On strongly continuous one-parameter groups of automorphisms of multilinear functionals[☆]

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Abstract

We prove a structure theorem for strongly continuous one-parameter groups formed by surjective isometries of the space of bounded N-linear functionals over complex Hilbert spaces. As a consequence, the strongly continuous one-parameter automorphism groups of Cartan factors of type I are classified.

Key words: strongly continuous one-parameter group, N-linear functional, automorphism, Hilbert space
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1. Introduction

Throughout this work let $\mathbf{H}, \mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)}$ be arbitrarily fixed complex Hilbert spaces. Our chief aim will be to study the structure of the *strongly continuous* one-parameter automorphism groups of the space $\mathcal{B} = \mathcal{B}(\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)})$ of all bounded N-linear functionals $\mathbf{H}^{(1)} \times \cdots \times \mathbf{H}^{(N)} \to \mathbb{C}$ that is the maps

$$\mathbf{U} : \mathbb{R} \to \mathfrak{A} := \{\text{surjective linear isometries } \ \mathcal{B} \to \mathcal{B}\}$$

with the group property $\mathbf{U}(t+h) = \mathbf{U}(t)\mathbf{U}(h)$ $(t, h \in \mathbb{R})$ and being such that the functions $t \mapsto \mathbf{U}(t)\Phi$ are continuous for all fixed $\Phi \in \mathcal{B}$. The case $N = 1$ is covered by Stone's classical theorem [8, 12]: given a strongly continuous one-parameter subgroup $\mathbf{U} : \mathbb{R} \to \mathcal{U}(\mathbf{H}) :=\{\text{unitary operators } \mathbf{H} \to \mathbf{H}\} \simeq \mathfrak{A}$, there exists a possibly unbounded self-adjoint linear operator A on some dense linear submanifold of $\mathbf{H}$ such that $\mathbf{U}(t) = \exp(itA)$ $(t \in \mathbb{R})$. In the case $N = 2$, as a simple consequence of the theory of unbounded C^*-algebra derivations [2], in $\mathcal{L}(\mathbf{H}) \simeq \mathcal{B}(\mathbf{H}, \mathbf{H})$ we have a precise abstract description of the special one-parameter isometry groups of the form $\mathbf{U}(t)X = \exp(itA)X\exp(-itA)$ with a suitable possibly unbounded self-adjoint operator A. Our problems with $N = 2$ and $\mathcal{B}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}) \simeq \mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})$ are naturally associated with Jordan triple derivations [7, 10], and may have far reaching importance even for the description of all strongly continuous one-parameter automorphism groups of general

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JB*-triples (complex Banach spaces with symmetric unit ball). Namely, by the Hille-Yosida theorem [3, 12] the infinitesimal generator of a strongly continuous one-parameter group of automorphisms of a JB*-triple is a possibly unbounded Jordan triple derivation. As far as we know, the *bounded* JB*-triple derivations are well-understood [1]. However, no results seem to be concerned with the unbounded case even for Cartan factors. From a Jordan theoretical view point, $\mathcal{L}(\mathbf{H}^{(1)},\mathbf{H}^{(2)})$ is a typical Cartan factor of type I where the connected component of the automorphism group containing the identity consists of mappings of the form $X\mapsto UXV$ with suitable unitary operators $U\in\mathcal{U}(\mathbf{H}^{(2)})$ and $V\in\mathcal{U}(\mathbf{H}^{(1)})$ [6, 11]. Hence the structure of all *norm-continuous* one parameter groups $\mathbf{W}:\mathbb{R}\to \mathrm{Aut}(\mathcal{L}(\mathbf{H}^{(1)},\mathbf{H}^{(2)}))\simeq\mathfrak{A}$ is immediate: in this case $\mathbf{W}(t)X=[\exp(itA_2)]X[\exp(itA_1)]$ for a suitable couple of *bounded* self-adjoint operators $A_1\in\mathcal{L}(\mathbf{H}^{(1)}),\ B\in\mathcal{L}(\mathbf{H}^{(2)})$. It seems that even the strongly continuous one-parameter subgroups of $\mathrm{Aut}(\mathcal{L}(\mathbf{H}^{(1)},\mathbf{H}^{(2)}))$ are not fully described in the literature. One may expect that, for any number N of factors, the elements of the identity-component of $\mathfrak{A}$ should be mappings of the form

$$[U_1\otimes\cdots\otimes U_N]\Phi := \big[(\mathbf{x}_1,\dots,\mathbf{x}_N)\mapsto\Phi(U_1\mathbf{x}_1,\dots,U_N\mathbf{x}_N)\big]\qquad(\Phi\in\mathcal{B}).$$

This seems also not yet been established in full generality, and Jordan theoretical arguments cannot be expected to be suitable for the proof. Our main result, which is prompted by this conjecture, is the following seemingly plausible statement.

Theorem 1.1. *Let $\mathbf{U}:\mathbb{R}\to\mathfrak{A}(\mathbf{H}^{(1)},\dots,\mathbf{H}^{(N)})$ be a strongly continuous one-parameter group such that*

$$\mathbf{U}(t)=U_{1,t}\otimes\cdots\otimes U_{N,t}\qquad(t\in\mathbb{R})$$

with suitable unitary operators $U_{k,t}\in\mathcal{U}(\mathbf{H}^{(k)})$. Then there are possibly unbounded self-adjoint operators $A_k:\mathrm{dom}(A_k)\to\mathbf{H}^{(k)}$ $(k=1,\dots,N)$ defined on dense linear submanifolds in the respective spaces such that that

$$\mathbf{U}(t)=\big[\exp(itA_1)\big]\otimes\cdots\otimes\big[\exp(itA_N)\big]\qquad(t\in\mathbb{R}).$$

Corollary 1.2. *If $\mathbf{W}:\mathbb{R}\to\mathrm{Aut}\big(\mathcal{L}(\mathbf{H}^{(1)},\mathbf{H}^{(2)})\big)$ is a strongly continuous one-parameter group then $\mathbf{W}(t)X=\exp(tA_1)X\exp(tA_2)$ for a suitable couple of possibly unbounded self-adjoint operators $A_k:\mathrm{dom}(A_k)\to\mathbf{H}^{(k)}$.*

The main technical obstacle for the proof arises from the fact that an operator $U_1\otimes\cdots\otimes U_N$ admits alternative representations as $\big[(\kappa_1U_1)\big]\otimes\cdots\otimes\big[(\kappa_NU_N)\big]$ with $\kappa_1,\dots,\kappa_N\in\mathbb{T}:=\{\kappa\in\mathbb{C}:\ |\kappa|=1\}$ and $\prod_{k=1}^N\kappa_k=1$. Our considerations, which rely heavily upon complex Hilbert space structure, can be divided into three main steps. First we establish that, under the hypothesis of Theorem 1.1, there are multiplier functions $\kappa_k:\mathbb{R}\to\mathbb{T}$ with $\prod_{k=1}^N\kappa_k(t)=1$ $(t\in\mathbb{R})$ such that each component $t\mapsto\kappa_k(t)U_{k,t}$ is strongly continuous; that is, all the functions $t\mapsto\kappa_k(t)U_{k,t}\mathbf{h}_k$ $(\mathbf{h}_k\in\mathbf{H}^{(k)};\ k=1,\dots,n)$ are continuous from $\mathbb{R}$ into $\mathbf{H}^{(k)}$ with norm topology. Assuming then without loss of generality the strong continuity of the components $t \mapsto U_{k,t}$, we show that the families $\{U_{k,t} : \ t \in \mathbb{R}\}$ are Abelian and then, by means of their Gelfand representations we can choose the multipliers $\kappa_k : \mathbb{R} \to \mathbb{T}$ even in a manner such that we have $U_{k,t} = \kappa_k(t) U_k^t$ with some not necessarily strongly continuous one-parameter groups $t \mapsto U_k^t$. We finish the proof after a series of probabilistic arguments where we establish that this representations can be improved to the form $U_{k,t} = \chi_k(t) \widetilde{U}_k^t$ with strongly continuous one-parameter groups $t \mapsto \widetilde{U}_k^t$ and continuous functions $\chi_k : \mathbb{R} \to \mathbb{T}$, respectively.

2. Preliminaries, adjusted strong continuity

Throughout the paper $\mathbb{R}$ and $\mathbb{C}$ are the standard notations for the sets of real and complex numbers, respectively and $\mathbb{T} := \{\kappa \in \mathbb{C} : \ |\kappa| = 1\}$ denotes the unit circle. Without danger of confusion, in each of the spaces $\mathbf{H}, \mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)}$, we shall write $\langle . | . \rangle$ and $\|.\|$ for the inner product and the norm, respectively. The products $\langle . | . \rangle$ are supposed to be linear in their first and conjugate-linear in their second variables. With this convention, $\mathbf{h}^*$ will denote the linear functional $\mathbf{x} \mapsto \langle \mathbf{x} | \mathbf{h} \rangle$. Conveniently, we shall use the customary tensor product notations [9, Sec.1.3] in the space $\mathcal{B} = \mathcal{B}(\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)})$ of all bounded N-linear functionals $\mathbf{H}^{(1)} \times \cdots \times \mathbf{H}^{(N)} \to \mathbb{C}$ equipped with the usual operator norm $\|\Phi\| := \sup_{\|\mathbf{x}_1\| = \cdots = \|\mathbf{x}_N\| = 1} |\Phi(\mathbf{x}_1, \ldots, \mathbf{x}_N)|$. Given a family $\mathbf{h}_1 \in \mathbf{H}^{(1)}, \ldots, \mathbf{h}_N \in \mathbf{H}^{(N)}$ of vectors, we shall write $\mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^*$ for the *elementary* functionals

$$\mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^* : \ (\mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto \prod_{k=1}^N \mathbf{h}_k^*(\mathbf{x}_k) = \prod_{k=1}^N \langle \mathbf{x}_k | \mathbf{h}_k \rangle.$$

Also we shall write $A_1 \otimes \cdots \otimes A_N$ for the *composition* operators

$$[A_1 \otimes \cdots \otimes A_N]\Phi := \Big[(\mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto \Phi\big(A_1\mathbf{x}_1, \ldots, A_N\mathbf{x}_N\big)\Big] \qquad \big(\Phi \in \mathcal{B}\big)$$

if $A_k \in \mathcal{L}(\mathbf{H}^{(k)}) := \{\text{bounded linear operators } \mathbf{H}^{(k)} \to \mathbf{H}^{(k)}\}$. Notice that

$$[A_1 \otimes \cdots \otimes A_N]\mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^* = [A_1^*\mathbf{h}_1]^* \otimes \cdots \otimes [A_N^*\mathbf{h}_N]^*.$$

The factorization of non-trivial composition operators is unique up to constant coefficients: if $A_1, \ldots, A_N \neq 0$ we have $A_1 \otimes \cdots \otimes A_N = B_1 \otimes \cdots \otimes B_N$ if and only if $B_k = \beta_k A_k$ for constants with $\prod_{k=1}^N \beta_k = 1$.[1] In particular for unitary

[1]Indeed, evaluated at $\mathbf{y}_1^* \otimes \cdots \otimes \mathbf{y}_N^*$, the relation $A_1 \otimes \cdots \otimes A_N = B_1 \otimes \cdots \otimes B_N$ entails $\prod_{k=1}^N \langle A_k\mathbf{x}_k | \mathbf{y}_k \rangle = \prod_{k=1}^N \langle B_k\mathbf{x}_k | \mathbf{y}_k \rangle$ for any $\mathbf{x}_1, \ldots \mathbf{x}_N$. Given any index k, if $\mathbf{x}_\ell, \mathbf{y}_\ell \in \mathbf{H}^{(\ell)}$ ($\ell \neq k$) are so chosen that $\langle B_\ell \mathbf{x}_j | \mathbf{y}_\ell \rangle = 1$ then for any $\mathbf{x}, \mathbf{y} \in \mathbf{H}^{(k)}$ we have $\langle B_k\mathbf{x} | \mathbf{y} \rangle = \beta_k \langle A_k \mathbf{x} | \mathbf{y} \rangle$ $(\mathbf{x}, \mathbf{y} \in \mathbf{H}^{(k)})$ with $\beta_k := \prod_{\ell : \ell \neq k} \langle A_\ell \mathbf{x}_\ell | \mathbf{y}_\ell \rangle$. The converse implication is trivial.

operators $U_k, V_k \in \mathcal{U}(\mathbf{H}^{(k)})$,

$$U_1 \otimes \cdots \otimes U_N = V_1 \otimes \cdots \otimes V_N \quad \Longleftrightarrow \quad V_k = \kappa_k U_k,\ \kappa_k \in \mathbb{T} \text{ with } \prod_{k=1}^{N} \kappa_k = 1.$$

Lemma 2.1. *Assume* $\|\Phi - \Psi\| \leq \varepsilon$ *where* $\Phi := \mathbf{g}_1^* \otimes \cdots \otimes \mathbf{g}_N^*$ *and* $\Psi := \mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^*$ *with unit vectors* $\mathbf{g}_k, \mathbf{h}_k \in \mathbf{H}^{(k)}$. *Then*

$$\operatorname{dist}(\mathbb{T}\mathbf{g}_k, \mathbb{T}\mathbf{h}_k) := \min_{\kappa,\mu\in\mathbb{T}} \|\kappa \mathbf{g}_k - \mu \mathbf{h}_k\| \leq 2^{N-1}\varepsilon \qquad (k = 1, \ldots, N).$$

Proof. Fix any $1 \leq k \leq N$. Define $\kappa_k := \prod_{\ell:\ell\neq k} \overline{\kappa_\ell}$ where for $\ell \neq k$ we set $\kappa_\ell := \langle \mathbf{g}_\ell | \mathbf{h}_\ell \rangle / |\langle \mathbf{g}_\ell | \mathbf{h}_\ell \rangle|$ if $\mathbf{g}_\ell \not\perp \mathbf{h}_\ell$ and $\kappa_\ell := 1$ if $\mathbf{g}_\ell \perp \mathbf{h}_\ell$. With this choice

$$\kappa_1, \ldots, \kappa_N \in \mathbb{T}, \quad \prod_{m=1}^{N} \kappa_m = 1 \quad \text{and} \quad \langle \mathbf{g}_\ell | \kappa_\ell \mathbf{h}_\ell \rangle = |\langle \mathbf{g}_\ell | \mathbf{h}_\ell \rangle| \geq 0 \quad (\ell \neq k).$$

Observe that for the vectors $\mathbf{x}_\ell := \mathbf{g}_\ell + \kappa_\ell \mathbf{h}_\ell$ with the values $\rho_\ell := 1 + |\langle \mathbf{h}_\ell | \mathbf{g}_\ell \rangle|$ we have

$$\langle \mathbf{x}_\ell | \mathbf{g}_\ell \rangle = \langle \mathbf{x}_\ell | \kappa_\ell \mathbf{h}_\ell \rangle = \rho_\ell \in [1,2], \quad \|\mathbf{x}_\ell\| = \langle \mathbf{x}_\ell | \mathbf{x}_\ell \rangle^{1/2} = (2\rho_\ell)^{1/2} \in [\sqrt{2}, 2] \qquad (\ell \neq k).$$

Thus, since also $\Psi = [\kappa_1 \mathbf{h}_1]^* \otimes \cdots \otimes [\kappa_N \mathbf{h}_N]^*$, for any $\mathbf{x} \in \mathbf{H}^{(k)}$ we can write

$$\big[\Phi - \Psi\big](\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{x}, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_N) = \big\langle \mathbf{x} \,\big|\, \mathbf{g}_k - \kappa_k \mathbf{h}_k \big\rangle \prod_{\ell:\ell\neq k} \rho_\ell \ .$$

Therefore we have the norm estimate

$$\big|\big\langle \mathbf{x} \,\big|\, \mathbf{g}_k - \kappa_k \mathbf{h}_k \big\rangle\big| \prod_{\ell:\ell\neq k} \rho_\ell \leq \big\|\phi - \Psi\big\| \ \|\mathbf{x}\| \prod_{\ell:\ell\neq k} |\mathbf{x}_\ell\| \ .$$

Since here $\|\Phi - \Psi\| \leq \varepsilon$, $1 \leq \rho_\ell$ and $\|\mathbf{x}_\ell\| \leq 2$, it follows $|\langle \mathbf{x} \,|\, \mathbf{g}_k - \kappa_k \mathbf{h}_k \rangle| \leq 2^{N-1}\varepsilon$ for all vectors $\mathbf{x} \in \mathbf{H}^{(k)}$. Hence $\operatorname{dist}(\mathbb{T}\mathbf{g}_k, \mathbb{T}\mathbf{h}_k) \leq \|\mathbf{g}_k - \kappa_k \mathbf{h}_k\| \leq 2^{N-1}\varepsilon$. □

In particular, with $\varepsilon := 0$ we see that $\mathbf{g}_1^* \otimes \cdots \otimes \mathbf{g}_N^* = \mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^*$ implies $\mathbf{h}_k = \kappa_k \mathbf{g}_k$ for suitable $\kappa_q, \ldots, \kappa_N \in \mathbb{T}$ with $\prod_{k=1}^{N} \kappa_k = 1$ whenever the vectors $\mathbf{g}_k, \mathbf{h}_k$ have norm 1. For later use, notice also that if $\mathbf{g}, \mathbf{h} \in \mathbf{H}$ are unit vectors in a Hilbert space then

$$\begin{aligned} \operatorname{dist}(\mathbb{T}\mathbf{g}, \mathbb{T}\mathbf{h}) &= \operatorname{dist}(\mathbf{g}, \mathbb{T}\mathbf{h}) = \min_{|\kappa|=1} \big[2 - 2\operatorname{Re}\langle \mathbf{g} | \kappa \mathbf{h} \rangle\big]^{1/2} = \\ &= \sqrt{2}\,\big[1 - |\langle \mathbf{g} | \mathbf{h} \rangle|\big]^{1/2}. \end{aligned} \tag{2.2}$$

Lemma 2.3. *Suppose* $\mathbf{F} : \mathbb{R} \to \mathbb{P}(\mathbf{H}) := \{\mathbb{T}\mathbf{g} : \ \langle \mathbf{g} | \mathbf{g} \rangle = 1\}$ *is a continuous mapping with respect to the distance* (2.2). *Then* $\mathbf{F}(t) = \mathbb{T}\mathbf{h}_t$ $(t \in \mathbb{R})$ *for some continuous function* $t \mapsto \mathbf{h}_t \in \partial\mathrm{Ball}(\mathbf{H}) := \{\mathbf{g} : \ \langle \mathbf{g} | \mathbf{g} \rangle = 1\}$.

Proof. Since the real line $\mathbb{R}$ is σ-compact, it suffices to establish the local version of the statement: for every $s \in \mathbb{R}$ there is an open interval I_s around s where the set valued function $\mathbf{F}$ admits a continuous section say $I_s \ni t \mapsto \mathbf{h}_t^{[s]} \in \mathbf{f}(t)$. $\big[$Proof. In this case there is a strictly increasing double sequence $\big(T_n\big)_{n=-\infty}^{\infty}$ such that $\mathbb{R} = \bigcup_n [T_n, T_{n+1}]$ and each interval $[T_n, T_{n+1}]$ is contained in some I_{s_n}. Since $\mathbf{h}_{T_n}^{[s_{n-1}]}, \mathbf{h}_{T_n}^{[s_n]} \in \mathbf{F}(T_n)$, for each $n \in \{0, \pm 1, \pm 2, \ldots\}$, there is a (unique) constant $\kappa_n \in \mathbb{T}$ such that $\mathbf{h}_{T_n}^{[s_n]} = \kappa_n \mathbf{h}_{\mathbf{T_n}}^{[\mathbf{s_{n-1}}]}$. Then the function assembled as $\mathbf{h}_t := \mu_n \mathbf{h}_t^{[s_n]}$ for $T_n \leq t \leq T_n$ where $\mu_0 := 1$, $\mu_p := \prod_{k=1}^p \kappa_k$ and $\mu_{-p} := \prod_{k=-p+1}^0 \overline{\kappa_k}$ $(p = 1, 2, \ldots)$ suits our requirements.$\big]$

To prove the local statement, we may assume $s = 0$ without loss of generality. The continuity of $\mathbf{F}$ entails the continuity of function $(t, u) \mapsto \mathrm{dist}\big(\mathbf{F}(t), \mathbf{F}(u)\big)$. Hence we can choose I_0 to be an open interval around 0 such that $\sqrt{2} > \mathrm{dist}\big(\mathbf{F}(t), \mathbf{F}(u)\big)$ $(t, u \in I_0)$ that is

$$|\langle \mathbf{v} | \mathbf{w} \rangle| > 0 \quad \text{whenever} \quad \mathbf{v} \in \mathbf{F}(t), \mathbf{w} \in \mathbf{F}(u) \ \text{ and } \ t, u \in I_0.$$

Fix any vector $\mathbf{f}_0 \in \mathbf{F}(0)$. Since, by (2.2).

$$\mathrm{dist}\big(\mathbf{F}(t), \mathbf{F}(0)\big) = \sqrt{2}\ \big[1 - \max_{\mathbf{v} \in \mathbf{F}(t)} \mathrm{Re}\langle \mathbf{f}_0 | \mathbf{v} \rangle |\big]^{1/2},$$

for every $t \in I_0$ there is a unique unit vector $\mathbf{f}_t$ such that

$$\mathbf{f}_t \in \mathbf{F}(t) \quad \text{and} \quad \langle \mathbf{f}_0 | \mathbf{f}_t \rangle = \max_{\mathbf{v} \in \mathbf{F}(t)} \mathrm{Re}\langle \mathbf{f}_0 | \mathbf{v} \rangle = 1 - \frac{1}{2} \mathrm{dist}\big(\mathbf{F}(0), \mathbf{F}(t)\big)^2 > 0.$$

In particular, with suitable unit vectors $\mathbf{u}_t \perp \mathbf{f}_0$ and with suitable angle parameters $0 \leq \varphi_t < \pi/2$ we can write

$$\mathbf{f}_t = \cos\varphi_t\ \mathbf{f}_0 + \sin\varphi_t\ \mathbf{u}_t \qquad (t \in I_0).$$

Given any convergent sequence $t_n \to t$ in I_0, the continuity of $\mathbf{F}$ means that $|\langle \mathbf{f}_{t_n} | \mathbf{f}_t \rangle| = 1 - 2^{-1} \mathrm{dist}\big(\mathbf{F}(t_n), \mathbf{F}(t)\big)^2 \to 1$, that is

$$\big| \cos\varphi_{t_n} \cos\varphi_t + \sin\varphi_{t_n} \sin\varphi_t \langle \mathbf{u}_{t_n} | \mathbf{u}_t \rangle \big| \longrightarrow 1\ .$$

Since $\varphi_{t_n}, \varphi_t \in [0, \pi/2]$, we have $1 \geq \cos(\varphi_{t_n} - \varphi_t) = \cos\varphi_{t_n} \cos\varphi_t + \sin\varphi_{t_n} \sin\varphi_t \geq \big| \cos\varphi_{t_n} \cos\varphi_t + \sin\varphi_{t_n} \sin\varphi_t \langle \mathbf{u}_{t_n} | \mathbf{u}_t \rangle \big| \to 1$. Thus necessarily $\varphi_{t_n} \to \varphi_t$. Hence, for any cluster point ζ of the sequence $\big(\langle \mathbf{u}_{t_n} | \mathbf{u}_t \rangle\big)_{n=1}^{\infty}$, it follows $\big| \cos^2\varphi_t + \zeta \sin^2\varphi_t \big| = 1$ that is $\zeta = 1$ unless $\varphi_t = 0$. In any case we must have

$$\langle \mathbf{f}_{t_n} | \mathbf{f}_t \rangle = \cos\varphi_{t_n} \cos\varphi_t + \sin\varphi_{t_n} \sin\varphi_t \langle \mathbf{u}_{t_n} | \mathbf{u}_t \rangle \to 1$$

which implies $\|\mathbf{f}_{t_n} - \mathbf{f}_t\| = \big[2 - 2\mathrm{Re}\langle \mathbf{f}_{t_n} | \mathbf{f}_t \rangle\big]^{1/2} \to 0$ for any sequence $t_n \to t$ in I_0. $\square$

Proposition 2.4. *Assume* $\Psi : \mathbb{R} \to \mathcal{B}$ *is a continuous function of the form* $\Psi(t) = \mathbf{h}_{1,t}^* \otimes \cdots \otimes \mathbf{h}_{N,t}^*$ *with suitable unit vectors* $\mathbf{h}_{k,t} \in \mathbf{H}^{(k)}$*. Then there are functions* $\kappa_1, \ldots, \kappa_N : \mathbb{R} \to \mathbb{T}$ *such that* $\prod_{k=1}^N \kappa_k(t) \equiv 1$ *and the modified components* $t \mapsto \kappa_k(t)\mathbf{h}_{k,t}$ *are continuous* (*as mappings* $\mathbb{R} \to [\mathbf{H}^{(k)}, \text{norm topology}]$).

Proof. According to Lemma 2.1, the functions $\mathbf{F}_k : t \mapsto \mathbb{T}\mathbf{h}_{k,t}$ are continuous from $\mathbb{R}$ into the metric space $\left[\mathbb{P}(\mathbf{H}^{(k)}), \text{dist}\right]$ in the sense of (2.2). Thus, by Lemma 2.3, we can find functions $\mu_1, \ldots, \mu_N : \mathbb{R} \to \mathbb{T}$ such that the sections $t \mapsto \mathbf{f}_{k,t} := \mu_k(t)\mathbf{h}_{k,t} \in \mathbf{F}_k(t)$ $(k = 1, \ldots, N)$ are continuous. Then their product $t \mapsto \widetilde{\Psi}(t) := \mathbf{f}_{1,t}^* \otimes \cdots \otimes \mathbf{f}_{N,t}^*$ is also a continuous map $\mathbb{R} \to \mathcal{B}$. Observe that $\widetilde{\Psi}(t) = \mu(t)\Psi(t)$ $(t \in \mathbb{R})$ with the scalar valued function $\mu(t) := \prod_{k=1}^N \overline{\mu_k(t)}$. From the continuity of both Ψ and $\widetilde{\Psi}$ we infer the continuity of $\mu : \mathbb{R} \to \mathbb{T}$.[2] Hence also the functions $t \mapsto \widetilde{\mathbf{f}}_{k,t} := \mu(t)\mathbf{f}_{k,t}$ are continuous. Since $\widetilde{\mathbf{f}}_{k,t} = \prod_{\ell:\ell \neq k} \overline{\mu_\ell(t)}\mathbf{h}_{k,t}$ and since $\Psi(t) = \overline{\mu(t)}\widetilde{\Psi}(t) = \widetilde{\mathbf{f}}_{1,t}^* \otimes \mathbf{f}_{2,t}^* \otimes \cdots \otimes \mathbf{f}_{N,t}^*$, the choice $\kappa_1(t) := \prod_{j=2}^N \overline{\mu_j(t)}$ along with $\kappa_k(t) := \mu_k(t)$ for $k > 1$ suits our requirements. □

Conventions 2.5. To simplify notations for the proof of Theorem 1.1, henceforth let $\mathbf{U} : \mathbb{R} \to \mathfrak{A} = \mathfrak{A}(\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)})$ be a one-parameter subgroup of operators of the form

$$\mathbf{U}(t) = U_{1,t}^* \otimes \cdots \otimes U_{N,t}^* \ , \qquad U_{k,t} \in \mathcal{U}(\mathbf{H}^{(k)}) \ .$$

An application of Proposition 2.4 to functions of the form $\mathbf{U}(t)[\mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^*] = [U_{1,t}\mathbf{h}_1]^* \otimes \cdots \otimes [U_{N,t}\mathbf{h}_N]^*$ yields immediately the following.

Corollary 2.6. *Given any family* $\mathbf{h}_k \in \mathbf{H}^{(k)}$ $(k = 1, \ldots, N)$ *of unit vectors, there are functions* $\kappa_k : \mathbb{R} \to \mathbb{T}$ *such that* $\prod_{k=1}^N \kappa_k = 1$ *and the functions* $t \mapsto \kappa_k(t)U_{k,t}\mathbf{h}_k$ *are continuous.* □

As usual, we say that a net $\left(V_\alpha\right)_{\alpha \in \mathcal{A}}$ of bounded linear operators $\mathcal{B} \to \mathcal{B}$ is *strongly convergent* to V (notation: $V_\alpha \xrightarrow{s} V$) if $\|(V_\alpha - V)\Phi\| \to 0$ for all $\Phi \in \mathcal{B}$. Accordingly, a function $\mathbf{V} : \mathbb{R} \to \mathcal{L}(\mathcal{B})$ is *strongly continuous*, if $\mathbf{V}(t_\alpha) \xrightarrow{s} \mathbf{V}(t)$ whenever $t_\alpha \to t$ in $\mathbb{R}$.

Proposition 2.7. *For some functions* $\kappa_1, \ldots, \kappa_N : \mathbb{R} \to \mathbb{T}$*, the operator-valued functions* $t \mapsto \kappa_k(t)U_{k,t}$ *are strongly continuous.*

[2] In general, if $v_\alpha \to v \neq 0$ and $\mu_\alpha v_\alpha \to \mu v$ are convergent nets in a locally convex Hausdorff vector space $\mathcal{V}$, then necessarily $\mu_\alpha \to \mu$ for the scalar coefficients. Proof: there exists a continuous linear functional ϕ on $\mathcal{V}$ such that $\phi(v) = 1$. Beyond some index α_0 we have $\phi(v_\alpha) \neq 0$ and $\mu_\alpha = \phi(\mu_\alpha v_\alpha)/\phi(v_\alpha) \to \phi(\mu v)/\phi(v) = \mu$.

PROOF. Fix any family $\mathbf{h}_1 \in \mathbf{H}^{(1)}, \dots, \mathbf{h}_N \in \mathbf{H}^{(N)}$ of unit vectors along with a family $\kappa_1, \dots, \kappa_N : \mathbb{R} \to \mathbb{T}$ of scalar functions with $\prod_{j=1}^N \kappa_j(t) = 1$ such that the functions $t \mapsto \kappa_k(t) U_{k,t} \mathbf{h}_k$ are continuous. This is guaranteed by Corollary 2.6. Consider any index $k \in \{1, \dots, N\}$ and let $0 \neq \mathbf{x} \in \mathbf{H}^{(k)}$ be any vector. It suffices to see that the function $t \mapsto \kappa_k(t) U_{k,t} \mathbf{x}$ is continuous.

Applying Corollary 2.6 with the vectors $\mathbf{h}_1, \dots, \mathbf{h}_{k-1}, \mathbf{x}/\|\mathbf{x}\|, \mathbf{h}_{k+1}, \dots, \mathbf{h}_N$, we see the existence of functions $\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_N : \mathbb{R} \to \mathbb{T}$ with $\prod_{\ell=1}^N \widetilde{\kappa}_\ell(t) = 1$ such that the functions $t \mapsto \widetilde{\kappa}_k(t) U_{k,t} \mathbf{x}$, $t \mapsto \widetilde{\kappa}_\ell(t) U_{\ell,t} \mathbf{h}_\ell$ $(\ell \neq k)$ are continuous. Given any index $\ell \neq k$, it is a consequence of the continuity of both the functions $t \mapsto \widetilde{\kappa}_\ell(t) U_{\ell,t} \mathbf{h}_\ell$ and $t \mapsto \kappa_\ell(t) U_{\ell,t} \mathbf{h}_\ell$ that the coefficient ratio $t \mapsto \widetilde{\kappa}_\ell(t) \overline{\kappa_\ell(t)}$ is also continuous (see footnote 2). Hence we deduce the continuity of $t \mapsto \big[\prod_{\ell:\ell\neq k} \widetilde{\kappa}_\ell(t)\overline{\kappa_\ell(t)}\big] \widetilde{\kappa}_k(t) U_{k,t}^* \mathbf{x}$. However, here we have $\big[\prod_{\ell:\ell\neq k} \widetilde{\kappa}_\ell(t)\overline{\kappa_\ell(t)}\big] \widetilde{\kappa}_k(t) = \big[\prod_{m=1}^N \widetilde{\kappa}_m(t)\big]\big[\prod_{\ell:\ell\neq k} \overline{\kappa_\ell(t)}\big] = \kappa_k(t)$. □

Corollary 2.8. *In the setting of 2.7, the functions $t \mapsto \overline{\kappa_k(t)} U_{k,t}^*$ are also strongly continuous.*

PROOF. It is a well-known elementary fact [5] that the adjoints of the elements of a strongly convergent net of unitary operators in a Hilbert space form a strongly convergent net. □

3. Separate commutativity

In view of 2.7 and 2.8, we may use symmetric strongly continuous factors in the one-parameter group $\mathbf{U}$ by passing from $U_{k,t}$ to $\kappa_k(t) U_{k,t}$ if $t \geq 0$ and $\overline{\kappa_k(-t)} U_{k,-t}^*$ for $t \leq 0$ with suitable functions $\kappa_1, \dots, \kappa_N : \mathbb{R}_+ \to \mathbb{T}$.

Conventions 3.1. In addition to 2.5, henceforth we assume without loss of generality that, for $k = 1, \dots, N$,

1) $t \mapsto U_{k,t}$ is strongly continuous;
2) $U_{k,0} = \mathrm{Id}$, $U_{k,-t} = U_{k,t}^*$ $(t \in \mathbb{R})$.

Proposition 3.2. *The families $\{U_{k,t} : t \in \mathbb{R}\}$ $(k = 1, \dots, N)$ are Abelian.*

PROOF. Consider any $t > 0$. We can see by induction on $n = 1, 2, \dots$ that

$$U_{k,nt} = \kappa_k^{(n)} [U_{k,t}]^n \quad (1 \leq k \leq N), \qquad \textstyle\prod_{k=1}^N \kappa_k^{(n)} = 1 \tag{3.3}$$

for some family $\big\{\kappa_k^{(n)} : 1 \leq k \leq N,\ n = 1, 2, \dots\big\} \subset \mathbb{T}$ of constants. Indeed, for $n = 1$ the choice $\kappa_k^{(1)} := 1$ suits trivially. Assume (3.3). for some $n \geq 1$. We have then

$$\begin{aligned}
\mathbf{U}\big((n+1)t\big) &= \big[U_{1,(n+1)t}^* \otimes \cdots \otimes U_{N,(n+1)t}^*\big]^*, \\
\mathbf{U}(t)\ \mathbf{U}(nt) &= \big[U_{1,t}^* \otimes \cdots \otimes U_{N,t}^*\big]^* \Big[\Big(\overline{\kappa_1^{(n)}}\big(U_{1,t}^*\big)^n\Big) \otimes \cdots \otimes \Big(\overline{\kappa_N^{(n)}}\big(U_{N,t}^*\big)^n\Big)\Big]^* = \\
&= \Big[\Big(\overline{\kappa_1^{(n)}}\big(U_{1,t}^*\big)^{n+1}\Big) \otimes \cdots \otimes \Big(\overline{\kappa_N^{(n)}}\big(U_{N,t}^*\big)^{n+1}\Big)\Big]^*
\end{aligned}$$

Since factorizations of composition operators are unique up to constants, it follows

$$U_{k,(n+1)t} = \sigma_k^{(n)} \kappa_k^{(n)} U_{k,t}^{n+1} \quad (1 \le k \le N), \qquad \prod_{k=1}^{N} \sigma_k^{(n)} = 1$$

for some $\sigma_1^{(n)}, \dots, \sigma_N^{(n)} \in \mathbb{T}$. Thus (3.3) holds with $n+1$ in place of n for $\kappa_k^{(n+1)} := \sigma_k^{(n)} \kappa_k^{(n)}$, which completes the induction step. As a consequence of (3.3), the families

$$\mathcal{U}_{k,t} := \{U_{k,nt} : \ n = 0, \pm 1, \pm 2, \dots\}$$

are Abelian because $U_{k,-nt} = U_{k,nt}^{-1} = U_{k,nt}^{*}$ $(t \ge 0, \ 1 \le k \le N)$ and the powers of $U_{k,t}$ commute. Since

$$\mathcal{U}_{k,1} \subset \mathcal{U}_{k,1/2!} \subset \mathcal{U}_{k,1/3!} \subset \cdots$$

each family $\{U_{k,q} : \ q \in \mathbb{Q}\} = \bigcup_{n=0}^{\infty} \mathcal{U}_{k,1/n!}$ where $\mathbb{Q} := \{\text{rational numbers}\}$ is Abelian. Given any couple $s, t \in \mathbb{R}$, choose sequences (p_n), (q_n) in $\mathbb{Q}$ converging to s and t, respectively. Then the commutator $[U_{k,s}, U_{k,t}] (:= U_{k,s}U_{k,t} - U_{k,t}U_{k,s})$ is the strong limit of the commutators $[U_{k,p_n}, U_{k,q_n}] = 0$ because the product of two bounded strongly convergent sequences of normed space operators converges strongly to the product of their limits. □

Theorem 3.4. *Given a strongly continuous one-parameter group* $\mathbf{U} : \mathbb{R} \to \mathfrak{A}$ *of the form* 2.5*, there are functions* $\kappa_1, \dots, \kappa_N : \mathbb{R} \to \mathbb{T}$ *and there are (not necessarily strongly continuous) one-parameter groups* $t \mapsto U_k^t \in \mathcal{U}(\mathbf{H}^{(k)})$ *such that*

$$U_{k,t} = \kappa_k(t) U_k^t, \quad \kappa_k(0) = 1 \qquad (t \in \mathbb{R}, \ 1 \le k \le N).$$

Proof. By Proposition 3.2, the families $\mathcal{U}_k := \{U_{k,t} : \ t \in \mathbb{R}\}$ are necessarily Abelian. Thus, given any index k, the complex norm-span $\mathcal{A}_k$ of $\mathcal{U}_k$ is a commutative unital C^*-subalgebra in $\mathcal{L}(\mathbf{H}^{(k)})$. In particular, for some compact topological space Ω_k, $\mathcal{A}_k$ is isometrically isomorphic to the algebra $\mathcal{C}(\Omega_k)$ of all continuous functions $\Omega_k \to \mathbb{C}$ equipped with the spectral norm, and there is a surjective linear isometry $\mathbf{T}_k : \mathcal{C}(\Omega_k) \leftrightarrow \mathcal{A}_k$ along with a family of continuous functions $u_{k,t} : \Omega_k \to \mathbb{T}$ $(t \in \mathbb{R})$ such that

$$\mathbf{T}_k u_{k,t} = U_{k,t} \qquad (t \in \mathbb{R}).$$

Similarly as in the proof of (3.3), the relations $\mathbf{U}(s)\mathbf{U}(t) = \mathbf{U}(s+t)$ $(s, t \in \mathbb{R})$ imply that

$$U_{k,s} U_{k,t} = \lambda_k(s,t) U_{k,s+t} \qquad (s, t \in \mathbb{R})$$

with suitable functions $\lambda_1, \dots, \lambda_N : \mathbb{R}^2 \to \mathbb{T}$ satisfying $\prod_{k=1}^{N} \lambda_k(s,t) = 1$. Fix any index $k \in \{1, \dots, N\}$ along with an element $\omega_0 \in \Omega_k$ and define

$$\kappa_k(t) := u_{k,t}(\omega_0), \quad U_k^t := \overline{\kappa_k(t)} U_{k,t} \qquad (t \in \mathbb{R}).$$

By convention 3.1(2), $U_{k,0} = \mathrm{Id}_{\mathbf{H}^{(k)}}$ whence $u_{k,0} \equiv 1$ and $\kappa_k(0) = 1$. Since $\lambda_k(s,t)u_{k,s+t} = \mathbf{T}_k^{-1}[\lambda_k(s,t)U_{k,s+t}] = \mathbf{T}_k^{-1}[U_{k,s}U_{k,t}] = u_{k,s}u_{k,t}$, we have

$$\lambda_k(s,t) = u_{k,s}(\omega_0)u_{k,t}(\omega_0)\overline{u_{k,s+t}(\omega_0)}.$$

It follows

$$\begin{aligned} U_k^s U_k^t &= \overline{u_{k,s}(\omega_0)u_{k,t}(\omega_0)}U_{k,s}U_{k,t} = \overline{u_{k,s}(\omega_0)u_{k,t}(\omega_0)}\lambda_k(s,t)U_{k,s+t} = \\ &= \overline{u_{k,s+t}(\omega_0)}U_{k,s+t} = U_k^{s+t} \qquad (s,t \in \mathbb{R}). \quad \square \end{aligned}$$

Remark 3.5. In contrast with previous constructions, the product of the functions $\kappa_1, \ldots, \kappa_N$ in Theorem 3.4 may differ from 1 in general.

4. Local Gelfand-Neumark representations

Conventions 4.1. Throughout this section let $k \in \{1, \ldots, N\}$ be an arbitrarily fixed index and write $\mathbf{H} := \mathbf{H}^{(k)}$ for short. We shall consider a one-parameter group $t \mapsto U^t \in \mathcal{U}(\mathbf{H})$ of operators along with a function $\kappa : \mathbb{R} \to \mathbb{T}$ such that

1) $t \mapsto U_t := \kappa(t)U^t$ is strongly continuous,
2) $\kappa(-t) = \overline{\kappa(t)} \quad (t \in \mathbb{R}), \quad \kappa(0) = 1.$

For motivation recall the decomposition $\kappa_k(t)U_k^t = U_{k,t}$ of the strongly continuous factor $t \mapsto U_{k,t}$ of $\mathbf{U}(\cdot)$ in Theorem 3.4. As further standard notations, define

$$\begin{aligned} \mathcal{A} &:= \big\{\text{the } C^*\text{-subalgebra of } \mathcal{L}(\mathbf{H}) \text{ generated by } \{U^t : \ t \in \mathbb{R}\}\big\}, \\ \mathbf{T} &: \ \mathcal{C}(\Omega) \leftrightarrow \mathcal{A} \quad \text{the Gelfand representation of } \mathcal{A}, \\ u^t &:= \mathbf{T}^{-1}U^t \qquad (t \in \mathbb{R}). \end{aligned}$$

Representation 4.2. Modifying slightly a familiar construction [5], for any unit vector $\mathbf{x} \in \mathbf{H}$, let

$$\mathbf{H}_\mathbf{x} := \overline{\mathrm{Span}}\{A\mathbf{x} : \ A \in \mathcal{A}\}$$

be the closed (necessarily separable) subspace of $\mathbf{H}$ spanned by the range of the continuous function $t \mapsto \kappa(t)U^t\mathbf{x}$. Since $\mathcal{A}$ is spanned by its self-adjoint elements and since the orthocomplement of any eigensubspace of a self-adjoint operator is also an eigensubspace, we have a complete orthogonal decomposition

$$\mathbf{H} = \textstyle\bigoplus_{j \in J} \mathbf{H}_{\mathbf{x}_j} \tag{4.3}$$

with any maximal family $\big\{\mathbf{H}_{\mathbf{x}_j} : \ j \in J\big\}$ such that $\mathbf{H}_{\mathbf{x}_j} \perp \mathbf{H}_{\mathbf{x}_k} \ \ (j \neq k \in J)$ guaranteed by the Zorn Lemma. For later use we fix a decomposition (4.3). Given any index $j \in J$, the mapping

$$\phi_j(\varphi) := \big\langle\, [\mathbf{T}\varphi]\mathbf{x}_j \,\big|\, \mathbf{x}_j \,\big\rangle \qquad (\varphi \in \mathcal{C}(\Omega))$$

is a positive linear functional with $\phi_j\big(1_\Omega\big) = 1$. By the Riesz-Kakutani Representation Theorem, there is a unique probability Radon measure μ_j on Ω such that

$$\int_{\omega\in\Omega} \varphi(\omega)\ \mu_j(d\omega) = \phi_j(\varphi) \qquad \big(\varphi \in \mathcal{C}(\Omega)\big).$$

Since

$$\Big\langle U^t\,[\mathbf{T}\varphi]\mathbf{x}_j \,\Big|\, [\mathbf{T}\psi]\mathbf{x}_j \Big\rangle = \Big\langle [\mathbf{T}\psi]^* U^t\,[\mathbf{T}\varphi]\mathbf{x}_j \,\Big|\, \mathbf{x}_j \Big\rangle = \int_{\omega\in\Omega} \overline{\psi}(\omega) u^t(\omega)\varphi(\omega)\ \mu_j(d\omega),$$

the representation $\mathbf{T}$ extends to an isometric isomorphism

$$\mathbf{T}_j : L^2(\Omega,\mu_j) \leftrightarrow \mathbf{H}_{\mathbf{x}_j}$$

with the property

$$\Big\langle\, U^t \mathbf{T}_j f \,\Big|\, \mathbf{T}_j g \,\Big\rangle = \int_{\omega\in\Omega} u^t(\omega) f(\omega)\overline{g(\omega)}\ \mu_j(d\omega) \qquad \big(t\in\mathbb{R},\ f,g\in L^2(\Omega,\mu_j)\big).$$

Notice that the restricted operator $U^t|\mathbf{H}_{\mathbf{x}_j} \in \mathcal{U}(\mathbf{H}_{\mathbf{x}_j})$ is unitarily equivalent to the multiplication operator

$$\mathbf{M}^{(j)}_{u^t} f := u^t f \qquad \big(f \in L^2(\Omega,\mu_j)\big).$$

Namely we have $\mathbf{M}^{(j)}_{u^t} = \mathbf{T}_j^{-1} U_j^t \mathbf{T}_j$.

Remark 4.4. According to the usual convention, the space $L^2(\Omega,\mu_j)$ consists of equivalence classes of functions modulo zero sets with respect to μ_j. Actually such zero sets may be rather "large" in the sense that $\mu_j\big(\mathrm{supp}(\mu_k)\big) = 0$ $(j \neq k)$ in general.

Example. Let $\mathbf{H} = \ell^2\big(\ = \big\{[\xi_n]_{n=1}^\infty :\ \sum_n |\xi_n|^2 < \infty\big\}\big)$ and $U^t[\xi_n]_{n=1}^\infty := [e^{int}\xi_n]_{n=1}^\infty$. We can take $J = \Omega = \{1,2,\ldots\}$, $\mathbf{x}_j = [\delta_{jn}]_{n=1}^\infty$. In this case, each measure μ_j is supported by the single point $\{j\}$, and $\mu_j(\Omega\setminus\{j\}) = 0$.

Remark 4.5. Recall the following simple fact concerning the strong convergence of bounded sequences (or even nets) of operators: *Given an orthogonal decomposition* $\mathbf{H} = \bigoplus_{j\in J} \mathbf{H}_j$*, a sequence* $A_1, A_2, \ldots \in \mathcal{L}(\mathbf{H})$ *with* $\sup_n \|A_n\| < \infty$ *converges strongly to* 0 *if and only if it converges to* 0 *strongly* componentwise *that is if* $\lim_n \|A_n\mathbf{x}\| = 0$ $(j \in J,\ \mathbf{x} \in \mathbf{H}_j)$.

In terms of the representation 4.2, we can interpret Remark 4.5 as follows.

Lemma 4.6. *Given a mapping* $t \mapsto w_t$ *from* $\mathbb{R}$ *into* $\mathcal{C}(\Omega)$ *such that* $\max|w_t| \leq 1$ $(t \in \mathbb{R})$ *the statements below are equivalent:*

(i) *the operator valued function* $t \mapsto W_t := \mathbf{T}w_t$ *is strongly continuous;*

(ii) *all the restrictions* $t \mapsto W_t|\mathbf{H}_{\mathbf{x}_j}$ $(j \in J)$ *are strongly continuous;*

(iii) *all the multiplication operator valued functions* $t \mapsto \mathbf{M}^{(j)}_{w(t)}$ $(j \in J)$ *with* $\mathbf{M}^{(j)}_{w(t)} := [f \mapsto w(t)f] \in \mathcal{L}(L^2(\Omega, \mu_j))$ *are strongly continuous.* □

Remark 4.7. The main step in our proof of Theorem 1.1 will be to show that, given any index $j \in J$, we have

$$\kappa(t)U^t|\mathbf{H}_{\mathbf{x}_j} = \chi_j(t)\widetilde{U}^t_j \qquad (t \in \mathbb{R})$$

with a suitable continuous function $\chi_j : \mathbb{R} \to \mathbb{T}$ and a strongly continuous one-parameter subgroup $t \mapsto \widetilde{U}^t_j$ of $\mathcal{U}(\mathbf{H}_{\mathbf{x}_j})$.

5. Probabilistic arguments

We are going to carry out the program of Remark 4.7.

Conventions 5.1. Throughout this section let Ω denote a compact topological space and let μ be a probability Radon measure on it (i.e. $\mu(\Omega) = 1$). Given any bounded μ-measurable function $a : \Omega \to \mathbb{C}$, we shall write $\mathbf{M}_a$ for the multiplication operator $\mathbf{M}_a : f \mapsto af$ on $L^2(\Omega, \mu)$. Furthermore let $[u^t : \ t \in \mathbb{R}]$ be a one-parameter family of continuous functions $\Omega \to \mathbb{T}$ in the sense that $u^{t+h}(\omega) = u^t(\omega)u^h(\omega)$ for all $t, h \in \mathbb{R}$ and $\omega \in \Omega$. Finally we assume that $\kappa : \mathbb{R} \to \mathbb{T}$ is a function such that

$$\kappa(0) = 1, \quad \kappa(-t) = \overline{\kappa(t)} \qquad (t \in \mathbb{R})$$

and the mapping $t \mapsto \kappa(t)\mathbf{M}_a$ is *strongly* continuous that is

$$\begin{aligned} &\lim_{h\to 0} \big\| \big[\kappa(t+h)\mathbf{M}_{u^{t+h}} - \kappa(t)\mathbf{M}_{u^t}\big] f\big\|^2 = \\ &= \lim_{h\to 0} \int_{\omega\in\Omega} \big|\kappa(t+h)u^{t+h}(\omega)f(\omega) - \kappa(t)u^t(\omega)]f(\omega)\big|^2 \ \mu(d\omega) = 0 \end{aligned} \tag{5.2}$$

for any $t \in \mathbb{R}$ and $f \in L^2(\Omega, \mu)$.

In terms of the representation 4.2, given any index $j \in J$ and, by taking $\mu := \mu_j$, for the existence of a decomposition required in Remark 4.5 we have to prove that $\kappa(t)u^t = \chi(t)\widetilde{u}^t$ $(t \in \mathbb{R})$ with some continuous function $\chi : \Omega \to \mathbb{T}$ and a suitable one-parameter group $[\widetilde{u}^t : \ t \in \mathbb{R}]$ of continuous functions $\Omega \to \mathbb{T}$ such that the operator valued function $t \mapsto \mathbf{M}_{\widetilde{u}^t}$ be strongly continuous.

Lemma 5.3. *Given a sequence* $a_1, a_2, \ldots : \Omega \to \mathbb{C}$ *of* μ*-measurable functions such that* $\sup_n \sup |a_n| < \infty$*, the multiplication operators* $\mathbf{M}_{a_1}, \mathbf{M}_{a_2}, \ldots \in \mathcal{L}\big(L^2(\Omega, \mu)\big)$ *converge strongly to* 0 *if and only if the functions* a_n *converge stochastically to* 0 *with respect to the measure* μ*; that is, if*

$$\lim_n \mu\{\omega \in \Omega : \ |a_n(\omega)| > \varepsilon\} = 0 \qquad (\varepsilon > 0). \tag{5.4}$$

Proof. If $a_n \not\to 0$ stochastically then $\liminf_n \mu\{\omega : \ |a_n(\omega)| > \varepsilon\} > 0$. Since $\|\mathbf{M}_{a_n} 1_\Omega\|^2 = \int |a_n|^2 \ \mu(d.) \geq \varepsilon^2 \mu\{\omega : \ |a_n(\omega)| > \varepsilon\}$, in this case we have $\liminf_n \|\mathbf{M}_{a_n} 1_\Omega\|^2 > 0$ that is $\mathbf{M}_{a_n} 1_\Omega \not\to 0$ in $L^2(\Omega, \mu)$.

Assume (5.4) and let $M := \sup_n \sup |a_n|$. Let $\varepsilon > 0$ and a function $f \in L^2(\Omega, \mu)$ be given. By the Markov inequality, $\mu\{\omega : \ |f(\omega)| > y\} \leq \int |f|^2 \ \mu(d.)/y$ $(y > 0)$. Thus we can choose a value $y > 1$ such that $\int_{\omega \in S} |Mf(\omega)|^2 \ \mu(d\omega) < \varepsilon/3$ with the set $S := \{\omega : \ |f(\omega)| > y\}$. As a consequence of (5.4), there exists an index N such that, with the sets $\Omega_n := \{\omega : \ |a_n(\omega)y|^2 > \varepsilon/3\}$ we have $y^2\mu(\Omega_n) < \varepsilon/3$ whenever $n > N$. Then, for any $n > N$, $\int_S |a_n f|^2 \ \mu(d.) \leq \varepsilon/3$ and also $\int_{\Omega_n} |a_n f|^2 \ \mu(d.) \leq \varepsilon/3$. For the remaining points $\omega \in \Omega \setminus (\Omega_n \cup S)$ we have $|a_n(\omega)|^2 \leq \varepsilon/(3y^2)$ and $|f(\omega)|^2 \leq y^2$. Therefore $\|\mathbf{M}_{a_n} f\|^2 = \int |a_n f|^2 \ \mu(d.) \leq \varepsilon$ for the indices $n > N$. Thus (5.4) entails $\|\mathbf{M}_{a_n} f\| \to 0 \ \ (f \in L^2(\Omega, \mu))$. ☐

Proposition 5.5. *We have*

$$\lim_{h \to 0} \int_{\omega_1, \omega_2 \in \Omega} \left| u^h(\omega_1) - u^h(\omega_2) \right|^2 \ \mu(d\omega_1)\mu(d\omega_2) = 0.$$

Proof. By assumption $u^0 = 1_\Omega$ is the unit element in $[u^t : \ t \in \mathbb{R}]$. Also $\kappa(0) = 1$. Thus, according to Lemma 5.3,

$$\lim_{h \to 0} \mu\{\omega : \ |\kappa(h)u^h(\omega) - 1| > \varepsilon\} = 0 \qquad (\varepsilon > 0). \tag{5.4'}$$

Notice that

$$\begin{aligned} \left|u^h(\omega_1) - u^h(\omega_2)\right| &= \left|\kappa(h)\big(u^h(\omega_1) - u^h(\omega_2)\big)\right| \leq \\ &\leq \ \left|\kappa(h)u^h(\omega_1) - 1\right| + \left|\kappa(h)u^h(\omega_2) - 1\right|. \end{aligned}$$

Hence, with the product measure $\mu \otimes \mu$, from (5.4′) it follows

$$\lim_{h \to 0} \mu \otimes \mu\big\{(\omega_1, \omega_2) \in \Omega^2 : \ \left|u^h(\omega_1) - u^h(\omega_2)\right| > \varepsilon\big\} = 0 \qquad (\varepsilon > 0). \tag{5.4''}$$

Since always $\left|u^h(\omega_1) - u^h(\omega_2)\right| \leq \text{diameter}(\mathbb{T}) = 2$, given any $\varepsilon > 0$, with the abbreviation $S_{h,\varepsilon} := \big\{(\omega_1, \omega_2) \in \Omega^2 : \ \left|u^h(\omega_1) - u^h(\omega_2)\right| > \varepsilon\big\}$ we have the estimate

$$\begin{aligned} \int_{\omega_1, \omega_2 \in \Omega} & \left|u^h(\omega_1) - u^h(\omega_2)\right|^2 \ \mu(d\omega_1)\mu(d\omega_2) \leq \\ & \leq \ \varepsilon^2\big[1 - \mu \otimes \mu\big(S_{h,\varepsilon}\big)\big] + 2^2 \mu \otimes \mu\big(S_{h,\varepsilon}\big). \end{aligned}$$

Then (5.4″) implies $\limsup_{h \to 0} \int_{\omega_1, \omega_2 \in \Omega} \left|u^h(\omega_1) - u^h(\omega_2)\right|^2 \ \mu(d\omega_1)\mu(d\omega_2) \leq \varepsilon^2$ for any $\varepsilon > 0$. ☐

Remark 5.6: By the aid of Euler's identity $2i\sin kx = (e^{ix})^k - (e^{-ix})^k$ and the closed formula for sums of geometric sequences, we get

$$\sum_{k=1}^{n} \sin^2 kx = \frac{2n+1}{4} - \frac{1}{4}\,\frac{\sin(2n+1)x}{\sin x}.$$

In the standard reference [4, p.36] we find $\sum_{k=1}^{n} \sin^2 kx = \frac{n}{2} - \frac{\cos(n+1)x\sin nx}{2\sin x}$. The form above is obtained hence by the aid of the identity $\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = \frac{1}{2}(\sin\alpha - \sin\beta)$ with $\alpha := (2n+1)x$ and $\beta := x$.

Notation 5.7. Henceforth we write

$$\phi_n(x) := \frac{1}{n}\sum_{k=1}^{n} \sin^2 kx.$$

Lemma 5.8. *For any index $n = 1, 2, \ldots$ we have*

$$\phi_n(x) \geq \frac{4}{\pi^2}\,\frac{(n+1)(2n+1)}{6}x^2 \geq \frac{4n^2}{3\pi^2}\,\sin^2 x \qquad \text{for} \quad 0 \leq x \leq \frac{\pi}{2n},$$
$$\phi_n(x) \geq \frac{1}{4} \qquad \text{for} \quad \frac{\pi}{2n} \leq x \leq \frac{\pi}{2}.$$

Proof. Recall that $y \geq \sin y \geq 2y/\pi$ for $0 \leq y \leq \pi/2$. Hence

$$0 < x \leq \frac{\pi}{2n} \implies \quad \tfrac{1}{n} \sum_{k=1}^{n} \sin^2 kx \geq \frac{1}{n}\sum_{k=1}^{n}\Big(\frac{2}{\pi}kx\Big)^2 =$$
$$= \frac{4}{\pi^2}\,\frac{(n+1)(2n+1)}{6}x^2 \geq \frac{4n^2}{3\pi^2}x^2 \ .$$

On the other hand,

$$\frac{\pi}{2n} \leq x \leq \frac{\pi}{2} \implies 4n\sin x \geq \frac{8n}{\pi}x \geq \frac{8n}{\pi}\,\frac{\pi}{2n} = 4 \implies$$
$$\frac{1}{n}\sum_{k=1}^{n}\sin^2 kx = \frac{2n+1}{4n} - \frac{\sin(2n+1)x}{4n\sin x} \geq$$
$$\geq \frac{2n+1}{4n} - \frac{1}{4} \geq \frac{1}{4} \ . \quad \Box$$

Definition 5.9. On $\mathbb{T}$ we introduce the *arc length distance*

$$d(\kappa_1, \kappa_2) := 2\arcsin\frac{|\kappa_1 - \kappa_2|}{2} \qquad (\kappa_1, \kappa_2 \in \mathbb{T}).$$

Furthermore we define

$$\Delta(\delta) := \sup_{|t|\leq\delta} \int_{\omega_1,\omega_2\in\Omega} |u^t(\omega_1) - u^t(\omega_2)|^2 \ \mu(d\omega_1)\mu(d\omega_2) \ ,$$
$$\Omega^{(2)}_{t,r} := \big\{(\omega_1,\omega_2) \in \Omega^2 : \ d\big(u^t(\omega_1), u^t(\omega_2)\big) < r\big\} \ .$$

Remark 5.10. 1) From Proposition 5.5 we know already that

$$\Delta(\delta) \searrow 0 \qquad (\delta \searrow 0).$$

2) It is a simple fact from elementary geometry that

$$d(\kappa_1, \kappa_2) \leq \frac{\pi}{n} \Longrightarrow \quad d(\kappa_1^k, \kappa_2^k) = kd(\kappa_1, \kappa_2) \quad (k = 1, \ldots, n).$$

3) Since $u^{t+s} = u^t u^s$ $(s, t \in \mathbb{R})$, for any pair of positive integers m, n and for any $t \in \mathbb{R}$ we have $u^{mnt} = [u^{nt}]^m$ and hence

$$\Omega^{(2)}_{t/(mn),\pi/(mn)} \subset \Omega^{(2)}_{t/n,\pi/n} \ .$$

4) In terms of the *principal branch* of the complex logarithm

$$\log_*(re^{i\varphi}) := \log r + i\varphi \qquad (r > 0,\ -\pi < \varphi < \pi),$$

in case of $\kappa \in \mathbb{T}$ with $d(\kappa, 1) < \pi/n$ we have $\log_*(\kappa^k, 1) = k \log_*(\kappa, 1) = \pm kd(\kappa, 1)$ for $k = 1, \ldots, n$. Hence

$$\begin{aligned}
&(\omega_1, \omega_2) \in \Omega^{(2)}_{t/n,\pi/n} \Longrightarrow \\
&\log_* \left[u^{kt/n}(\omega_1)/u^{kt/n}(\omega_2)\right] = \frac{k}{n} \log_* \left[u^t(\omega_1)/u^t(\omega_2)\right] \quad (k = -n, \ldots, n), \\
&u^{kt/n}(\omega_1)/u^{kt/n}(\omega_2) = \exp\Big(\frac{k}{n} \log_* \left[u^t(\omega_1)/u^t(\omega_2)\right]\Big) \quad (k \in \mathbb{Z}).
\end{aligned}$$

Proposition 5.11. *For any $t \in \mathbb{R}$ we have*

$$\mu \otimes \mu\Big(\bigcap_{n=1}^{\infty} \Omega^{(2)}_{t/n!,\pi/n!} \Big) \geq 1 - \Delta(|t|) \ .$$

Proof. Let $t \in \mathbb{R}$ be arbitrarily fixed and write $\varepsilon := \Delta(|t|)$. According to Remark 5.10(3),

$$\Omega^{(2)}_{t,\pi} \supset \Omega^{(2)}_{t/2,\pi/2} \supset \Omega^{(2)}_{t/3!,\pi/3!} \supset \cdots$$

that is the sets $\Omega^{(2)}_{t/n!,\pi/n!}$ form a shrinking sequence. Thus it suffices to establish that $\mu \otimes \mu\big(\Omega^{(2)}_{t/n,\pi/n}\big) \geq 1 - \varepsilon$ $(n = 1, 2, \ldots)$. Fix also $n > 0$ arbitrarily. Then

$$\int_{\omega_1,\omega_2 \in \Omega} |u^{kt/n}(\omega_1) - u^{kt/n}(\omega_2)|^2 \ \mu(d\omega_1)\mu(d\omega_2) \leq \varepsilon \qquad (k = 1, \ldots, n).$$

It follows

$$\begin{aligned}
\varepsilon \quad &\geq \quad \frac{1}{n} \sum_{k=1}^{n} \int_{\omega_1,\omega_2 \in \Omega} |u^{kt/n}(\omega_1) - u^{kt/n}(\omega_2)|^2 \ \mu(d\omega_1)\mu(d\omega_2) = \\
&= \quad \frac{1}{n} \sum_{k=1}^{n} \int_{\omega_1,\omega_2 \in \Omega} 4 \sin^2 \frac{1}{2} d\big(u^{kt/n}(\omega_1), u^{kt/n}(\omega_2)\big) \ \mu(d\omega_1)\mu(d\omega_2) = \\
&= \quad 4 \int_{\omega_1,\omega_2 \in \Omega} \phi_n\Big(\frac{1}{2} d\big(u^{t/n}(\omega_1), u^{t/n}(\omega_2)\big)\Big) \ \mu(d\omega_1)\mu(d\omega_2) \ .
\end{aligned}$$

By Lemma 5.8 we have $\phi_n\big(\frac{1}{2}\,d\big(u^{t/n}(\omega_1),u^{t/n}(\omega_2)\big)\big)\geq 1/4$ for all $(\omega_1,\omega_2)\in\Omega^2\setminus\Omega^{(2)}_{t/n,\pi/n}$. Therefore

$$\begin{aligned}\frac{\varepsilon}{4}\;&\geq\;\int_{(\omega_1,\omega_2)\in\ \Omega^2\setminus\Omega^{(2)}_{t/n,\pi/n}}\phi_n\Big(d\big(\frac{1}{2}u^{t/n}(\omega_1),u^{t/n}(\omega_2)\big)\Big)\ \mu(d\omega_1)\mu(d\omega_2)\geq\\ &\geq\;\frac{1}{4}\ \mu\otimes\mu\Big(\Omega^2\setminus\Omega^{(2)}_{t/n,\pi/n}\Big)=\frac{1}{4}\ \Big[1-\mu\otimes\mu\Big(\Omega^{(2)}_{t/n,\pi/n}\Big)\Big]\end{aligned}$$

whence the statement is immediate. □

Corollary 5.12. *There is a* $\mu\otimes\mu$*-measurable function* $\alpha:\Omega^2\to\mathbb{R}$ *such that we have*

$$u^q(\omega_1)/u^q(\omega_2)=\exp\big(i\,q\,\alpha(\omega_1,\omega_2)\big)\qquad(q\in\mathbb{Q}:=\{\text{rational numbers}\})$$

for $\mu\otimes\mu$*-almost every* $(\omega_1,\omega_2)\in\Omega^2$.

Proof. Let $t_m:=1/m$ and consider the pairwise disjoint $\mu\otimes\mu$-measurable sets

$$S_m:=\bigcap_{n=1}^{\infty}\Omega^{(2)}_{t_m/n!,\pi/n!}\qquad(m=1,2,\ldots),$$

$$D_\ell:=S_\ell\setminus\bigcup_{m=1}^{\ell}S_m\qquad(\ell=1,2,\ldots).$$

Since $\Delta(t)\searrow 0$ for $t\searrow 0$, we have $\mu\otimes\mu\big(\bigcup_{\ell=1}^m D_\ell\big)\geq\mu\otimes\mu\big(S_m\big)\geq 1-\Delta(t_m)\nearrow 1$ $(m\to\infty)$. Thus the set $\Omega^2\setminus\bigcup_{N=1}^{\infty}D_N$ has $\mu\otimes\mu$-measure zero. Let $(\omega_1,\omega_2)\in D_\ell$ and $q\in\mathbb{Q}$ be any rational number. For some pair of integers k and n with $n>0$ we can write $q=k/(\ell n!)=(k/n!)t_\ell$. Thus, according to Remark 5.10(4),

$$\begin{aligned}&(\omega_1,\omega_2)\in D_\ell\Longrightarrow\quad(\omega_1,\omega_2)\in\Omega^{(2)}_{t_\ell/n!,\pi/n!}\Longrightarrow\\ &\frac{u^q(\omega_1)}{u^q(\omega_2)}=\frac{u^{(k/n!)t_\ell}(\omega_1)}{u^{(k/n!)t_\ell}(\omega_2)}=\exp\Big(\frac{k}{n!}\log_*\big[u^{t_\ell}(\omega_1)/u^{t_\ell}(\omega_2)\big]\Big)=\\ &\qquad=\exp\Big(\frac{q}{t_\ell}\log_*\big[u^{t_\ell}(\omega_1)/u^{t_\ell}(\omega_2)\big]\Big).\end{aligned}$$

Therefore the real-valued function

$$\alpha(\omega_1,\omega_2):=\frac{1}{i\,t_\ell}\log_*\big[u^{t_\ell}(\omega_1)/u^{t_\ell}(\omega_2)\big]\qquad\big((\omega_1,\omega_2)\in D_\ell;\ \ell=1,2,\ldots\big)$$

is well-defined $\mu\otimes\mu$-almost everywhere and suits the requirements of 5.12. □

Theorem 5.13. *In the setting of* 5.1, *there is a continuous function* $\chi:\mathbb{R}\to\mathbb{T}$ *along with a* μ*-measurable function* $\widetilde{\alpha}:\Omega\to\mathbb{R}$ *such that*

$$\kappa(t)u^t(\omega)=\chi(t)\exp\big(it\widetilde{\alpha}(\omega)\big)$$

for all $t\in\mathbb{R}$ *and* μ*-almost every* $\omega\in\Omega$.

Proof. With the function $\alpha : \Omega^2 \to \mathbb{R}$ constructed above,

$$\mu \otimes \mu\Big\{(\omega_1,\omega_2) :\ u^q(\omega_1)/u^q(\omega_2) = \exp\big(iq\alpha(\omega_1,\omega_2)\big) \quad (q \in \mathbb{Q})\Big\} = 1.$$

Thus, since μ is a probability measure, there is a point $\omega_0 \in \Omega$ (moreover ω_0 can be chosen μ-almost everywhere in Ω) such that

$$\mu\big\{\omega \in \Omega :\ u^q(\omega) = u^q(\omega_0)\exp\big(iq\alpha(\omega,\omega_0)\big) \quad (q \in \mathbb{Q})\big\} = 1.$$

Fix ω_0 with this property. Recall that there is a function $\kappa : \mathbb{R} \to \mathbb{T}$ such that the mapping $t \mapsto \kappa(t)u^t$ is continuous from $\mathbb{R}$ into $\mathcal{C}(\Omega)$ equipped with the topology of stochastic convergence. Then there is a (unique) function $\chi_0 : \mathbb{Q} \to \mathbb{T}$ such that

$$\chi_0(q)\exp\big(iq\alpha(\omega,\omega_0)\big) = \kappa(q)u^q(\omega) \quad \text{for all } q{\in}\mathbb{Q} \text{ and } \mu\text{-almost every } \omega.$$

Define

$$\widetilde{\alpha}(\omega) := \alpha(\omega,\omega_0) \qquad (\omega \in \Omega).$$

Observe that the mapping $t \mapsto \exp\big(it\widetilde{\alpha}(\omega)\big)$ is continuous from $\mathbb{R}$ into the space $\mathcal{S}(\Omega,\mu)$ of all μ-measurable functions $\Omega \to \mathbb{C}$ equipped with the topology of stochastic convergence (because $t_n \to t$ implies the convergence $\exp\big(it_n\widetilde{\alpha}(\omega)\big) \to \exp\big(it\widetilde{\alpha}(\omega)\big)$ μ-almost everywhere in $\omega{\in}\Omega$). It is well-known that the product of bounded stochastically continuous maps is stochastically continuous. Hence $t \mapsto u^t\exp(-iq\widetilde{\alpha})$ as a mapping $\mathbb{R} \to \mathcal{S}(\Omega,\mu)$ is stochastically continuous. However, the functions $\kappa(t)u^t\exp(-iq\widetilde{\alpha})$ $(t \in \mathbb{R})$ are constant μ-almost everywhere. Therefore the function

$$q \mapsto \chi_0(q) = \big[\mu\text{-almost everywhere value of } \kappa(q)u^q\exp(-iq\widetilde{\alpha})\big]$$

ranging in $\mathbb{T}$ must admit a continuous extension χ from $\mathbb{Q}$ to $\mathbb{R}$. □

Corollary 5.14. *There is a point $\omega_0 \in \Omega$ such that for all $q \in \mathbb{Q}$ and for μ-almost every $\omega \in \Omega$ we have $u^q(\omega) = u^q(\omega_0)\exp\big(iq\alpha(\omega,\omega_0)\big)$ and the function $q{\mapsto}\kappa(q)u^q(\omega_0)$ admits a continuous extension from $\mathbb{Q}$ to $\mathbb{R}$.* □

6. Proof of Theorem 1.1

Remark 6.1. According to Theorem 5.13 and in view of Remark 4.7, we can represent each operator valued function $t \mapsto U_{k,t} = \kappa_k(t)U_k^t$ in Theorem 3.4 in the form

$$U_{k,t} = \bigoplus_{j\in J_k} \chi_j^{(k)}(t)\exp\big(itA_j^{(k)}\big) \qquad (t \in \mathbb{R}) \tag{6.2}$$

with suitable families $[\chi_j^{(k)} :\ j \in J_k]$ of continuous functions $\mathbb{R} \to \mathbb{T}$, respectively not necessarily bounded self-adjoint operators

$$A_j^{(k)} : \mathbf{D}_j^{(k)} \to \mathbf{H}_j^{(k)}, \qquad \mathbf{D}_j^{(k)} \text{ dense linear submanifold} \subset \mathbf{H}_j^{(k)}$$

for some orthogonal decompositions

$$\mathbf{H}^{(k)} = \bigoplus_{j\in J_k} \mathbf{H}_j^{(k)} \qquad (k=1,\dots,N).$$

Thus, with the operators

$$M_t^{(k)} := \bigoplus_{j\in J_k} \chi_j^{(k)}(t)\mathrm{Id}_{\mathbf{H}_j^{(k)}}, \quad \widetilde{U}_k^t := \bigoplus_{j\in J_k} \exp\left(itA_j^{(k)}\right)$$

we can represent our object the semigroup $\mathbf{U}(\cdot)$ in Theorem 1.1 in the form

$$\mathbf{U}(t) = \mathbf{B}^t\mathbf{C}^t \text{ where } \mathbf{B}^t := \left[\widetilde{U}_1^t\right]^*\otimes\cdots\otimes\left[\widetilde{U}_N^t\right]^*, \quad \mathbf{C}^t := \left[M_t^{(1)}\right]^*\otimes\cdots\otimes\left[M_t^{(N)}\right]^*.$$

According to Lemma 4.6, the Hilbert space operator-valued functions $t\mapsto M_t^{(k)}$ respectively $t\mapsto \widetilde{U}_k^t$ are strongly continuous. Recalling the elementary fact [5] that the product of bounded strongly convergent nets of linear operators is strongly convergent, we see that both the operator-valued functions $t\mapsto\mathbf{B}^t$, $t\mapsto\mathbf{C}^t$ are also strongly continuous. Since the restrictions of the operators $M_t^{(k)}$ are multiples of the identity on the subspaces $\mathbf{H}_j^{(k)}$ $(j\in J_k)$, the family $\left\{\mathbf{U}(t),\mathbf{B}^t,\mathbf{C}^t:\ t\in\mathbb{R}\right\}$ is Abelian. Consequently

$$\mathbf{C}^{t+h} = \mathbf{U}(t+h)\mathbf{B}^{-t-h} = \mathbf{U}(t)\mathbf{U}(h)\mathbf{B}^{-t}\mathbf{B}^{-h} = \mathbf{U}(t)\mathbf{B}^{-t}\mathbf{U}(h)\mathbf{B}^{-h} = \mathbf{C}^t\mathbf{C}^h$$

for all $t,h\in\mathbb{R}$.

Lemma 6.3. *Assume* (*as in the setting described in* 6.1) *that we have the orthogonal decompositions* $\mathbf{H}^{(k)} = \bigoplus_{j\in J_k}\mathbf{H}_j^{(k)}$, *and the operators*

$$\mathbf{C}^t := \left[\oplus_{j\in J_1}\ \chi_j^{(1)}(t)\mathrm{id}_{\mathbf{H}_j^{(1)}}\right]^*\otimes\cdots\otimes\left[\oplus_{j\in J_N}\ \chi_j^{(N)}(t)\mathrm{id}_{\mathbf{H}_j^{(N)}}\right]^* \qquad (t\in\mathbb{R})$$

form a strongly continuous one-parameter group $\mathbb{R}\to\mathfrak{A}(\mathbf{H}^{(1)},\dots,\mathbf{H}^{(N)})$ *with some* (*not necessarily continuous*) *functions* $\chi_j^{(k)}:\mathbb{R}\to\mathbb{T}$ $(j\in J_k;\ k=1,\dots,N)$. *Then, with suitable real constants* $a_j^{(k)}$ $(j\in J_k;\ k=1,\dots,N)$, *we can write*

$$\mathbf{C}^t = \left[\oplus_{j\in J_1}\exp\left(ia_j^{(1)}t\right)\mathrm{id}_{\mathbf{H}_j^{(1)}}\right]^*\otimes\cdots\otimes\left[\oplus_{j\in J_N}\exp\left(ia_j^{(N)}t\right)\mathrm{id}_{\mathbf{H}_j^{(N)}}\right]^* \quad (t\in\mathbb{R}).$$

Proof. Given any index $1\le k\le N$, for any $j\in J_k$ choose a unit vector $\mathbf{h}_j^{(k)}\in\mathbf{H}_j^{(k)}$. Notice that

$$\mathbf{C}^t\left([\mathbf{h}_{j_1}^{(1)}]^*\otimes\cdots\otimes\mathbf{h}_{j_N}^{(N)}]^*\right) = \prod_{k=1}^N \chi_{j_k}^{(k)}(t)\ \left[\mathbf{h}_{j_1}^{(1)}\right]^*\otimes\cdots\otimes\left[\mathbf{h}_{j_N}^{(N)}\right]^* \tag{6.4}$$

for all $t\in\mathbb{R}$, and $j_1\in J_1,\dots,j_N\in J_N$. Since $t\mapsto\mathbf{C}^t$ is a strongly continuous one-parameter group $\mathbb{R}\to\mathfrak{A}(\mathbf{H}^{(1)},\dots,\mathbf{H}^{(N)})$, each coefficient function

$$\psi_{j_1,\dots,j_N}(t) := \prod_{k=1}^N \chi_{j_k}^{(k)}(t)$$

in (6.4) must be a continuous homomorphism $(\mathbb{R},+) \to (\mathbb{T},\cdot)$. Therefore

$$\psi_{j_1,\dots,j_N}(t) = \exp\left(i\, c_{j_1,\dots,j_N}\, t\right) \qquad (t\in\mathbb{R};\ j_1\in J_1,\dots,j_N\in J_N)$$

with suitable constants $c_{j_1,\dots,j_N} \in \mathbb{R}$. Fix any tuple $(j_1^*,\dots,j_N^*) \in J_1 \times \dots \times J_N$ of indices. We can write

$$\begin{aligned} \chi_{j_k}^{(k)}(t) &= \frac{\psi_{j_1^*,\dots,j_{k-1}^*,j_k,j_{k+1}^*,\dots,j_N^*}(t)}{\psi_{j_1^*,\dots,j_N^*}(t)}\ \chi_{j_k^*}^{(k)}(t) = \\ &= \chi_{j_k^*}^{(k)}(t) \exp\left(i\left[c_{j_1^*,\dots,j_{k-1}^*,j_k,j_{k+1}^*,\dots,j_N^*} - c_{j_1^*,\dots,j_N^*}\right] t\right). \end{aligned}$$

Thus the statement of the lemma is fulfilled with the choice

$$\begin{aligned} a_{j_k}^{(k)} &:= c_{j_1^*,\dots,j_{k-1}^*,j_k,j_{k+1}^*,\dots,j_N^*} - c_{j_1^*,\dots,j_N^*} \quad && (k < N;\ j_1 \in J_1,\dots,j_{N-1} \in J_{N-1}), \\ a_{j_N}^{(N)} &:= c_{j_1^*,\dots,j_{k-1}^*,j_k,j_{k+1}^*,\dots,j_N^*} && (j_N \in J_N)\ . \quad \square \end{aligned}$$

6.5. Finish of the proof of Theorem 1.1

In the setting established in 6.1, it suffices to see that in (6.2) we can also write

$$U_{k,t} = \bigoplus_{j\in J_k} \exp\left(it\widehat{A}_j^{(k)}\right)$$

with suitable possibly unbounded self-adjoint operators $\widehat{A}_j^{(k)} : \mathbf{D}_j^{(k)} \to \mathbf{H}_j^{(k)}$ $(j \in J_k;\ k = 1,\dots,N)$. This is possible in view of Lemma 6.3 with the choices

$$\widehat{A}_j^{(k)} := A_j^{(k)} + a_j^{(k)} \mathrm{id}_{\mathbf{H}_j^{(k)}}\ . \quad \square$$

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