19] on the possible lack of symmetry of continuous products of discs with different radius, in [16] we introduced the concept of *continuous Reinhardt domains* (CRD for short). By definition, *a CRD is a bounded complete Reinhardt domain in the C*-algebra of all bounded continuous functions over some topological space* or which is the same, in a commutative C*-algebra. In [16] we have shown that a symmetric CRD is a continuous mixture of finite dimensional Euclidean balls, essentially more involved than direct sums of topological products of balls. In [7] we found matrix representations for linear isomorphisms

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between two symmetric CRDs. To achieve these results we intensively used the Jordan theory of the bidual embedding of symmetric domains. However, the main points of both Sunada's and Vigué's papers concern the non-symmetric case. Recently, based upon the Lie theory of Hermitian operators in the dual space, in [17] we managed to extend the matrix representations of [7] to a Banach-Stone type theorem on the isomorphisms of general Banach lattice normed commutative C*-algebras. This result includes implicitly the description of all the possible linear isomorphisms between CRSs because the convex hull of any CRD can be regarded as the unit ball of some lattice norm in a commutative C*-algebra and linear isomorphisms preserve convex hulls. The aim of this paper is a description of all possible holomorphic equivalences of CRDs. According to a classical result of [4], every bounded circular domain and hence even a non-symmetric CRD admits a natural partial Jordan*-triple structure, a so-called partial JB*-triple, which gives rise to the description of its complete holomorphic vector fields – a crucial piece of information about its holomorphic geometry. In particular, every holomorphic isomorphism between two bounded circular domains is the composition of a linear isomorphism with the exponential of a suitable complete holomorphic vector field over one of the two domains.

In Section 2 we review the basic material [4,13,14] concerning partial JB*-triples. The bidual embedding arguments used in [16] to treat the Jordan structure in the symmetric case are not available for general CRDs. Although Dineen [5] and Barton–Timoney [1] established a satisfactory bidual Jordan theory for all bounded convex circular domains already in 1986, it is still one of the fundamental open questions in geometric Jordan theory without known interesting partial results as far, if the canonical partial triple product associated with any *non-convex* bounded circular domain extends in a weak*-continuous manner to the canonical partial triple product of some bounded circular domain in the bidual. Instead, in Section 3 we develop an alternative approach for determining the partial JB*-triple product associated with a CRD. The conclusion, Theorem 3.5 is an integral representation of this triple product. In classical finite dimensional complex analysis, Reinhardt domains are popular test objects for conjectures. In the second half of the paper we use this integral representation as a starting point to solve the special case of two open problems on bounded circular domains in the setting of CRDs.

The first problem we treat takes its origin in a work of Panou [10] where it is shown that every inner derivation of the Jordan-triple associated with the symmetric part of a finite-dimensional bounded circular domain admits a unique extension to an inner derivation of the partial Jordan triple associated with the whole domain. Though it is natural to expect that the analog holds in general Banach spaces, the only known infinite-dimensional results concern domains with nearly atomic symmetric part [15]. On the basis of Theorem 3.5 along with the fine structure description of the Jordan triple product associated with a symmetric CRD [7], in Section 4 we can establish immediately that the partial Jordan triple of a CRD has the unique extension property of inner derivations.

The second question we solve for CRDs is the mentioned open problem of the Jordan structure of second dual of a partial JB*-triple. First, in Section 5 we refine Theorem 3.5 into a natural extension of the results for symmetric CRDs given in [7] whose proofs there relied upon some bidual considerations in [16]. By proceeding the opposite way, in Section 6 we apply the fine structure description obtained in Section 5 along with function representations of $\mathcal{C}_0(\Omega)''$ spaces to establish that the Jordan triple product associated with a CRD admits a separately weak*-continuous bidual extension which can be regarded as the canonical Jordan triple of some not necessarily unique CRD.

In course of the investigations in Section 5 we applied a Riesz type representation theorem for positive multilinear functionals on products of $\mathcal{C}_0$ -spaces which seems to be never stated explicitly in the literature. Actually the result we need is contained implicitly in a recent work of Villanueva [20]. We close the paper with an Appendix including a short direct proof.

2. Preliminaries on partial JB*-triples

Recall [14] that given a complex Banach space E (with norm $\|.\|$), the tuple $(E, E_0, \{...\})$ is called a *partial Jordan*-triple* if E_0 is a closed complex subspace of E, and $\{\ldots\}$ is a continuous operation $E \times E_0 \times E \to E$ with the following properties:

(J1) $\{xay\}$ is symmetric bilinear in the variables $x, y (\in E)$, conjugate linear in $a (\in E_0)$ and $\{E_0 E_0 E_0\} \subset E_0$;
(J2) the Jordan identity holds, i.e. for all $a, b, c \in E_0$ and $x, y \in V$
$\{ab\{xcy\} = \{\{abx\}cy\} - \{x\{bac\}y\} + \{xc\{abz\}\}$;
(J3) we have the weak associativity
$\{\{xax\}bx\} = \{xa\{xbx\}\}, \quad a, b \in E_0, \; x \in E$.

Notice that in the case of full Jordan*-triples i.e. if $E_0 = E$, axiom (J3) is a consequence of (J2) (see e.g. [6, Ch. 10]). The geometric importance of partial Jordan*-triples relies upon the fact established first implicitly in [4,9,6] that given any bounded circular domain D in a Banach space E, there is a necessarily unique partial Jordan*-triple $(E, E_D, \{\ldots\}_D)$ called the *canonical partial Jordan*-triple of* D such that the figure $D \cap E_D$ consists of the centers of holomorphic symmetries of D and $\lim_{t \downarrow 0} S_{ta} S_0(x) = a - \{xax\}_D$ for all $a \in E_D$ and $x \in E_0$ where S_c denotes the holomorphic symmetry of D with the center $c \in D \cap E_D$. We say that $(E, E_0, \{\ldots\})$ is a *partial JB*-triple* if it is a subtriple in some canonical partial Jordan*-triple $(E, E_D, \{\ldots\}_D)$. In other words this means that all the vector fields $[a - \{xax\}]\partial/\partial x$, $a \in E_0$ are complete in a suitable bounded circular domain $D (\subset E)$. This terminology is in accordance with the customary use of the term JB*-triple for full Jordan triples. Indeed, by Kaup's Riemann mapping theorem [8], in the case $E = E_D$ the domain D is necessarily convex and hence the carrier space E can be renormed in a manner such that D becomes the unit ball and the usual C*- and hermitian positivity axioms be satisfied. By the results of [13,14], we have a complete axiomatic description of partial JB*-triples. A partial Jordan*-triple $(E, E_0, \{\ldots\})$ is a partial JB*-triple if and only if

(J4) the operators $L(a) : x \mapsto \{aax\}$, $a \in E_0$ have spectrum ≥ 0 with
$\inf_{\|a\|=1} \|L(a)a\| \neq 0$;
(J5) $L(a) \in \mathrm{Her}(B)$, $a \in E_0$ for some bounded circular domain B.

It is well-known that the domain B in (J5) can be chosen to be convex and such that his gauge function $\|.\|_B$ should satisfy the C*-axiom

(J4′) $\mathrm{Sp}L(a) \geq 0$ with $\|L(a)a\|_B = \|a\|_B^3$ for all $a \in E_0$.

Given any bounded circular domain B fulfilling (J5), there exists $\varepsilon > 0$ such that for any $\delta \in (0, \varepsilon)$, $(E, E_0, \{\ldots\})$ is a subtriple of $(E, E_{D_\delta}, \{\ldots\}_{D_\delta})$ with the bounded circular domain

$$D_\delta := \bigcup_{a \in E_0} \big[\exp\big((a - \{xax\})\partial/\partial x\big)\big](\delta B). \tag{2.1}$$

Our next aim will be to describe the canonical partial JB*-triples of CRDs. Recall [4] that the group $\mathrm{Aut}(E, E_0, \{\ldots\}) := \big\{L \in \mathcal{L}(E) : \ LE_0 \subset E_0,\ L\{xay\} = \{(Lx)(La)(Ly)\} \text{ for } a \in E_0,\ x, y \in E\big\}$ of all automorphisms of the triple $\mathbf{E} = (E, E_0, \{\ldots\})$ coincides with the set of all injective linear transformations $L : E \to E$ such that $LD = D$ whenever $\mathbf{E}$ is the canonical JB*-triple of a bounded circular domain D . In particular, if $D \subset \mathcal{C}_0(\Omega)$ is a CRD, all multiplications with continuous functions of absolute value one belong to $\mathrm{Aut}(E, E_D, \{\ldots\}_D)$. So first we consider the effect of linear automorphisms to the construction (2.1).

2.2 Lemma. *Let* $(E, E_0, \{\ldots\})$ *be a partial JB*-triple and* Ψ *a bounded subgroup of* $\mathrm{Aut}(E, E_0, \{\ldots\})$ *. Then there exists a* Ψ *-invariant bounded circular domain* $D \subset E$ *such that* $(E, E_0, \{\ldots\})$ *is a subtriple of* $(E, E_D, \{\ldots\}_D)$ *.*

Proof. Choose a bounded circular domain B in E satisfying axiom (J5). Define $B_1 := \bigcup_{\psi \in \Psi} \psi B$. Since Φ is a bounded group of linear mappings, B_1 is a bounded Ψ-invariant circular domain in E . Given any $a \in E_0$ and $\psi \in \Psi$, since $\psi \in \mathrm{Aut}(E, E_0, \{\ldots\})$, we have $L(\psi^{-1}a) = \psi^{-1}L(a)\psi$. Since $\psi^{-1}a \in E_0$, by axiom (J5) it follows $\exp\big(itL(\psi^{-1}a)\big)\psi B = \psi B$, $t \in \mathbb{R}$. Since $\psi^{-1}a$ can be any element in E_0 , we also get $\exp\big(itL(a)\big)\psi B = \psi B$ for all $a \in E_0$, $t \in \mathbb{R}$ and $\psi \in \Psi$. That is $L(a) \in \mathrm{Her}(\psi B)$, $\psi \in \Psi$ and hence $L(a) \in \mathrm{Her}\big(\bigcup_{\phi \in \Psi} \psi B\big) = \mathrm{Her}(B_1)$ for all $a \in E_0$. Thus the domain B_1 suits axiom (J5) and we can use it in the construction (2.1) instead of B with some $\delta > 0$. Given any $\psi \in \Psi$, it only remains to prove that $\psi \bigcup_{a \in E_0} \big[\exp\big((a - \{xax\})\partial/\partial x\big)\big](\delta B_1) = \bigcup_{a \in E_0} \big[\exp\big((a - \{xax\})\partial/\partial x\big)\big](\delta B_1)$. However this is again a direct consequence of the facts $\psi(\delta B_1) = \delta B_1$ and $\psi \exp\big((a - \{xax\})\partial/\partial x\big) = \big[\exp\big((\psi a - \{x(\psi a)x\})\partial/\partial x\big)\big]\psi$. Here the latter identity follows from the relation $\psi \in \mathrm{Aut}(E, E_0, \{\ldots\})$. $\square$

3. Integral formula of the canonical partial triple product for a CRD

Let $(E, E_0, \{\ldots\})$ denote a fixed partial JB*-triple over $E := \mathcal{C}_0(\Omega)$ with a locally compact topological Hausdorff space Ω . Throughout this section we assume its Reinhardt property

(R) $\qquad \Psi \subset \mathrm{Aut}(E, E_0, \{\ldots\})$ where $\Psi := \{\psi\cdot :\ \psi \in \mathcal{C}(\Omega),\ |\psi| = 1\}$

and $\psi\cdot$ denotes the multiplication operator $\mathcal{C}_0(\Omega) \ni f \mapsto \varphi f$. As we mentioned, the canonical partial JB*-triple $(E, E_D, \{\ldots\}_D)$ of any CRD D has property (R). Moreover, from Lemma 2.2 we know also that $(E, E_0, \{\ldots\})$ can be regarded as a subtriple in the canonical JB*-triple of some CRD in $\mathcal{C}_0(\Omega)$.

As a first consequence of (R), we have $e^{it\phi}E_0 \subset E_0$ and

$$e^{it\phi}\{xay\} = \{(e^{it\phi}x)(e^{it\phi}a)(e^{it\phi}y)\}, \qquad t \in \mathbb{R}$$

for any bounded continuous function $\phi : \Omega \to \mathbb{R}$ (with $a \in E_0,\ x, y \in E$). Hence derivation with respect to the variable t yields

$$\text{(3.1)} \qquad \psi E_0 \subset E_0\ , \quad \psi\{xay\} = \{(\psi x)ay\} - \{x(\overline{\psi}a)y\} + \{xa(\psi y)\}$$

for all bounded continuous functions $\psi : \Omega \to \mathbb{C}$. In particular E_0 is a closed ideal in $\mathcal{C}_0(\Omega)$ regarded as a commutative C^*-algebra with the pointwise product of functions. Therefore necessarily

$$E_0 = \mathcal{C}_0(\Omega_0) := \{f \in \mathcal{C}_0(\Omega) :\ f(\Omega \setminus \Omega_D) = 0\}$$

with the open set $\Omega_0 := \{\omega \in \Omega :\ \exists\ a \in E_0 \quad a(\omega) \neq 0\}$.

3.2 Lemma. $\{xay\}(\omega) = 0$ *whenever* $x(\omega) = y(\omega) = 0$.

Proof. By the symmetry (J1), it suffices to see the statement for the case $x = y$. Furthermore, by the continuity of the triple product and since continuous functions vanishing at $\omega (\in \Omega)$ can uniformly be approximated with continuous function vanishing on some neighborhood of ω, it suffices to see that $\{xax\}(\omega) = 0$ if $x(U) = 0$ for some neighborhood $U \subset \Omega$ of the point ω.

Assume $\omega \in U$ open $\subset \Omega$, $x \in E$, and $x(U) = 0$. Choose a compact neighborhood V of ω within U and let $\phi : \Omega \to [0,1]$ be a continuous function such that $\phi(\omega) = 1$ and $\phi(\Omega \setminus V) = 0$. Observe that if $c \in E_0$ is a function with $c(V) = 0$ then $\phi x = \phi c = 0$ and, by (5.1),

$$\{xcx\}(\omega) = \phi\{xcx\}(\omega) = 2\{(\phi x)cx\}(\omega) - \{x(\phi c)x\}(\omega) = 0.$$

Consider any $a \in E_0$. Choose a continuous function $\psi : \Omega \to [0,1]$ with $\psi(V) = 0$ and $\psi(\Omega \setminus U) = 1$. Since $\psi x = x$, by the aid of the function $c := \psi a$ vanishing on V we get

$$\begin{aligned} 0 = \psi\{xax\}(\omega) &= 2\{(\psi x)ax\}(\omega) - \{x(\psi a)x\}(\omega) = \\ &= 2\{xax\} - \{xcx\} = 2\{xax\}. \quad \square \end{aligned}$$

3.3 Corollary. *We have*

$$\{xay\} = \frac{1}{2}x\{z(\overline{y}a)z\} + \frac{1}{2}y\{z(\overline{x}a)z\} \qquad \text{if} \quad z \in E \ \text{ with } \ xz = x \ \text{ and } \ yz = z\ .$$

Proof. Suppose $xz = x$ and $yz = z$. Consider any point $\omega \in \Omega$ and apply Lemma 3.2 to the functions $x_\omega := x - x(\omega)z$ and $y_\omega := y - y(\omega)z$ satisfying $x_\omega(\omega) = y_\omega(\omega) = 0$. We get

$$\begin{aligned} 0 = \{x_\omega a y_\omega\}(\omega) = \{[x - x(\omega)z]a[y - y(\omega)z]\}(\omega) = \\ = \{xay\}(\omega) + x(\omega)y(\omega)\{zaz\}(\omega) - x(\omega)\{zay\}(\omega) - y(\omega)\{zax\}(\omega) \ . \end{aligned}$$

Thus, everywhere on Ω,

$$\{xay\} = -xy\{zaz\} + x\{zay\} + y\{zax\} \ .$$

Observe that, by (3.1) and since $yz = y$, here we have

$$xy\{zaz\} = x\big[2\{yaz\} - \{z(\overline{y}a)z\}\big]$$

and similarly $yx\{zaz\} = y[2\{xaz\} - \{z(\overline{x}a)z\}]$. Therefore

$$\begin{aligned} xy\{zaz\} &= x\{zay\} + y\{zax\} - \frac{1}{2}\{z(\overline{x}a)z\} - \frac{1}{2}\{z(\overline{y}a)z\} \ , \\ \{xay\} &= -xy\{zaz\} + x\{zay\} + y\{zax\} = \\ &= \frac{1}{2}x\{z(\overline{y}a)z\} + \frac{1}{2}y\{z(\overline{x}a)z\} \ . \quad \square \end{aligned}$$

3.4 Lemma. *The triple product* $\{...\}$ *is positive in the sense*

$$x, a, y \geq 0 \Rightarrow \{xay\} \geq 0 \ .$$

Proof. Fix $0 \leq x, y \in E$ and $0 \leq a \in E_0$ arbitrarily. Since functions with compact support are dense in $\mathcal{C}_0$-spaces, we may assume

$$\mathrm{supp}(x), \mathrm{supp}(y) \text{ compact} \subset \Omega \ , \quad \mathrm{supp}(a) \text{ compact} \subset \Omega_0 \ .$$

Then we can choose $0 \leq x_0, x_1, y_0, y_1, z \in \mathcal{C}_0(\Omega)$ with compact support such that

$$\begin{aligned} &x = x_0 + x_1 \ , \quad y = y_0 + y_1 \ , \\ &\mathrm{supp}(x_0), \mathrm{supp}(y_0) \subset \Omega_0 \ , \\ &\mathrm{supp}(x_1) \cap \mathrm{supp}(a) = \mathrm{supp}(y_1) \cap \mathrm{supp}(a) = \emptyset \ , \\ &\mathrm{supp}(x) \cup \mathrm{supp}(y) \cup \mathrm{supp}(a) \subset \{\zeta \in \Omega : \ z(\zeta) = 1\} \ . \end{aligned}$$

By (J1) we have $\{xay\} = \sum_{k,\ell=0}^{1} \{x_k a y_\ell\}$. Here we have $\{x_0 a y_0\} \geq 0$ for the following reasons. The subtriple $(E_0, E_0, \{\ldots\}|E_0^3)$ is a JB*-triple with $\Psi \subset \mathrm{Aut}(E_0, E_0, \{\ldots\}|E_0^3)$. Therefore it is necessarily the canonical JB*-triple of a bounded symmetric continuous Reinhardt domain in $E_0 = \mathcal{C}_0(\Omega_0)$. However, by [16, Theorem 2] the triple product is non-negative for non-negative functions for symmetric CRDs. Thus indeed $\{x_0 a y_0\} \geq 0$ since $a, x_0, y_0 \in E_0$. On the other hand, by Corollary 3.3,

$$\{x_1 a y_1\} = \frac{1}{2} x_1 \{z(\overline{y_1}a)z\} + \frac{1}{2} y_1 \{z(\overline{x_1}a)z\} = 0$$

because $\overline{x_1}a = \overline{y_1}a = 0$. It only remains to see $\{x_0ay_1\} \geq 0$ (since the proof of $\{x_1ay_0\} = \{y_0ax_1\} \geq 0$ is analogous). Define

$$c := \sqrt{x_0a} \ .$$

Since $\mathrm{supp}(c) \subset \mathrm{supp}(a) \subset \Omega_0$, we have $c \in E_0$ and $cz = c$. By Corollary 3.3 (applied with y_1 instead of y and first with c instead both of a and x and then with x_0 instead of x),

$$\{ccy_1\} = \frac{1}{2}c\{z(y_1c)z\} + \frac{1}{2}y_1\{z(c^2)z\} = \frac{1}{2}y_1\{z(c^2)z\} \ ,$$
$$\{x_0ay_1\} = \frac{1}{2}x_0\{z(y_1a)z\} + \frac{1}{2}y_1\{z(x_0a)z\} = \frac{1}{2}y_1\{z(x_0a)z\}$$

because $y_1a = y_1c = 0$. That is

$$\{x_0ay_1\} = \{ccy_1\} = \frac{1}{2}y_1\{z(x_0a)z\} \ .$$

According to (J4), $\mathrm{Sp}(L(c)) \geq 0$. However, it is a basic fact about the spectra of multipliers in commutative Banach algebras that

$$u(\Omega) \subset \mathrm{Sp}\big[\mathcal{C}_0(X) \ni f \mapsto uf\big] \quad \text{if } X \text{ open} \subset \Omega \text{ and } u \in \mathcal{C}_0 \ .$$

In particular, by taking $X := \Omega \setminus \mathrm{supp}(a)$ and $u := \{z(x_0a)z\}$ we have

$$\frac{1}{2}\{z(x_0)az\}\big(\Omega \setminus \mathrm{supp}(a)\big) \subset L(c) \subset [0,\infty).$$

Thus $\{z(x_0a)z\} \geq 0$ on $\mathrm{supp}(y_1)$ and hence $2\{x_0ay_1\} = y_1\{z(x_0a)z\} \geq 0$. □

We can summarize the results of this section in the following theorem.

3.5 Theorem. *Let Ω be a locally compact space, $E := \mathcal{C}_0(\Omega)$ and suppose $(E, E_0, \{\ldots\})$ is a partial JB*-triple with the Reinhardt property* (R) *. Then there exists an open subset Ω_0 in Ω such that $E_0 = \{f \in E : \ f(\Omega \setminus \Omega_0) = 0\}$. Given any point $\omega \in \Omega$, there is a* (*unique*) *positive Radon measure μ_ω on Ω_0 with total mass $\leq M := \sup_{0 \leq x,a,y \leq 1} \max\{xay\}$ and*

$$\text{(3.6)} \qquad \{xay\}(\omega) = \frac{1}{2}x(\omega)\int \overline{a}y \ d\mu_\omega + \frac{1}{2}y(\omega)\int \overline{a}x \ d\mu_\omega, \qquad x,y \in E, \ a \in E_0.$$

Proof. We have established already the relation $E_0 = \mathcal{C}_0(\Omega_0)$ and the positivity of the triple product in the sense of Lemma 3.4. Fix $\omega \in \Omega$ arbitrarily. According to [20] *, the positivity of the bounded 3-linear functional $(x,a,y) \mapsto \{x\overline{a}y\}(\omega)$

* This fact is implicit in [20]. For the sake of completeness, we include a short direct proof in the Appendix.

implies the existence of a positive Radon measure ν_ω of finite total variation on $\Omega\times\Omega_0\times\Omega$ such that

$$\{x\overline{a}y\}(\omega)=\int x\otimes a\otimes y\ d\nu_\omega,\qquad x,y\in E,\ a\in E_0$$

where $x\otimes a\otimes y$ denotes the function $(\xi,\alpha,\eta)\mapsto x(\xi)a(\alpha)y(\eta)$ on $\Omega\times\Omega_0\times\Omega$. It is well-known that $\nu_\omega(\Omega\times\Omega_0\times\Omega)=\sup\big\{\int x\otimes a\otimes y\ d\nu_\omega:\ a\in\mathcal{C}_0(\Omega_0),\ x,y\in\mathcal{C}_0(\Omega),\ 0\le a,x\le 1\big\}=M$. By the inner compact regularity of Radon measures, given any functions $x,y\in E$ with compact support and and $a\in E_0$, we can choose an increasing sequence $K_1\subset K_2\subset\ldots\subset\Omega$ of compact sets such that $\mathrm{supp}(x)\cup\mathrm{supp}(y)\subset K_1$ and $\lim_{n\to\infty}\nu_(\Omega\setminus K_n)=0$. Also we can choose a sequence of functions $z_1,z_2,\ldots\in\mathcal{C}_0(\Omega)=E$ such that $0\le z_1\le z_2\le\ldots\le 1$ and $z_n(K_n)=1$ $(n=1,2\ldots)$. Then, by Corollary 3.3, we have

$$\begin{aligned}\{xay\}(\omega)&=\frac12 x(\omega)\{z_n(\overline{y}a)z_n\}+\frac12 y(\omega)\{z_n(\overline{x}a)z_n\}=\\&=\frac12 x(\omega)\int z_n\otimes(\overline{a}y)\otimes z_n\ d\nu_\omega+\frac12 y(\omega)\int z_n\otimes(\overline{a}x)\otimes z_n\ d\nu_\omega\to\\&\to\frac12 x(\omega)\int 1_\Omega\otimes(\overline{a}y)\otimes 1_\Omega\ d\nu_\omega+\frac12 y(\omega)\int 1_\Omega\otimes(\overline{a}x)\otimes 1_\Omega\ d\nu_\omega\end{aligned}$$

where 1_Ω denotes the function identically 1 on Ω. Thus with the measure

$$\mu_\omega(X):=\nu_\omega(\Omega\times X\times\Omega)\qquad(X\ \text{Borel}\subset\Omega)$$

we have the stated relation for $x,y\in E$ with compact support. The statement follows by the uniform density of functions with compact support in E. ∎

3.7 Remark. It would be tempting to conjecture that every partial JB*-triple satisfying the hypothesis of Theorem 3.5 is the canonical JB*-triple of some CRD. However there is a counterexample even in 2 dimensions.

Let $\Omega:=\{1,2\}$, $\Omega_0:=\{1\}$ *and* $\{xay\}:=[1\mapsto x(1)\overline{a(1)}y(1),\ 2\mapsto x(1)\overline{a(1)}y(2)/2+x(2)\overline{a(1)}y(1)/2]$. *Then any CRD* D *over* Ω *such that the vector fields* $\big[a-\{xax\}\big]\partial/\partial x$ *are complete in* D *must be an ellipsoid of the form* $D=\{x:\ |x(1)|^2+\lambda|x(1)|^2<1\}$ *for some* $\lambda>0$ *with* $E_D=E\neq E_0$.

4. Extension from the symmetric part

Next we are going to study partial Jordan*-triples with triple product of the form obtained in Theorem 3.5 for canonical JB*-triples of CRDs. By the aid of bidual embedding, a technique not yet available for general partial JB*-triples, in [16] we have achieved a finer analog of Theorem 3.5 in the special case of symmetric CRDs. Applying [16, Theorem 2] to the restriction of the triple product to the symmetric part E_0 in Theorem 3.5, we see that the measures μ_ω, $\omega\in\Omega_0$ have finite support. Moreover there exists a partition $\{\Omega_i:\ i\in I\}$ of Ω_0 consisting of finite sets along with a function $m:\Omega_0\to\mathbb{R}$ such that, by writing $i(\omega)$ for the unique index $i\in I$ with $\omega\in\Omega_i$,

$$\mu_\omega=\sum_{\eta\in\Omega_{i(\omega)}} m(\eta)\delta_\eta,\quad \omega\in\Omega_0,\quad 0<\inf m\le\sup_{i\in I}\sum_{\eta\in\Omega_i} m(\eta)<\infty. \tag{4.1}$$

4.2. Theorem. *Let* $(E,E_0,\{\ldots\})$ *be a partial Jordan*-triple where* $E:=\mathcal{C}_0(\Omega)$ *with a locally compact topological space* Ω, $E_0:=\{f\in E:\ f(\Omega\setminus\Omega_0)=0\}$ *with a non-empty open subset* $\Omega_0\subset\Omega$ *and the triple product* $\{\ldots\}$ *has the form* (3.6). *If the measures* μ_ω *are all positive and* (4.1) *holds then* $(E,E_0,\{\ldots\})$ *is a subtriple of the canonical partial JB*-triple of some CRD.*

Proof. By assumption, the triple product $\{\ldots\}$ satisfies axioms (J1),(J2),(J3) in Section 2. Thus to see that $(E,E_0,\{\ldots\})$ is a partial JB*-triple, we have to verify axioms (J4),(J5). According to [16, Theorem 2], the restriction of $\{\ldots\}$ to the symmetric part $E_0\times E_0\times E_0$, is the canonical JB*-triple product of the symmetric CRD $D_0:=\big\{f\in E_0:\ \sup_{i\in I}\sum_{\eta\in\Omega_i} m(\eta)|f(\eta)|^2<1\big\}$ in E_0. Hence $\inf_{\|a\|_\infty=1}\|L(a)a\|_\infty>0$. Consider the set

$$B:=\Big\{f\in E:\ [\inf m]\max|f|^2<1,\ \sup_{i\in I}\sum_{\eta\in\Omega_i} m(\eta)|f(\eta)|^2<1\Big\}.$$

This is clearly a bounded convex CRD in E. We have $B=\bigcap_{\omega\in\Omega\setminus\Omega_0}B_\omega\cap\bigcap_{i\in I}B^{(i)}$ with the (unbounded) CRDs $B_\omega:=\big\{f\in E:\ |f(\omega|<1/\inf m\big\}$ and $B^{(i)}:=\big\{f\in E:\ \big\langle[f|\Omega_i]\big|[f|\Omega_i]\big\rangle<1\big\}$ where $\langle.|.\rangle_i$ denotes the scalar product $\langle\varphi|\psi.\rangle_i:=\sum_{\eta\in\Omega_i} m(\eta)\varphi(\eta)\overline{\psi}(\eta)$. Given any function $a\in E_0$, we have

$$\begin{aligned} L(a)x(\omega)&=\{aax\}(\omega)=\frac{1}{2}x(\omega)\int_{\eta\in\Omega_0}|a(\eta)|^2\,d\mu_\omega(\eta),\quad \omega\in\Omega\setminus\Omega_0;\\ L(a)x\big|\Omega_i&=\frac{1}{2}\big\langle[a|\Omega_i]\big|[a|\Omega_i]\big\rangle_i[x|\Omega_i]+\frac{1}{2}\big\langle[x|\Omega_i]\big|[a|\Omega_i]\big\rangle_i[a|\Omega_i],\quad i\in I.\end{aligned}$$

For any fixed $\omega\in\Omega\setminus\Omega_0$, $\tau\in\mathbb{C}$, it follows $\exp\big(i\tau L(a)\big)x(\omega)=e^{i\tau\int|a|^2\ d\mu_\omega}x(\omega)$. As a consequence, $\exp\big(i\tau L(a)\big)B_\omega\subset B_\omega$ whenever $\operatorname{Im}\tau\ge 0$. Similarly, $\exp\big(i\tau L(a)\big)B^{(i)}\subset B^{(i)}$ for $\operatorname{Im}\tau\ge0$ and $i\in I$ because the any mapping $\varphi\mapsto\frac12\langle\alpha|\alpha\rangle_i\varphi+\frac12\langle\varphi|\alpha\rangle_i\alpha$ is a positive linear operator with respect to the scalar product $\langle.|.\rangle_i$. Therefore $\exp\big(i\tau L(a)\big)B\subset B$ if $\operatorname{Im}\tau\ge0$. Hence axioms (J4),(J5) are immediate. Thus $(E,E_0,\{\ldots\})$ is a partial JB*-triple.

Due to the form (3.6) of the triple product, the group Ψ of the multiplications with functions of modulus 1 consists of automorphisms of $(E,E_0,\{\ldots\}$. In view of Lemma 2.2 we conclude that $(E,E_0,\{\ldots\})$ is a subtriple of the canonical triple of some bounded domain being invariant under the multiplications with continuous functions with modulus 1 that is a CRD. □

We can also apply the structural descriptions of Section 3 with Theorem 4.2 to testing if all inner derivations of the canonical partial JB*-triple of a CRD can be extended in a uniformly continuous manner from the symmetric part to the whole space. As we shall see, this category does not give any counterexample.

4.3 Theorem. *On the space* $E := \mathcal{C}_0(\Omega)$ *, let* $(E, E_0, \{\ldots\})$ *be a partial JB*-triple such that* $\{\psi\cdot :\ \psi \in \mathcal{C}(\Omega),\ |\psi| = 1\} \subset \mathrm{Aut}(E, E_0, \{\ldots\})$ *. Then there exists a finite constant* M *such that* $\|\Delta\| \leq M\|\Delta|E_0\|$ *for all inner derivations* Δ *of* $(E, E_0, \{\ldots\})$ *.*

Proof. We know there exists an open subset Ω_0 of Ω with $E_0 = \{f \in \mathcal{C}_0(\Omega_0) : f(\Omega| \setminus \Omega_0) = 0\}$. Furthermore, for any point $\omega \in \Omega$, there is a positive Radon measure μ_ω on Ω_0 such that

$$\{xax\}(\omega) = x(\omega) \int_{\Omega_0} x\overline{a}\ d\mu_\omega, \qquad x \in E,\ a \in E_0,\ \omega \in \Omega.$$

Since $\mathcal{C}_0(\Omega_0) = \{x\overline{a}|\Omega_0 :\ x \in E,\ a \in E_0\}$, the function $\omega \mapsto \int_{\Omega_0} f\ d\mu_\omega$ is necessarily continuous for all fixed $f \in \mathcal{C}_0(\Omega_0)$. Finally we may assume the measures μ_ω , $\omega \in \Omega_0$ to be in the form (4.1). Thus, by writing $S(\omega) := \Omega_{i(\omega)}$ for short,

$$\int_{\Omega_0} f\ d\mu_\omega = \sum_{\eta \in S(\omega)} m(\eta) f(\eta), \quad \omega \in \Omega_0,\ f \in \mathcal{C}_0(\Omega_0).$$

Notice also that $\omega \in S(\omega)$ for all $\omega \in \Omega_0$ and $0 < \inf m \leq \sup m < \infty$ and $\sup_{\omega \in \Omega_0} \#S(\omega) < \infty$. Consider an inner derivation Δ of $(E, E_0, \{\ldots\})$. That is

$$\Delta x = \sum_{k=1}^{N} \{a_k b_k x\}, \qquad x \in E$$

for some finite sequence $a_1, b_1, \ldots, a_N, b_N \in E_0$. In particular, given any function $x \in E = \mathcal{C}_0(\Omega)$,

$$2\Delta x(\omega) = \sum_{\eta \in S(\omega)} \sum_{k=1}^{N} \big[a_k(\eta)\overline{b_k(\eta)} x(\omega) + x(\eta)\overline{b_k(\eta)} a_k(\omega) \quad \text{for } \omega \in \Omega_0,$$

$$\Delta x(\omega) = \int_{\Omega_0} \sum_{k=1}^{N} a_k(\zeta)\overline{b_k(\zeta)}\ d\mu_\omega(\zeta) x(\omega) \quad \text{for } \omega \in \Omega \setminus \Omega_0.$$

The continuity $\|\{xay\}\| \leq K\|x\|\|a\|\|y\|$ of the partial triple product implies that $\sup_{\omega \in \Omega} \mu_\omega(\Omega_0) \leq K < \infty$. Hence it suffices to see that

$$\text{(4.4)} \qquad \sup_{\zeta \in \Omega_0} \Big| \sum_{k=1}^{N} a_k(\zeta)\overline{b_k(\zeta)} \Big| \leq \frac{4\|\Delta|E_0\|}{\inf m}.$$

For the proof of this inequality, fix any point $\zeta \in \Omega_0$. Since the set $S(\zeta)$ is finite, for each point $\omega \in S(\zeta)$ we can find a function $e_\omega \in E_0 \equiv \mathcal{C}_0(\Omega_0)$ such that

$1 = e_\omega(\zeta) = \sup|e_\omega(.)|$ but $e_\omega(\eta) = 0$ for $\zeta \neq \eta \in S(\zeta)$. Then

$$2\Delta e_\omega = \Big[\sum_{k=1}^N \sum_{\eta\in S(\zeta)} m(\eta) a_k(\eta)\overline{b_k(\eta)}\Big] e_\omega + \sum_{k=1}^N m(\omega)\overline{b_k(\omega)} a_k,$$

$$2\sum_{\omega\in S(\zeta)} [\Delta e_\omega](\omega) = \sum_{k=1}^N \#S(\zeta) \sum_{\omega\in S(\zeta)} m(\omega) a_k(\omega)\overline{b_k(\omega)} + \sum_{k=1}^N \sum_{\omega\in S(\zeta)} m(\omega) a_k(\omega)\overline{b_k(\omega)} =$$

$$= \big[\#S(\zeta)+1\big] \sum_{\omega\in S(\zeta)} \sum_{k=1}^N m(\omega) a_k(\omega)\overline{b_k(\omega)}.$$

It follows

$$m(\zeta)\sum_{k=1}^N \overline{b_k(\zeta)} a_k(\zeta) = 2[\Delta e_\zeta](\zeta) - \sum_{\omega\in S(\zeta)} m(\omega) \sum_{k=1}^N a_k(\omega)\overline{b_k(\omega)},$$

$$\sum_{\omega\in S(\zeta)} m(\omega) \sum_{k=1}^N a_k(\omega)\overline{b_k(\omega)} = \frac{2}{\#S(\zeta)+1} \sum_{\omega\in S(\zeta)} [\Delta e_\omega](\omega).$$

Notice that $\|e_\omega\| = \max|e_\omega(.)| = 1$ and hence $|[\Delta e_\omega](\omega)| \leq \|\Delta|E_0\|$ for all $\omega \in \Omega$. Therefore

$$m(\omega)\Big|\sum_{k=1}^N a_k(\omega)\overline{b_k(\omega)}\Big| \leq 2\|\Delta|E_0\| + \frac{2}{\#S(\zeta)+1} \sum_{\omega\in S(\zeta)} \|\Delta|E_0\| \leq 4\|\Delta|E_0\|.$$

This completes the proof of (4.4) and hence the theorem. □

5. The fine structure of the canonical partial JB*-triple of a CRD

Throughout this section Ω denotes a locally compact Hausdorff space, $\Omega_0 \neq \emptyset$ is a fixed open subset of Ω and we write $E := \mathcal{C}_0(\Omega)$, $E_0 := \{f \in E :\ f(\Omega\backslash\Omega_0)\}$. Also we reserve the notations $\big[\mu_\omega :\ \omega\in\Omega\big]$, $\{\Omega_i :\ i \in I\}$ respectively m for a given measure valued map $\Omega \to \mathcal{M}(\Omega_0)_+$, a partition of Ω_0 into finite non-empty sets and a function $m : \Omega_0 \to \mathbb{R}_+$ such that (4.1) holds. We know from Theorems 3.5 and 4.2 that the canonical triple product of a CRD has necessarily the form (3.6) in terms of these objects.

Our purpose will be to find a description in terms of the topological properties of the partition $\{\Omega_i :\ i \in I\}$ and the for a triple product of the form (3.6) to be the canonical triple product of some CRD. It is clear that there are plenty of mappings $\omega \mapsto \mu_\omega$ even satisfying (4.1) for which the operation (3.6) is no partial JB*-triple product. Indeed, the following observation is an immediate but fundamental consequence of Theorem 3.5 and its proof. *Given a bounded Reinhardt domain D in E , the canonical triple product $\{\dots\} := \{\dots\}_D$ has the form*

$$(5.1)\qquad \{xay\} = \frac{1}{2}xA(\overline{a}y) + \frac{1}{2}yA(\overline{a}x), \qquad a \in E_0,\ x,y \in E$$

with some positive linear map $A\colon E_0 \to \mathcal{C}_b(\Omega) := \{\text{bounded cont. functions } \Omega \to \mathbb{C}\}$. It is well-known [11] that the positivity of A entails its boundedness automatically. Notice also that, by the Riesz-Kakutani representation theorem, any positive linear mapping $A : \mathcal{C}_0(\Omega_0) \to \mathcal{C}_b(\Omega)$ has the form $Af(\omega) = \int_{\Omega_0} f \, d\mu_\omega$ with a uniquely determined mapping $\Omega \ni \omega \to \mu_\omega \in \mathcal{M}(\Omega_0)_+$.

5.2. Lemma. *Suppose* $A : E_0 \to \mathcal{C}_b(\Omega)$ *is a positive linear mapping. Then the structure* $(E, E_0, \{\ldots\})$ *with the operation* (5.1) *is a partial Jordan*-triple if and only if*

$$(5.3) \qquad A\big(fA(g)\big) = A\big(gA(f)\big), \quad A\big(fA(g)\big)\big|\Omega_0 = A(f)A(g)\big|\Omega_0 \qquad f, g \in E_0.$$

Proof. Since E_0 and E are closed ideals in $C_b(\Omega)$ with respect to the pointwise product of functions, the operation (5.1) is a well-defined positive continuous sesquitrilinear map $E \times E_0 \times E \to E$. It satisfies the identities

$$\{xa\{xbx\}\} = \frac{1}{2} xA\big(\bar{a}xA(\bar{b}x)\big) + \frac{1}{2} xA(\bar{b}x)A(\bar{b}x),$$
$$\{xa\{xbx\}\} - \{xb\{xax\}\} = \frac{1}{2} xA\big[\bar{a}xA(\bar{b}x)\big) - \bar{b}xA(\bar{a}x)\big].$$

Hence, by taking $f := \bar{a}x$ and $g := \bar{b}x$, we see that (5.3) implies axiom (J3). Assume (J3) holds. Then $xA\big[\bar{a}xA(\bar{b}x)\big) - \bar{b}xA(\bar{a}x)\big] = 0$ for $a, b \in E_0$ and $x \in E$. Consider any functions $f, g \in E_0$ with compact support. Then, given any point $\omega \in \Omega$, we can choose a function $x_\omega \in E$ with compact support such that the interior of $\text{supp}(x_\omega)$ contains $\{\omega\} \cup \text{supp}(f) \cup \text{supp}(g)$. Then we can write $f = \overline{a_\omega} x_\omega$ and $g = \overline{b_\omega} x_\omega$ with some $a_\omega, b_\omega \in E_0$ and hence (J3) implies $0 = A\big[fA(g) - gA(f)\big](\omega)$. Thus, since functions with compact supports are dense in E_0 , axiom (J3) is equivalent to the identity $A\big(fA(g)\big) = A\big(gA(f)\big)$ in (5.3).

Let us now proceed to the axiom (J2) of the Jordan identity. By polarization, (J2) is equivalent to its special case

$$(\text{J2}') \qquad \big\{aa\{xbx\}\big\} = 2\big\{\{aax\}bx\big\} - \big\{x\{aab\}x\big\}, \quad a, b \in E_0, \ x \in E.$$

In terms of the operation A , this identity (multiplied by 2) can be stated as

$$aA\big(\bar{a}xA(\bar{b}x)\big) + xA(\bar{b}x)A(|a|^2) =$$
$$= \big[aA(\bar{a}x) + xA(|a|^2)\big]A(\bar{b}x) + xA\big(\bar{b}\big[aA(\bar{a}x) + xA(|a|^2)\big]\big) -$$
$$- xA\big(\big[aA(\bar{a}b) + bA(|a|^2)\big]^{-} x\big).$$

By the positivity of A , we can write $-xA\big(\big[\bar{a}A(a\bar{b}) + \bar{b}A(|a|^2)\big]x\big)$ for the last term above. Thus, by the linearity of A , axiom $(\text{J2}')$ is equivalent to

$$aA\big(\bar{a}xA(\bar{b}x)\big) + xA(\bar{b}x)A(|a|^2) =$$
$$= aA(\bar{a}x)A(\bar{b}x) + xA(|a|^2)A(\bar{b}x) + xA\big(\bar{b}aA(\bar{a}x)\big) + xA\big(\bar{b}xA(|a|^2)\big) -$$
$$- xA\big(\bar{a}xA(a\bar{b})\big) - xA\big(\bar{b}xA(|a|^2)\big).$$

Here the terms $xA(\bar{b}x)A(|a|^2)$ and $xA\big(\bar{b}xA(|a|^2)\big)$ cancel, whence we get

$$\text{(J2}''\text{)} \qquad aA\big(\bar{a}xA(\bar{b}x)\big) = aA(\bar{a}x)A(\bar{b}x) + xA\big(\bar{b}aA(\bar{a}x)\big) - xA\big(\bar{a}xA(a\bar{b})\big).$$

Observe that (5.3) implies (J2$''$) immediately. To finish the proof, assume (J2)+(J3). As we have shown, this is nothing else as the identity $A\big(fA(g)\big) = A\big(gA(f)\big)$ along with (J2$''$). By substituting $f := \bar{a}x$ and $g := a\bar{b}$ in (J2$''$), we see that two terms cancel and the remaining identity $aA\big(\bar{a}xA(\bar{b}x)\big) = aA(\bar{a}x)A(\bar{b}x)$ is equivalent to its polarized form

$$a_1A\big(\overline{a_2}xA(\bar{b}x)\big) = a_1A(\overline{a_2}x)A(\bar{b}x), \qquad a_1, a_2, b \in E_0,\ x \in E.$$

Since each function $a_1 \in E_0$ vanishes outside Ω_0 but for any point $\omega \in \Omega_0$ there is a function $a_{1,\omega} \in E_0$ with $a_{1,\omega}(\omega) \neq 0$, the polarized identity is further equivalent to

$$A\big(\bar{a}xA(\bar{b}x)\big)\big|\Omega_0 = A(\bar{a}x)A(\bar{b}x)\big|\Omega_0, \qquad a, b \in E_0,\ x \in E.$$

As we have seen, any functions $f, g \in E_0$ with compact support can be written in the form $f = \bar{a}x$, $g = \bar{b}x$ for suitable functions $a, b \in E_0$ and $x \in E$ with compact support. This implies the second identity in (5.3) for functions with compact support, and statement follows by a standard density argument. □

5.4. Remark. An application of the results in [7] concerning symmetric CRDs to the symmetric part of the canonical partial JB*-triple of a CRD yields the following observation. If $(E, E_0, \{\ldots\})$ is a partial JB*-triple with a triple product of the form (3.6) and having property (4.1), then the set-valued function $\omega \mapsto \Omega_{i(\omega)} \cup \{\infty\}$ (where $i(\omega)$ denotes the unique index $i \in I$ with $\omega \in \Omega_i$) is continuous with respect to the Hausdorff topology of the non-empty compact subsets of $\Omega_0 \cup \{\infty\}$. As a consequence, *given a relatively closed subset F of Ω_0 and a point $\omega \in F$ such that $\#[F \cap \Omega_{i(\omega)}] = N_F := \max_{\eta \in F} \#[F \cap \Omega_{i(\eta)}]$, there are disjoint open sets $U_1, \ldots, U_{N_F} \subset \Omega_0$ such that $\omega \in U_1$ and $\#[U_k \cap F \cap \Omega_{i(\eta)}] = 1$ for any $\eta \in U_1 \cup \cdots \cup U_{N_F}$ and $k = 1, \ldots, N_F$.*

5.5. Lemma. *Assume the mapping $\omega \to \Omega_{i(\omega)} \cup \{\infty\}$ is Hausdorff continuous in the sense of 5.4. Then given any point $\omega \in \Omega$, there exists a finite family of disjoint Borel subsets $G_1, \ldots, G_N \subset \Omega_0$ such that $\mu_\omega\big(\Omega_0 \setminus \bigcup_{k=1}^N G_k\big) = 0$ and $\#[\Omega_i \cap G_k] \leq 1$ for all $i \in I$ and $k = 1, \ldots, N$.*

Proof. We use countable transfinite exhaustion to construct the sets $G_1, \ldots, G_N$. For starting, let $N := N_{\Omega_0}$, $F^{(0)} := \Omega_0$ and $U_1^{(0)}, \ldots, U_N^{(0)} := \emptyset$. For any countable ordinal $r \succ 0$, until each set $U_k^{(s)}$ with $s \prec r$ and $1 \leq k \leq N$ is open and we have $\mu_\omega\big(\bigcup_{s \prec r} \bigcup_{k=1}^N U_k^{(s)}\big) < \mu_\omega(\Omega_0)$, define $F^{(r)} := \Omega_0 \setminus \bigcup_{s \prec r} \bigcup_{k=1}^N U_k^{(s)}$, $N_r := \max_{\eta \in F^{(r)}} \#[F^{(r)} \cap \Omega_{i(\eta)}]$. We also choose some point $\omega_r \in F^{(r)}$ with $\#[F^{(r)} \cap \Omega_{i(\omega_r)}] = N_r$ along with a finite disjoint family $U_1^{(r)}, \ldots, U_{N_r}^{(r)}$ such that $\mu_\omega(U_1^{(r)}) > 0$ and $\#[U_k^{(r)} \cap \Omega_{i(\eta)}] = 1$ for all $\eta \in U_1^{(r)} \cup \cdots \cup U_{N_r}^{(r)}$ and

$k=1,\ldots,N_r$. Finally we set $U_k^r := \emptyset$ for the indices $N_r < k \le N$. This can be well-done in view of Remark 5.4 and the fact that trivially $N_r \le N$. Since the measure μ_ω is finite, in this manner, for some countable ordinal r^*, we get a family $\{U_k^{(r)} : r \prec r^*, \ k=1,\ldots,N\}$ of open subsets of Ω_0 such that $\mu_\omega\big(\Omega_0 \setminus \bigcup_{s\prec r^*}\bigcup_{k=1}^N U_k^{(s)}\big)=0$ and $\#[U_k^{(r)}\cap F^{(r)}\cap\Omega_{i(\eta)}]\le 1$, $1\le k\le N$ but $\Omega_{i(\eta)}\subset\bigcup_{k=1}^N[U_k^{(r)}\cap F^{(r)}]$ for all $\eta\in\bigcup_{k=1}^N[U_k^{(r)}\cap F^{(r)}]$ for all ordinals $r\prec r^*$. Therefore the choice $G_k := \bigcup_{s\prec r^*}[U_k^{(s)}\cap F^{(s)}]$, $k=1,\ldots,N$ suits our requirements. □

5.6. Corollary. *Let* $\mathcal{K} := \{K\subset I : \ \bigcup_{i\in K}\Omega_i \text{ is Borel measurable}\}$ *and define* $\widetilde{\mu}_\omega(K) := \mu_\omega\big(\bigcup_{i\in K}\Omega_i\big)$, $K\in\mathcal{K}$. *Then (under the hypothesis of Lemma* 5.5) *there is a Borel function* $p_\omega : \Omega_0\to[0,1]$ *such that* $\sum_{\eta\in\Omega_i}p_\omega(\eta)=1$, $i\in I$ *and for all bounded Borel functions* $f:\Omega_0\to\mathbb{C}$ *we have*

$$\int_{\Omega_0} f\ d\mu_\omega = \int_{i\in I}\sum_{\eta\in\Omega_i} f(\eta)p_\omega(\eta)\ d\widetilde{\mu}_\omega(i).$$

Proof. As we have noted, the sets $G_k^{(r)} := U_k^{(r)}\cap F^{(r)} = U_k^{(r)}\setminus\bigcup_{s\prec r}\bigcup_{k=1}^{N_r}U_k^{(s)}$, $r\prec r^*$, $1\le k\le N_r$ form a disjoint covering of Ω_0 up to a set of μ_ω-measure 0. Let $\widetilde{p}_k^{(r)}$ denote the Radon-Nikodým derivative $d\widetilde{\mu}_{\omega,k}^{(r)}/d\widetilde{\mu}_\omega$ with the measure $\widetilde{\mu}_{\omega,k}^{(r)}(K) := \mu_\omega\big(G_k^{(r)}\cap\bigcup_{i\in K}\Omega_i\big)$, $K\in\mathcal{K}$. These are functions $I\to\mathbb{R}$ defined up to a set of $\widetilde{\mu}_\omega$-measure 0, and we can choose Borel measurable representatives with $0\le\widetilde{p}_k^{(r)}\le 1$ and $\sum_{k=1}^{N_r}\widetilde{p}_k^{(r)}=1$ on $I^{(r)} := \{i\in I : \ \Omega_i\subset\bigcup_{k=1}^{N_r}G_k^{(r)}\}$ and vanishing outside $I^{(r)}$. This can be done because every partition member Ω_i meets any set $G_k^{(r)}$ in at most one point and for the sets $G^{(r)} := \bigcup_{k=1}^{N_r}G_k^{(r)}$ either we have $\Omega_i\subset G^{(r)}$ or $\Omega_i\cap G^{(r)}=\emptyset$. Hence the statement holds with the function $p(\eta) := \sum_{r\prec r^*}\sum_{k=1}^{N_r}\widetilde{p}_k^{(r)}\big(i(\eta)\big)$, $\eta\in\Omega_0$. □

5.7. Corollary. *Suppose we have* (4.1) *with a weight function* $m>0$ *and let the mapping* $A:\mathcal{C}_0(\Omega_0)\to\mathcal{C}_b(\Omega)$ *have the form* $Af(\omega)=\int_{\eta\in\Omega_0}f\ d\mu_\omega$ *with suitable Radon measures* μ_ω, $\omega\in\Omega$. *Then the identity* $A\big(fA(g)\big)=A\big(gA(f)\big)$ *is equivalent to the fact that*

$$\mu_\omega(X) = \int_{i\in I}\sum_{\eta\in X\cap\Omega_i} m(\eta)\ d\kappa_\omega(i), \qquad X\subset\Omega_0 \tag{5.8}$$

with suitable measures $\kappa_\omega : \{K\subset I : \bigcup_{i\in K}\Omega_i \text{ is Borel measurable}\}\to\mathbb{R}_+$, $\omega\in\Omega$.

Proof. Using the results of Corollary 5.6, we can write

$$\begin{aligned}\big[A\big(fA(g)\big)\big](\omega) &= \int_{i\in I}\sum_{\eta\in\Omega_i} f(\eta)\big[A(g)\big](\eta)\ p_\omega(\eta)\ d\widetilde{\mu}_\omega(i) = \\ &= \int_{i\in I}\sum_{\eta\in\Omega_i} f(\eta)\sum_{\zeta\in\Omega_{i(\eta)}} g(\zeta)m(\zeta)p_\omega(\eta)\ d\widetilde{\mu}_\omega(i) = \\ &= \int_{i\in I}\sum_{\zeta,\eta\in\Omega_i} f(\eta)g(\zeta)m(\zeta)p_\omega(\eta)\ d\widetilde{\mu}_\omega(i)\end{aligned}$$

because we have $i(\eta)=i$ for the points $\eta\in\Omega_i$. Thus the identity $A\big(fA(g)\big)=A\big(gA(f)\big)$ is equivalent to

$$(5.9)\qquad 0=\int_{i\in I}\sum_{\zeta,\eta\in\Omega_i} f(\eta)g(\zeta)\big[m(\zeta)p_\omega(\eta)-m(\eta)p_\omega(\zeta)\big]\ d\widetilde{\mu}_\omega(i)$$

for all $f,g\in\mathcal{C}_0(\Omega_0)$ and $\omega\in\Omega$. By passing to limits of monotone sequences, we see that (5.9) holds for all $f,g\in\mathcal{C}_0(\Omega_0)$ if and only if it holds for all bounded Borel measurable functions $f,g:\Omega_0\to\mathbb{C}$. Consider (5.9) with the partition $\Omega_0=\bigcup_{r\prec r^*}\bigcup_{k=1}^{N_r}G_k^{(r)}\cup\big[\mu_\omega\text{-zero-set}\big]$ constructed in the proof of Corollary 5.6. By writing $\zeta_{i,k}^{(r)}$ for the unique element of the intersection $\Omega_i\cap G_k^{(r)}$, we get

$$(5.9')\quad 0=\sum_{r\prec r^*}\sum_{k,\ell=1}^{N_r}\int_{i\in I^{(r)}} f(\zeta_{i,k}^{(r)})g(\zeta_{i,\ell}^{(r)})\big[m(\zeta_{i,k}^{(r)})p_\omega(\zeta_{i,\ell}^{(r)})-m(\zeta_{i,\ell}^{(r)})p_\omega(\zeta_{i,k}^{(r)})\big]\ d\widetilde{\mu}_\omega(i).$$

This holds for all bounded Borel functions $f,g:\Omega_0\to\mathbb{C}$ if and only if, given any $r\prec r^*$, for $\widetilde{\mu}_\omega$ -almost every $i\in I^{(r)}$ we have

$$m(\zeta_{i,k}^{(r)})p_\omega(\zeta_{i,\ell}^{(r)})-m(\zeta_{i,\ell}^{(r)})p_\omega(\zeta_{i,k}^{(r)})=0,\qquad 1\le k,\ell\le N_r.$$

Indeed, if we just consider functions f,g vanishing outside the sets $G_k^{(r)}$ respectively $G_\ell^{(r)}$ (with fixed $r\prec r^*$ and $1\le k,\ell\le N_r$), we obtain $(5.9')$ without the summations $\sum_{r\prec r^*}$ and $\sum_{k,\ell=1}^{N_r}$, whence the statement is immediate. Thus, since $\sum_{\zeta\in\Omega_i}p_\omega(\zeta)=1$ for $\widetilde{\mu}_\omega$ -almost every $i\in I$, $(5.9')$ holds for all bounded Borel functions if and only if

$$p_\omega(\eta)=m(\eta)\Big[\sum_{\zeta\in\Omega_i}m(\zeta)\Big]^{-1}\quad\text{for } \widetilde{\mu}\text{-almost every } i\in I \text{ and } \eta\in\Omega_i\,.$$

This observation establishes the statement of 5.7 with the measures $\kappa_\omega(K):=\int_{i\in K}\big[\sum_{\zeta\in\Omega_i}m(\zeta)\big]^{-1}d\widetilde{\mu}_\omega(i)$, $K\in\mathcal{K}$. □

6. Bidual of the canonical JB*-triple of a CRD

On the basis of the previous section, first we give an exhaustive parametric description of the canonical JB*-triples of continuous Reihardt domains. Also we answer in the affirmative the question if the bidual of the canonical JB*-triple of a continuous Reihardt domain can be regarded as the canonical JB*-triple of a continuous Reihardt domain in the bidual commutative C*-algebra.

As in the previous sections, Ω denotes an arbitrarily fixed locally compact Hausdorff space, Ω_0 is a non-empty open subset of Ω , m is a function $\Omega_0\to\mathbb{R}$ and $\Pi=\{\Omega_i:\ i\in I\}$ is a partition of Ω_0 . We shall write Ω_0/Π for the index set I of Π equipped with the topology inherited from the Hausdorff topology of $\widetilde{\Omega_0}:=\big\{\Omega_i\cup\{\infty\}:\ i\in I\big\}$. That is a set $J\subset I$ is open if $\big\{\Omega_i\cup\{\infty\}:\ i\in J\big\}$

is an open subset of $\widetilde{\Omega_0}$ with respect to the Hausdorff topology of the compact subsets of $\Omega_0 \cup \{\infty\}$ restricted to $\widetilde{\Omega_0}$.

6.1 Definition. (cf. [7, 1.1-2]) We say that the couple (m, Π) is *admissible* if $\sup_{i\in I} \#\Omega_i < \infty$, $0 < \inf m \le \sup m < \infty$ and all the functions $\Omega_0 \ni \omega \mapsto \sum_{\eta\in\Omega_{i(\omega)}} m(\eta)f(\eta)$, $f \in \mathcal{C}_0(\Omega_0)$ are continuous.

According to [7, 1.2], the couple (m, Π) is admissible if and only if the function space $\mathcal{C}_0(\Omega_0)$ endowed with the triple product polarized from $\{xax\}(\omega) := \sum_{\zeta\in\Omega_{i(\omega)}} m(\zeta)x(\zeta)\overline{a(\zeta)}x(\omega)$ (where $i(\omega)$ denotes the (unique) index with $\omega \in \Omega_{i(\omega)}$) is the canonical triple of some symmetric Reinhardt domain in $\mathcal{C}_0(\Omega_0)$. Furthermore, as a consqeence of [7, 1.3(iii)], given an admissible couple (m, Π), the topological space Ω_0/Π is locally compact and Hausdorff.

6.2. Lemma. *Let (m, Π) be an admissible couple.*

1) *A function $\phi : I \to \mathbb{C}$ belongs to $\mathcal{C}_0(\Omega_0/\Pi)$ if and only if $f_\phi := \big[\omega \mapsto \phi\big(i(\omega)\big)\big]$ is a bounded continuous function on Ω_0 being constant along the sets Ω_i, $i \in I$ and being such that for any $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \in \Omega_0$ with $|f_\phi(\Omega_i)| < \varepsilon$ whenever $\Omega_i \cap K_\varepsilon = \emptyset$.*
2) *The range of the operator $\widetilde{A}_0$ on $\mathcal{C}_0(\Omega_0)$ defined by*

$$\widetilde{A}_0 f(i) := \sum_{\zeta\in\Omega_{i(\omega)}} m(\zeta)f(\zeta), \qquad i \in I,\ f \in \mathcal{C}_0(\Omega_0) \tag{6.3}$$

is a uniformly dense multiplicative ideal in $\mathcal{C}_0(\Omega_0/\Pi)$.

Proof. 1) Let $\phi \in \mathcal{C}_0(\Omega_0/\Pi)$. By construction, the function f_ϕ is constant along the sets Ω_i, $i \in I$. Also the ranges of ϕ and f_ϕ coincide, thus f_ϕ is necessarily bounded. Consider a convergent net $\omega_j \to \omega_0$ in Ω_0. According to [7, 1.2(iv)], we have $\Omega_{i(\omega_j)} \cup \{\infty\} \to \Omega_{i(\omega_0)} \cup \{\infty\}$ with respect to Hausdorff topology. Therefore $f_\phi(\omega_j) \to f_\phi(\omega_0)$ showing that $f_\phi \in \mathcal{C}_b(\Omega_0)$. The stated vanishing property of f_ϕ at infinity is straightforward. Conversely, assume that $\phi : I \to \mathbb{C}$ is a function such that $f_\phi \in \mathcal{C}_b(\Omega_0)$ with the behavior at infinity in the sense of the statement 1). Then ϕ vanishes at infinity in the sense of the locally compact inherited Hausdorff topology of Ω_0/Π. We show the continuity of ϕ as follows. Let $[i_j : j \in J]$ be a net in I such that $\Omega_{i_j} \cup \{\infty\} \to \Omega_{i_0} \cup \{\infty\}$ in Hausdorff sense. By [7, 1.3(i)] we can find a convergent net $\omega_j \to \omega_0$ in Ω_0 with $\omega_j \in \Omega_{i_j}$, $j \in J$ and $\omega_0 \in \Omega_{i_0}$. Hence $\phi(i_j) = f_\phi(\Omega_{i_j}) = f_\phi(\omega_j) \to f_\phi(\omega_0) = f_\phi(\Omega_{i_0}) = \phi(i_0)$.

2) As we have noted, for each function $f \in \mathcal{C}_0(\Omega_0)$ the function $A_0(f) := \big[\Omega_0 \ni \omega \mapsto \sum_{\zeta\in\Omega_{i(\omega)}} m(\zeta)f(\zeta)\big]$ is continuous. Obviously, A_f is constant along the sets Ω_i, $i \in I$. Given a net $[i_j : j \in J]$ of indices such that $\Omega_{i_j} \to \{\infty\}$ in Hausdorff sense (i.e. $\forall\ K$ compact $\subset \Omega_0$ $\exists\ j_K \in J$ $K \cap \Omega_{i_j} = \emptyset$ for$j \ge j_K$), we have $A_0 f(\Omega_{i_j} \to 0$ because $|A_0 f(\Omega_{i_j})| \le \sup_\omega m(\omega) \max_i \#\Omega_i \max_{\zeta\in\vee} |f(\zeta|$ and $\max_{\zeta\in\vee} |f(\zeta| \to 0$. By 1), $A_0 f = f_\phi$ for some $\phi \in \mathcal{C}_0(\Omega_0/\Pi)\}$. Thus $\mathrm{ran}\widetilde{A}_0 \subset \mathcal{C}_0(\Omega_0/\Pi)\}$. Observe that, for any $\psi \in \mathcal{C}_0(\Omega_0/\Pi)$ we have $\psi[\widetilde{A}_0 f] = \widetilde{A}_0(f_\psi f)$. Thus $\mathrm{ran}\widetilde{A}_0$ is an ideal in $\mathcal{C}_0(\Omega_0/\Pi)$. For any $i \in I$, there exists $\phi \in \mathrm{ran}\widetilde{A}_0$

with $\phi(i)\neq 0$. Indeed, by choosing any element $\omega\in\Omega_i$, there exists a function $f\in\mathcal{C}_0(\Omega_0)$ with $f(\omega)=1$ and $f(\zeta)=0$ for $\zeta\in\Omega_i\setminus\{\omega\}$ and $\widetilde{A}_0 f(i)=\sum_{\zeta\in\Omega_i} m(\zeta)f(\zeta)=m(\omega)f(\omega)>0$. Hence, by the Stone-Weierstrass theorem, the ideal $\mathrm{ran}\widetilde{A}_0$ is uniformly dense in $\mathcal{C}_0(\Omega_0/\Pi)$. □

6.4. Definition. Given an admissible couple (m,Π) and a non-negative measure valued mapping $\omega\mapsto\kappa_\omega$ from Ω to $\mathcal{M}(\Omega/\Pi)$, write

$$\mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$$

for the structure $(\mathcal{C}_0(\Omega),\{f\in\mathcal{C}_0(\Omega: f(\Omega\backslash\Omega_0)=0\},\{\ldots\})$ where the triple product $\{\ldots\}$ is the polarized form of

$$\text{(6.5)}\quad \{xax\}(\omega)=x(\omega)\int_{i\in I}\sum_{\zeta\in\Omega_i} m(\zeta)\overline{a(\zeta)}x(\zeta)\; d\kappa_\omega(i),\quad \omega\in\Omega,\ a\in E_0,\ x\in E.$$

We say that the tuple $(\Omega,\Omega_0,m,\Pi,\kappa)$ is *admissible* if (m,Π) is an admissible couple and the measure valued mapping $\omega\mapsto\kappa_\omega$ is weakly continuous * and such that $\kappa_\omega=\delta_{i(\omega)}$ whenever $\omega\in\Omega_0$.

6.6 Theorem. *Let Ω be a locally compact Hausdorff space and $\emptyset\neq\Omega_0\subset\Omega$ an open subset. By setting $E:=\mathcal{C}_0(\Omega)$, $E_0:=\{f\in E:\ f(\Omega\setminus\Omega_0)=0\}$ the triple $(E,E_0,\{\ldots\})$ is a subtriple in the canonical JB*-triple of some Reinhardt domain in E if and only if it is of the form $\mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$ with an admissible tuple $(\Omega,\Omega_0,m,\Pi,\kappa)$.*

The canonical JB-triple of any Reinhardt domain in $\mathcal{C}_0(\Omega)$ with non-zero symmetric part has the form $\mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$ with a suitable admissible tuple $(\Omega,\Omega_0,m,\Pi,\kappa)$.*

Proof. We know already from Theorem 3.5 and Corollary 5.7 the following facts. The canonical JB*-triple of any Reinhardt domain with non-zero symmetric part in $E:=\mathcal{C}_0(\Omega)$ coincides with $\mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$ for some open $\emptyset\neq\Omega_0\subset\Omega$ and an admissible couple (Π,m). Moreover any partial Jordan*-triple $(E,E_0,\{\ldots\})$ with $E_0=\{f\in E:\ f(\Omega\setminus\Omega_0)=0\}$ for some $\emptyset\neq\Omega_0\subset\Omega$ and being such that all multiplications with continuous functions $\Omega\to\mathbb{T}\ (=\{z\in\mathbb{C}:\ |z|=1\})$ belong to $\mathrm{Aut}(E,E_0,\{\ldots\})$ must have the form $\mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$ with suitable open $\emptyset\neq\Omega_0\subset\Omega$ and an admissible couple (Π,m). Finally, by Lemma 2.2 and Corollary 5.7, each $\mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$ is a subtriple in the canonical JB*-triple of some Reinhardt domain in E if and only if the triple product maps $E\times E_0\times E$ to E. Thus it remains to prove only that, in a structure of the form $\mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$, the triple product maps $E\times E_0\times E$ into E (where $E:=\mathcal{C}_0(\Omega)$ and $E_0:=\{f\in E:\ f(\Omega\setminus\Omega_0)=0\}$) if and only if the mapping $\omega\mapsto\kappa_\omega$ is weakly continuous. The sufficiency of the weak continuity of κ for $\{EE_0E\}\subset E$ is immediate. Conversely, suppose (m,Π) is an admissible couple and the triple product (6.5) is continuous and satisfies $\{EE_0E\}\subset E$. Then, by Corollary 5.7,

* That is $\omega\mapsto\int_{i\in I}\phi(i)\; d\kappa_\omega(i)$ is continuous for every $\phi\in\mathcal{C}_0(\Omega/\pi)$.

$(E,E_0,\{\ldots\}) = \mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$ is a partial JB*-triple and, in particular, the operation

$$Af(\omega) = \int_{\eta\in\Omega_0} \sum_{\zeta\in\Omega_{i(\eta)}} m(\zeta)f(\zeta)\ d\mu_\omega(\eta) = \int \widetilde{A}_0 f\ d\kappa_\omega, \quad \omega\in\Omega,\ f\in E_0$$

ranges in the space $\mathcal{C}_b(\Omega)$ of all bounded continuous functions $\Omega\to\mathbb{C}$. Therefore also the operation $T_0 g := \big[\Omega\ni\omega\mapsto \int\psi\ d\kappa_\omega\big]$, $\psi\in\mathrm{ran}\widetilde{A}_0$ ranges in $\mathcal{C}_b(\Omega)$. We know that the measures μ_ω, $\omega\in\Omega$ have total mass bounded by the norm $M := \sup_{a\in E_0,\ x,y\in E}\|\{xay\}\|$ of the triple product. It follows $\kappa_\omega(I)\le M$, $\omega\in\Omega$ and hence T_0 is bounded with norm $\le M$ (i.e. $\sup_{\omega\in\Omega}|T_0\psi(\omega)|\le M\sup_{i\in I}|\psi(i)|$, $\psi\in\mathrm{ran}\widetilde{A}_0$). Therefore T_0 admits a continuous extension $T:[\mathrm{ran}\widetilde{A}_0]^{\urcorner}\to\mathcal{C}_b(\Omega)$ to the closure of the range of $\widetilde{A}_0$ with $T\phi(\omega)=\int\phi\ d\kappa_\omega$, $\phi\in[\mathrm{ran}\widetilde{A}_0]^{\urcorner}$. By Lemma 6.2, we have $\mathcal{C}_0(\Omega_0/\Pi)=[\mathrm{ran}\widetilde{A}_0]^{\urcorner}$. This fact implies the weak continuity of the mapping $\omega\mapsto\kappa_\omega$. □

Next we proceed to the bidualization of the partial triple $(E,E_0,\{\ldots\}) := \mathbf{E}(\Omega,\Omega_0,m,\Pi,\kappa)$. As usually, we shall regard the commutative C*-algebra $E := \mathcal{C}_0(\Omega)$ with the spectral norm as a weak*-dense subspace of the bidual $\mathbf{E} := E^{**}\equiv\mathcal{C}(\boldsymbol{\Omega})$ where $\boldsymbol{\Omega}$ is the hyperstonian compact topological space of all norm continuous multiplicative functionals with respect to the jointly weak*-continuous extension of the product in E equipped with the weak*-topology inherited from E^{***}. That is we identify any element $\mathbf{a}\in\mathbf{E}$ canonically with the evaluation function $\omega\mapsto\omega(\mathbf{a})$ on $\boldsymbol{\Omega}$.

6.7 Theorem. *Let D be a bounded Reinhardt domain in $E:=\mathcal{C}_0(\Omega)$. Then there exists a bounded Reinhardt domain $\mathbf{D}$ in $\mathbf{E}:=E^{**}\equiv\mathcal{C}(\boldsymbol{\Omega})$ such that the canonical JB*-triple $(E,E_D,\{\ldots\}_D)$ is a subtriple of $\mathbf{E},\mathbf{E_D},\{\ldots\}_\mathbf{D}$ and $\mathbf{E_D}$ is the weak*-closure of E in $\mathbf{E}$ and the triple product $\{\ldots\}_\mathbf{D}$ is the jointly weak*-continuous extension of $\{\ldots\}_D$.*

Proof. According to Lemma 5.2, there is a positive and hence norm-continuous mapping $A:E_D\to F$ satisfying the identities (5.3) such that $2\{xay\}_D = xA(\overline{a}y)+yA(\overline{a}x)$, $a\in E_D;\ x,y\in E$. To study the bidual continuation of A, let us regard the commutative C*-algebra $F:=\mathcal{C}_b(\Omega)$ of all bounded continuous functions over Ω as a weak*-dense subspace of the bidual $\mathbf{F}:=F^{**}\equiv\mathcal{C}(\widehat{\boldsymbol{\Omega}})$ where $\widehat{\boldsymbol{\Omega}}$ is a suitable compact hyperstonian topological space. Since $E=\mathcal{C}_0(\Omega)$ is a closed multiplicative ideal in F and E_D is a closed multiplicative ideal in E, also the weak*-closures $\mathbf{E}=\overline{E}^{w*}$ and $\mathbf{E}_0:=\overline{E_0}^{w*}$ are weak*-closed M-ideals in $\mathbf{F}$. Hence we may assume without loss of generality that

$$\mathbf{E}=\big\{\mathbf{f}\in\mathbf{F}:\ \mathbf{f}(\widehat{\boldsymbol{\Omega}}\setminus\boldsymbol{\Omega})=0\big\},\quad \mathbf{E}_0=\big\{\mathbf{f}\in\mathbf{F}:\ \mathbf{f}(\widehat{\boldsymbol{\Omega}}\setminus\boldsymbol{\Omega}_0)=0\big\}$$

for some open-closed subsets $\boldsymbol{\Omega}_0\subset\boldsymbol{\Omega}\subset\widehat{\boldsymbol{\Omega}}$ and the biadjoint A^{**} maps $\mathbf{E}_0$ into $\mathbf{F}$. Consider the operation

$$\text{(6.7)}\qquad \{\mathbf{xay}\}_{**} := \frac{1}{2}\big[A^{**}(\mathbf{x}\overline{\mathbf{a}})\big]\mathbf{y}+\frac{1}{2}\big[A^{**}(\mathbf{y}\overline{\mathbf{a}})\big]\mathbf{x},\quad \mathbf{a}\in\mathbf{E}_0,\ \mathbf{x},\mathbf{y}\in\mathbf{E}.$$

Since the biadjoint of any positive linear operator (between Banach lattices) is weak*-continuous and positive and since the product in $\mathbf{F}$ is separately weak*-continuous, the product (6.7) is a separately weak*-continuous extension of the triple product $\{\ldots\}_D$. From (5.3) it also follows that $A^{**}\big(\mathbf{f}A^{**}(\mathbf{g})\big) = A^{**}\big(\mathbf{g}A^{**}(\mathbf{f})\big)$ and $\mathbf{a}A^{**}\big(\mathbf{f}A^{**}(\mathbf{g})\big) = \mathbf{a}A^{**}(\mathbf{f})A^{**}(\mathbf{g})$ for all $\mathbf{a},\mathbf{f},\mathbf{g} \in \mathbf{E}_0$. Thus $A^{**}: \mathbf{E}_0 \to \mathbf{F}$ is a positive linear operator with the property (5.3) and, by Lemma 5.2, the operation $\{\ldots\}_{**}$ is a partial Jordan*-triple product.

To complete the proof, it remains to verify axioms (J4),(J5) for the product $\{\ldots\}_{**}$ with some bounded circular domain $\mathbf{B} \subset \mathbf{E}$. The weak*-closure of the domain D seems a tempting but technically unsuitable choice for $\mathbf{B}$ in our setting. Instead we proceed as follows. Let $\boldsymbol{\Omega}_1 := \boldsymbol{\Omega} \setminus \boldsymbol{\Omega}_0$ and regard $\mathbf{E}$ as the ℓ^∞-direct sum of the weak*-closed ideals $\mathbf{E}_0$ and $\mathbf{E}_1 := \big\{\mathbf{f} \in \mathbf{F} :\ \mathbf{f}(\widehat{\boldsymbol{\Omega}} \setminus \boldsymbol{\Omega}_1) = 0\big\}$. Define

$$\mathbf{B} := \mathbf{B}_0 + \mathbf{B}_1 \quad \text{where } \mathbf{B}_0 := \mathrm{Interior}_{\mathbf{E}_0} \overline{D \cap E_D}^{w*}, \ \ \mathbf{B}_1 := \{\mathbf{x} \in \mathbf{E}_1 :\ \max|\mathbf{x}| < 1\}.$$

Recall [5, 1] that the bidual of a (full) JB*-triple is a JB*-triple with the separately weak*-continuous extension of the triple product. Hence, since the set $B_0 := D \cap E_D$ is the open unit ball of the canonical norm $\|a\|_{\{\ldots\}_D} := [\max \mathrm{Sp}[E_D \ni c \mapsto \{aac\}_D]]^{1/2}$ on E_D its weak*-closure $\mathbf{B}_0$ is the norm closure of open the unit ball of the norm $\|.\|_{\{\ldots\}_{**}}$ on $\mathbf{E}_0$. We show that actually $\mathbf{B}_0$ is a (bounded symmetric) complete Reinhardt domain in in the function space $\mathbf{E}_0 \simeq \mathcal{C}(\boldsymbol{\Omega}_0)$. Indeed, by Lemma 5.2 we have $A\big(fA(g)\big)|\Omega_0 = A(f)A(g)|\Omega_0$ for $f,g \in E_D$. Hence $A^{**}\big(\mathbf{f}A^{**}(\mathbf{g})\big)|\boldsymbol{\Omega}_0 = A^{**}(\mathbf{f})A^{**}(\mathbf{g})|\boldsymbol{\Omega}_0$ for $\mathbf{f},\mathbf{g} \in \mathbf{E}_0$. Since $\mathbf{B}_0$ is the canonical unit ball of the triple product $\{\ldots\}_{**}$ restricted to $\mathbf{E}_0^3$, Lemma 5.2 implies the Reihardt property of $\mathbf{B}_0$. On the other hand, $\mathbf{B}_1$ is trivially a (bounded symmetric) complete Reinhardt domain in $\mathbf{E}_1 \simeq \mathcal{C}(\boldsymbol{\Omega}_1)$. Since $(\mathbf{E}_0, \mathbf{E}_0, \{\ldots\}_{**}|\mathbf{E}_0^3)$ is a (full) JB*-triple, for each element $\mathbf{a} \in \mathbf{E}_0$, the operator $L(\mathbf{a})\mathbf{x} := \{\mathbf{aax}\}_{**}$, $\mathbf{x} \in \mathbf{E}$ is $\mathbf{B}_0$-hermitian. On the other had, the positiveness of A^{**} (in the sense that it preserves the cone of all non-negative functions) entails the positiveness of the operators $L(\mathbf{a})$, $\mathbf{a} \in \mathbf{E}_0$. Hence (J4) is immediate for the partial Jordan*-triple $(\mathbf{E}, \mathbf{E}_0, \{\ldots\}_{**})$ with the set $\mathbf{B}$ in the role of B there. To estabish (J5), we only have to see that given any function $\mathbf{a} \in \mathbf{E}_0$, the operator $L(\mathbf{a})$ is $\mathbf{B}$-hermitian. We have $L(\mathbf{a}) = \frac{1}{2}L_0(\mathbf{a}) + \frac{1}{2}L_1(\mathbf{a})$ where $L_0(\mathbf{a})\mathbf{x} := A^{**}\big(|\mathbf{a}|^2\big)\mathbf{x}$ and $L_1(\mathbf{a})\mathbf{x} := A^{**}\big(\mathbf{x}\overline{\mathbf{a}}\big)\mathbf{a}$. The operator $L_1(\mathbf{a})$ is a multiplication with a non-negative function in $\mathbf{E}$ and hence necessarily both $\mathbf{B}_0$- and $\mathbf{B}_0$-hermitian. For the operator $L_0(\mathbf{a})$ we have $L_0(\mathbf{a})\mathbf{E} \subset \mathbf{Fa} \subset \mathbf{FE_0} = \mathbf{E}_0$ and $L_0(\mathbf{a})\mathbf{E}_1 = A^{**}(\mathbf{E}_1\overline{\mathbf{a}})\mathbf{a} = A^{**}(0)\mathbf{a} = 0$. Thus the complementary ideals $\mathbf{E}_0$ and $\mathbf{E}_1$ are invariant subspaces of the operator $L(\mathbf{a})$ which acts on $\mathbf{E}_k$ as a $\mathbf{B}_k$-hermitian operator both for $k = 0,1$. Therefore $L(\mathbf{a})$ is $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$-hermitian.
□

7. Appendix

7.1 Theorem. *Let Ω be a locally compact Hausdorff space and $\phi : \mathcal{C}_0(\Omega)^N \to \mathbb{R}$ a continuous positive N-linear form (that is $\Phi(f_1,\ldots,f_N) \geq 0$ for $f_1,\ldots,f_N \geq 0$). Then, with the functions $f_1 \otimes \cdots \otimes f_N : (\omega_1,\ldots,\omega_N) \mapsto \prod_{k=1}^N f_k(\omega_k)$ we have*

$$\Phi(f_1,\ldots,f_N) = \int f_1 \otimes \cdots \otimes f_N \ d\mu, \qquad f_1,\ldots,f_N \in \mathcal{C}_0(\Omega)$$

for some bounded Radon measure μ *on* Ω^N *.*

Proof. Consider the family $\mathbf{U}$ of all finite minimal open coverings of Ω including at most one non-precompact member. That is each term $\mathcal{U} \in \mathbf{U}$ can be written in the form $\mathcal{U} = \{U_1, \ldots, U_m\}$ where $\Omega = \bigcup_{k=1}^m U_k$ with open sets U_k such that the the members $U_1, \ldots, U_{m=1}$ have compact closure in Ω and $\bigcup_{i \in I} U_i \neq \Omega$ whenever I is a proper subset of $\{1, \ldots, m\}$. The latter property means that the covering $\mathcal{U}$ is minimal. This minimality property guarantees that for any covering $\mathcal{U} \in \mathbf{U}$ we can fix a system $\{\mathcal{U}\omega_U : U \in \mathcal{U}\}$ of points such that

$$^{\mathcal{U}}\omega_U \in U \setminus \bigcup_{U \neq V \in \mathcal{U}} V, \qquad U \in \mathcal{U} .$$

Since locally compact spaces are precompact, also we can choose a partition of unity $\{^{\mathcal{U}}\varphi_U : U \in \mathcal{U}\}$ subordinated to the covering $\mathcal{U}$. That is $\sum_{U \in \mathcal{U}} {}^{\mathcal{U}}\varphi_U = 1$ where $0 \leq {}^{\mathcal{U}}\varphi_U \in \mathcal{C}(\Omega)$ with ${}^{\mathcal{U}}\varphi_U(\Omega \setminus U) = 0$. Notice that necessarily ${}^{\mathcal{U}}\varphi_U({}^{\mathcal{U}}\omega_V) = \delta_{UV}(= 1 \text{ if } U = V, \ \ 0 \text{ else}])$. Hence the linear operator

$$P_{\mathcal{U}} f := \sum_{\substack{U \in \mathcal{U} \\ U \text{ precompact} \subset \Omega}} f({}^{\mathcal{U}}\omega_U)\, {}^{\mathcal{U}}\varphi_U, \qquad f \in \mathcal{C}_0(\Omega)$$

is a projection of $\mathcal{C}_0(\Omega))$ onto its finite dimensional subspace with linear basis $\{^{\mathcal{U}}\varphi_U : U \in \mathcal{U}, \ U \text{ precompact} \subset \Omega\}$.

The class $\mathbf{U}$ has the natural net ordering $\mathcal{U} \prec \mathcal{V}$ of being finer. That is $\mathcal{U} \prec \mathcal{V}$ if for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ with $V \subset U$. It is well-known that, given any function $f \in \mathcal{C}_0(\Omega)$ and $\varepsilon > 0$, there exists $\mathcal{U} \in \mathbf{U}$ such that $\sup_{\omega_1, \omega_2 \in U} |f(\omega_1) - f(\omega_2)| \leq \varepsilon$ for all $U \in \mathcal{U}$. This means that

$$\lim_{\mathcal{U} \in \mathbf{U}} \|P_{\mathcal{U}} f - f\| = 0, \qquad f \in \mathcal{C}_0(\Omega) .$$

Consider the linear functionals

$$\widehat{\Phi}_{\mathcal{U}} \widehat{f} := \sum_{\substack{U_1, \ldots, U_N \in \mathcal{U} \\ U_1, \ldots, U_N \text{ precompact} \subset \Omega}} \widehat{f}({}^{\mathcal{U}}\omega_{U_1}, \ldots, {}^{\mathcal{U}}\omega_{U_N})\, \Phi({}^{\mathcal{U}}\varphi_{U_1}, \ldots, {}^{\mathcal{U}}\varphi_{U_N})$$

on the space $\mathcal{C}_0(\Omega^N)$. Observe that, for $f_1, \ldots, f_N \in \mathcal{C}_0(\Omega)$,

$$\Phi(P_{\mathcal{U}} f_1, \ldots, P_{\mathcal{U}} f_N) = \widehat{\Phi}_{\mathcal{U}}(f_1 \otimes \cdots \otimes f_N) .$$

Since the form Φ is assumed to be positive, if $-1 \leq \widehat{f} \leq 1$,

$$\sum_{\substack{U_1, \ldots, U_N \in \mathcal{U} \\ U_1, \ldots, U_N \text{ precompact} \subset \Omega}} (-1)\Phi({}^{\mathcal{U}}\varphi_{U_1}, \ldots, {}^{\mathcal{U}}\varphi_{U_N}) \leq \widehat{\Phi}_{\mathcal{U}} \widehat{f} \leq \sum_{\substack{U_1, \ldots, U_N \in \mathcal{U} \\ U_1, \ldots, U_N \text{ precompact} \subset \Omega}} \Phi({}^{\mathcal{U}}\varphi_{U_1}, \ldots, {}^{\mathcal{U}}\varphi_{U_N})$$

which shows that

$$\|\widehat{\Phi}_{\mathcal{U}}\| \leq \sum_{\substack{U_1, \ldots, U_N \in \mathcal{U} \\ U_1, \ldots, U_N \text{ precompact} \subset \Omega}} \Phi({}^{\mathcal{U}}\varphi_{U_1}, \ldots, {}^{\mathcal{U}}\varphi_{U_N}), \qquad \mathcal{U} \in \mathbf{U} .$$

On the other hand, the functions ${}^{\mathcal{U}}f := \sum_{U\in\mathcal{U}:\ U \text{ precompact}\subset\Omega} {}^{\mathcal{U}}\varphi_U$ satisfy

$$0 \le {}^{\mathcal{U}}f \le 1\ , \quad 0 \le \Phi\big({}^{\mathcal{U}}f,\dots,{}^{\mathcal{U}}f\big) = \sum_{\substack{U_1,\dots,U_N\in\mathcal{U} \\ U_1,\dots,U_N \text{ precompact}\subset\Omega}} \Phi\big({}^{\mathcal{U}}\varphi_{U_1},\dots,{}^{\mathcal{U}}\varphi_{U_N}\big), \qquad \mathcal{U}\in\mathbf{U}\ .$$

Hence we deduce

$$\|\widehat{\Phi}_{\mathcal{U}}\| = \sum_{\substack{U_1,\dots,U_N\in\mathcal{U} \\ U_1,\dots,U_N \text{ precompact}\subset\Omega}} \Phi\big({}^{\mathcal{U}}\varphi_{U_1},\dots,{}^{\mathcal{U}}\varphi_{U_N}\big) \le$$
$$\le \|\phi\|\Big(:= \sup_{\|f_1\|=\cdots=\|f_N\|=1} |\Phi(f_1,\dots,f_N)|\Big)\ .$$

By the continuity of Φ we have $\|\Phi\| < \infty$. According to the Alaoglu-Bourbaki theorem, the bounded net $\big(\widehat{\Phi}_{\mathcal{U}}\big)_{\mathcal{U}\in\mathbf{U}}$ admits cluster points in the dual of $\mathcal{C}_0(\Omega)$ in weak* sense. (Actually one could even proof its weak*-convergence but we do not need this finer argument). By taking any cluster point $\widehat{\Phi}$ of $\big(\widehat{\Phi}_{\mathcal{U}}\big)_{\mathcal{U}\in\mathbf{U}}$, for all $f_1,\dots,f_N\in\mathcal{C}_0(\Omega)$ we have

$$\Phi(f_1,\dots,f_N) = \lim_{\mathcal{U}\in\mathbf{U}} \Phi(P_{\mathcal{U}}f_1,\dots,P_{\mathcal{U}}f_N) =$$
$$= \lim_{\mathcal{U}\in\mathbf{U}} \widehat{\Phi}_{\mathcal{U}}(f_1\otimes\cdots\otimes f_N) = \widehat{\Phi}(f_1\otimes\cdots\otimes f_N)\ .$$

The proof is complete. $\square$

References

[1] T.J. Barton and R.M. Timoney, *Weak*-continuity of Jordan triple products and applications*, Math. Scand. 59 (1986), 177-191.

[2] T. Barton, *Biholomorphic equivalence of bounded Reinhardt domains*, Annali Scuola Normale Sup. Pisa Cl. Sci 13(4) (1986), 1-13.

[3] T. Barton, S. Dineen and R.M. Timoney, *Bounded Reinhardt domains in Banach spaces*, Compositio Mathematica 59 (1986), 265-321..

[4] R. Braun, W. Kaup and H. Upmeier, *On the automorphisms of symmetric and Reinhardt domains in complex Banach spaces*, Manuscripta Math. 25 (1978), 97-133.

[5] S. Dineen, *Complete holomorphic vector fields on the second dual of a Banach space*, Math. Scand. 59 (1986), 131-142.

[6] J.M. Isidro and L.L. Stachó, *Holomorphic Automorphism Groups in Banach Spaces*, North Holland Math. Studies 105, Elsevier, Amsterdam, 1985.

[7] J.M. Isidro and L.L. Stachó, *Holomorphic invariants of continuous bounded symmetric Reinhardt domains*, Acta Sci. Math. (Szeged) 71 (2004), 105-119.

[8] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. 138 (1983), 503-509.

[9] W. Kaup and J.-P. Vigué, *Symmetry and local conjugacy on complex manifolds*, Math. Ann. 286 no. 1-3 (1990), 329-340.

[10] D.-P. Panou, *Bounded bicircular domains in* $\mathbf{C}^n$, Manuscr. Math. 68 No.4 (1990), 373-390.

[11] H.H. Schaefer, *Banach Lattices of Positive Operators*, Grundlehren der Math. Wiss. 215, Springer Verlag, Berlin, 1974.

[12] L.L. Stachó, *A projection principle concerning biholomorphic automorphisms*, Acta Sci. Math. 44 (1982), 99-124.

[13] L.L. Stachó, *On the spectrum of inner derivations in partial Jordan triples*, Math. Scandinavica 66 (1990), 242-248.

[14] L.L. Stachó, *On the classification of bounded circular domains*, Proc. R. Ir. Acad. 91 A No. 2 (1991), 219-238.

[15] L.L. Stachó, *On the structure of inner derivations in partial Jordan-triple algebras*, Acta Sci. Math. 60 (1995), 619-636.

[16] L.L. Stachó and B. Zalar, *Symmetric continuous Reinhardt domains*, Archiv der Math. (Basel) 81 (2003), 50-61.

[[17]] L.L. Stachó, *Banach–Stone type theorem for lattice norms in* C_0 *-spaces*, Acta Sci. Math. (Szeged) 73 (2007), 193-208.

[18] T. Sunada, *On bounded Reinhardt domains*, Proc. Japan Acad. 50 (1974), 119-123.

[19] J.-P. Vigué, *Automorphismes analytiques des domaines produits*, Ark. Mat. 36 (1998), 177-190.

[20] I. Villanueva, *Integral multilinear forms on* $C(K,X)$ *spaces*, Czechoslovak Math. J. 54 No. 2 (2004), 373–378.

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