Math. Nachr. **166** (1994) 305–315

Weakly Continuous JB*-triples

By WILHELM KAUP of Tübingen and LÁSZLÓ STACHÓ [1]) of Szeged

(Received January 28, 1993)
(Revised Version August 19, 1993)

1. Introduction

By [12] the geometry of a complex Banach space E is completely encoded in the holomorphic structure of its open unit ball B – more precisely: Two complex Banach spaces are isometrically isomorphic if, and only if, their open unit balls are biholomorphically equivalent. From this it seems natural to study those Banach spaces E for which B has a rich holomorphic structure. In the special case where B is homogeneous under the group of all biholomorphic automorphisms many authors have studied the situation. The corresponding Banach spaces then are called JB*-triples since they generalize the JB*-algebras (a Jordan version of C*-algebras) and are characterized by a certain ternary product $(x, y, z) \mapsto \{xyz\}$ on E. For instance, if E is a C*-algebra this ternary product is given by $\{xyz\} = (xy^*z + zy^*x)/2$, the so called Jordan triple product on E.

For a given JB*-triple E the group G of all those biholomorphic automorphisms of B that are continuous with respect to the weak topology on B was studied in [9]. In genereal, this group is no longer transitive on B. As shown in [9] the situation is completely described by the closed linear subspace $\mathsf{Cont}_w(E) \subset E$ defined as the set of all $a \in E$ for which the squaring map $z \mapsto \{zaz\}$ is weakly continuous: A biholomorphic automorphism g of B is weakly continuous if, and only if, $g(0)$ is in $\mathsf{Cont}_w(E)$. In particular, G is transitive on B if, and only if, $E = \mathsf{Cont}_w(E)$ holds (in which case E is called a weakly continuous JB*-triple).

In [9] the space $\mathsf{Cont}_w(E)$ has been computed for various JB*-triples E. But it was not possible there to handle the building blocks of JB*-triples, that is, the commutative C*-algebras of type $\mathscr{C}_0(S)$ with $S > 0$ a locally compact subset of $\mathbb{R}$. Our main result in section 3 is that $\mathsf{Cont}_w(\mathscr{C}_0(I)) = 0$ holds for $I \subset \mathbb{R}$ being the open unit interval and hence that every weakly continuous biholomorphic automorphism of the open unit ball of $\mathscr{C}_0(I)$ is linear. From this we show that a commutative JB*-triple is weakly continuous if, and only if, its spectrum is scattered. In section 4 we give a characterization of weakly continuous JB*-triples in terms of representations into Cartan factors.

---

[1]) Supported by Humboldt Foundation.

Notation. For complex Banach spaces E, F we denote by $\mathscr{L}(E, F)$ the Banach space of all bounded linear operators $E \to F$. In particular, $\mathscr{L}(E) := \mathscr{L}(E, E)$ is the Banach algebra of all bounded endomorphisms of E. The operator $\lambda \in \mathscr{L}(E)$ is called hermitian if $e^{it\lambda}$ is an isometry of E for every real number t. E^* is the dual and E^{**} is the bidual of E. Throughout we call the topologies $\sigma(E, E^*)$ and $\sigma(E^{**}, E^*)$ the weak and the weak* topology, or shorter, the w- and w^*-topology on E and E^{**}. We consider E always as a (norm closed) linear subspace of its bidual via the canonical embedding. For every family $\mathscr{F} = (E_i)_{i \in I}$ of complex Banach spaces we denote by $\oplus^\infty \mathscr{F}$ or $\oplus^\infty\{E_i : i \in I\}$ the Banach space $\{(z_i) \in \prod_i E_i : \sup_i \|z_i\| < \infty\}$ and call it the l^∞-sum of $\mathscr{F}$.

To simplify notation we adopt the following convention for the rest of the paper (but only in case of complex Banach spaces with name E): For every subset $A \subset E^{**}$ the w^*-closure will be denoted by the corresponding bold face letter, i.e., $\boldsymbol{A}$ in case of the subset A. In this sense $\boldsymbol{E} = E^{**}$ is the bidual itself. For every closed linear subspace $F \subset E$, the w^*-closure $\boldsymbol{F}$ can be identified with the bidual of F and satisfies $E \cap \boldsymbol{F} = F$.

2. Preliminaries

Let us start with some basic definitions and known facts on JB*-triples, for details see [11]. A JB*-triple is a complex Banach space E equipped with a continuous ternary operation $(x, y, z) \mapsto \{xyz\}$ which is symmetric bilinear in the outer variables x, z and conjugate linear in the inner variable y such that, by writing $x \square y$ for the linear operator $z \mapsto \{xyz\}$, it satisfies the axioms

(J_1) $$[x \square x, y \square y] = \{xxy\} \square y + y \square \{yxx\},$$

(J_2) $x \square x$ is hermitian and has spectrum ≥ 0,

(J_3) $$\|x \square x\| = \|x\|^2,$$

for all $x, y \in E$ and $[,]$ being the commutator product of linear operators. The triple product on E admits a separate w^*-continuous extension (denoted by the same symbol) to the bidual $\boldsymbol{E} = E^{**}$ with which it is also a JB*-triple. The algebraic properties of the triple product determine completely the geometry of a JB*-triple and vice versa. In particular, every injective triple morphism is automatically isometric and surjective isometries between JB*-triples are necessarily triple isomorphisms [11]. A complex subspace U of E with $\{UUU\} \subset U$ is called a subtriple. For every $a \in E$ denote by E_a the smallest closed subtriple of E containing a and by $S_a \subset \mathbb{R}$ the spectrum of the operator $(a \square a) | E_a$. Then E_a is isomorphic to $\mathscr{C}_0(S_a)$, the JB*-triple of all $\mathbb{C}$-valued continuous functions vanishing at inifinity on S_a. A linear subspace U of E is called an ideal if $\{EEU\} + \{EUE\} \subset U$ holds. Closed ideals of JB*-triples are precisely the kernels of triple morphisms. Recently the following simple characterization of ideals was established in [6] (compare also [9]): A closed subspace $U \subset E$ is an ideal of E if, and only if, $\{EEU\} \subset U$ holds, or equivalently, if U is invariant under all inner automorphisms of E. Later we need the following consequence thereof.

Lemma 2.1. *Let E be a JB*-triple, U a closed ideal in E and V a closed ideal in U. Then V is also an ideal in E.*

Proof. It suffices to show that V is invariant under the linear operator $a \square a$ for each $a \in E$. Fix $a \in E$ and $b \in V$ arbitrarily. Since V_a is isomorphic to $\mathscr{C}_0(S_a)$, there exists $v \in V$ such that $b = \{vvv\}$. Since $v \in U$ and since U is an ideal in E, we have $u := \{aav\} \in U$. Therefore with (J_1) we derive

$$(a \square a)\, b = (a \square a)\, \{vvv\} = 2\{uvv\} - \{vuv\} \in V . \quad \square$$

In the following we shall write $(a)_E$ or simply (a) for the closed ideal of E generated by the element $a \in E$.

Corollary 2.2. $(a)_U = (a)_E$ *holds for every closed ideal U of the* JB*-*triple E and for every $a \in U$.*

An element $e \in E$ is called a tripotent if $\{eee\} = e$ holds. Every tripotent splits E into the direct sum $E_1 \oplus E_{1/2} \oplus E_0$ where $E_k = E_k(e)$ is the k-eigen space of the operator $e \square e$ in E, called the Peirce-k-space of e. A tripotent $e \in E$ is said to be an atom or a minimal tripotent if $E_1(e) = \mathbb{C}e$. We denote the set of all atoms in E by $\mathrm{at}(E)$.

Lemma 2.3. $\mathrm{at}(U) = U \cap \mathrm{at}(\boldsymbol{E})$ *holds for every closed ideal $U \subset E$ and the bidual $\boldsymbol{E}$ of E.*

Proof. Suppose that e is an atom of U. The operator $\theta := 2(e \square e)^2 - (e \square e)$ is a w^*-continuous projection on $\boldsymbol{E}$ with $\theta(U) = \mathbb{C}e$. For every $z \in \boldsymbol{E}_1(e)$ the identity $\{eez\} = z$ implies $z \in U$ since U is an ideal in $\boldsymbol{E}$, i.e., $\theta(U) = \theta(\boldsymbol{E})$. This implies $\theta(\boldsymbol{E}) = \mathbb{C}e$ and hence $e \in \mathrm{at}\,(\boldsymbol{E})$. $\square$

An important class of JB*-triples are the Cartan factors (compare [10] p. 473). These come in 6 types: The first one consists of all spaces $\mathscr{L}(H, K)$ where H, K are complex Hilbert spaces with $\dim(H) \leq \dim(K)$ and the triple product is given by $\{xyz\} = (xy^*z + zy^*x)/2$. Type 2 and 3 are the spaces $E = \{z \in \mathscr{L}(H) : z' = \varepsilon z\}$ where $z \mapsto z'$ is a transposition on $\mathscr{L}(H)$ and $\varepsilon = -1$ and $\varepsilon = 1$ respectively. Type 4 consists of all spin factors and types 5 and 6 are the two exceptional JB*-triples of dimensions 16 and 27. By [10] p. 471, at $(E) \subset W$ holds for every (non necessarily closed) ideal $W \neq 0$ in a Cartan factor E. Therefore the closed linear span K of $\mathrm{at}(E)$ is the unique minimal closed ideal in E. For the types $1-3$, K is just the subtriple of all compact operators in E. Furthermore $K = E$ holds if, and only if, E has finite rank, or equivalently, if E is not of type $1-3$ with $\dim(H) = \infty$. In any case, E is the bidual of K and K is simple, that is, it does not contain a closed ideal different from 0 and K. Following [4] we call a JB*-triple elementary if it is isomorphic to the minimal closed ideal of a Cartan factor, or equivalently, if its bidual is a Cartan factor.

3. Bidual characterization of w-continuity

Let E be an arbitrary JB*-triple with closed unit ball $B \subset E$. According to our convention in section 1, $\boldsymbol{B}$ is the closed unit ball of $\boldsymbol{E}$. Denote by $\mathrm{Cont}_{w^*}(\boldsymbol{E})$ the space of all $a \in \boldsymbol{E}$ such that the corresponding squaring mapping

$$q_a : \boldsymbol{B} \to \boldsymbol{E}$$

defined by $q_a(x) = \{xax\}$ is w^*-continuous (compare [9] and also [16], where this spac has been denoted by $\mathrm{comp}_{w^*}(\boldsymbol{E})$). In the same way denote by $\mathrm{Cont}_w(E)$ the space of a $a \in E$ with $q_a : B \to E$ w-continuous. Since the restriction of the w^*-topology on $\boldsymbol{E}$ to E i the weak topology, we have $E \cap \mathrm{Cont}_{w^*}(\boldsymbol{E}) \subset \mathrm{Cont}_w(E)$. Actually, equality holds, i.e.:

Lemma 3.1. *We have* $E \cap \mathrm{Cont}_{w^*}(\boldsymbol{E}) = \mathrm{Cont}_w(E)$.

Proof. A neighbourhood basis of the origin in $\boldsymbol{E}$ for the w^*-topology can be given b the system of all polar sets $M^0 := \{x \in \boldsymbol{E} : |\langle x, M\rangle| \le 1\}$, where M runs through all finit subsets of E^*. So, fix an element $a \in \mathrm{Cont}_w(E)$ and a finite subset $M \subset E^*$. The weak continuity of $q_a : B \to E$ implies the existence of another finite subset $N \subset E^*$ with

$$q_a(B \cap N^0) \subset M^0 .$$

Consider an arbitrary element $x \in \boldsymbol{B} \cap N^0$ and choose a net $(x_\alpha)_{\alpha \in A}$ in B with $w^*\text{-}\lim x_\alpha = x$ Then

$$\lim_\alpha |\langle x_\alpha, \psi\rangle| = |\langle x, \psi\rangle| \le 1$$

for all $\psi \in N$ implies

$$\lim_\alpha r_\alpha = 1 \quad \text{where} \quad r_\alpha := \max\{1, |\langle x_\alpha, N\rangle|\} .$$

Therefore $(x_\alpha/r_\alpha)_{\alpha \in A}$ is a net in $B \cap N^0$ with w^*-limit x, i.e., the w^*-closure of $B \cap N^0$ in $\boldsymbol{E}$ contains the set $\boldsymbol{B} \cap N^0$. Since M^0 is w^*-closed in $\boldsymbol{E}$ and the triple product on $\boldsymbol{E}$ is separately w^*-continuous [2], we drive

$$q_a(\boldsymbol{B} \cap N^0) \subset M^0 .$$

This means that q_a is w^*-continuous at $0 \in \boldsymbol{B}$ and hence on all of $\boldsymbol{B}$, i.e., $a \in \mathrm{Cont}_{w^*}(\boldsymbol{E})$. □

As a consequence we get:

Proposition 3.2. *Let E be a* JB**-triple and $a \in E$ an arbitrary element. Then a is weakly continuous (i.e., $a \in \mathrm{Cont}_w(E)$) if, and only if, the closed ideal $(a) \subset E$ generated by a in E is weakly continuous as a* JB**-triple.*

Proof. Assume that $a \in \mathrm{Cont}_w(E)$. Since $\mathrm{Cont}_w(E)$ is a closed weakly continuous ideal in E (compare [9]), we have that also $(a) \subset \mathrm{Cont}_w(E)$ is weakly continuous.

Conversely, assume that $F := (a)$ is weakly continuous. By (3.1) it is enough to show that $a \in \mathrm{Cont}_{w^*}(\boldsymbol{E})$, where $\boldsymbol{E} = E^{**}$ according to our convention. The ideal $\boldsymbol{F} \subset \boldsymbol{E}$ is complemented by a w^*-closed ideal in $\boldsymbol{E}$ (compare [8]). Denote by $P : \boldsymbol{E} \to \boldsymbol{F}$ the canonical projection along this complement. Then P is a w^*-continuous JB*-homomorphism. Since $\boldsymbol{F}$ is isometrically isomorphic to F^{**} we know already from (3.1) that $a \in \mathrm{Cont}_{w^*}(\boldsymbol{F})$. Therefore, if $(z_\alpha)_{\alpha \in A}$ is an arbitrary bounded net in $\boldsymbol{E}$ with $w^* - \lim z_\alpha = 0$, also $w^* - \lim w_\alpha = 0$ holds, where $w_\alpha := q_a(z_\alpha) = Pq_a(z_\alpha) = q_a(P(z_\alpha))$ for all $\alpha \in A$. □

Corollary 3.3. *The weakly continuous closed ideals of E are precisely the closed ideals of* $\mathrm{Cont}_w(E)$.

Proof. Let $F \subset E$ be a weakly continuous closed ideal and $a \in F$ an arbitrary element. Then (2.2) and (3.2) imply $a \in \operatorname{Cont}_w(E)$, i.e., $F \subset \operatorname{Cont}_w(E)$. On the other hand every closed ideal $F \subset \operatorname{Cont}_w(E)$ is a weakly continuous ideal of E by (2.1). □

4. The commutative case

Throughout this section let E be an arbitrary commutative JB*-triple, i.e., $E \,\square\, E \subset \mathscr{L}(E)$ is a commutative set of linear operators. Then, by [11], E can be realized as

$$(4.1) \qquad E = \mathscr{C}_0^{\mathbb{T}}(S) := \{f \in \mathscr{C}_0(S) : f(ts) = tf(s)\ \forall t \in \mathbb{T}\}$$

where $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$ and $\pi : S \to \Omega$ is a principal $\mathbb{T}$-fibre bundle of the locally compact spaces S, Ω. The base space Ω can be identified with the space of maximal ideals in E and is called the spectrum of E. In case $\pi : S \to \Omega$ is a trivial bundle the JB*-triple E can be identified with the space $\mathscr{C}_0(\Omega)$. The main goal of this section is the precise description of the closed characteristic ideal $\operatorname{Cont}_w(E)$ in E (cf. [9]). We start with the special case $E = \mathscr{C}_0(I)$ where $I := \{t \in \mathbb{R} : 0 < t < 1\}$ is the open unit interval in $\mathbb{R}$.

Lemma 4.2. *For $E := \mathscr{C}_0(I)$ there is a net $(T_\alpha)_{\alpha \in A}$ of isometric C*-homomorphisms $T_\alpha : E \to E$ such that*

$$(4.3) \qquad \lim_{\alpha \in A} \int T_\alpha(f)\, \mathrm{d}\mu = v(I) \cdot \int f \, \mathrm{d}\lambda$$

holds for every $f \in E$ and every regular complex Borel measure μ of bounded variation on I, where λ denotes the Lebesgue measure on $\mathbb{R}$ and v is the absolutely continuous part of μ with respect to λ.

Proof. Denote by A the set of all pairs $\alpha = (K, r)$ where $K \subset I$ is a compact subset with $\lambda(K) = 0$ and where $r > 0$ satisfies

$$I_\alpha := \{t \in I : |t - c| > r,\ \forall c \in K\} \neq \emptyset .$$

Then A becomes a directed set if we define

$$\alpha \leq \beta :\Leftrightarrow (K \subset L \text{ and } \lambda(I_\alpha) \leq \lambda(I_\beta))$$

for all $\alpha = (K, r)$ and $\beta = (L, s)$. Observe that $\lim_\alpha \lambda(I_\alpha) = 1$ holds.

Every connected component of $I \setminus I_\alpha$ is an interval of length $> r$. From this we derive immediately that I_α is a finite union of open sub-intervals of I. For the definition of the operators T_α consider E in the obvious way as a subalgebra of $\mathscr{C}(I')$, where $I' : I \cup \{\infty\}$ is the one-point compactification of I. For every $\alpha \in A$ there is a unique continuous mapping $h_\alpha : I \to I'$ such that $h_\alpha(I \setminus I_\alpha) = \infty$ and such that for every connected component J of I_α the restriction of h_α to J is an increasing affine mapping from J onto I. Then $T_\alpha(f) := f \circ h_\alpha$ defines a C*-homomorphism $T_\alpha : E \to E$ which clearly is isometric. Consider any compact interval $S \subset I$ and denote by $\mathbf{1}_S$ the indicator function of S. Choose an index $\beta = (L, s)$ with L containing the endpoints of S. For every $\alpha \geq \beta$ we have

$$\int T_\alpha(f)\, \mathbf{1}_S \, \mathrm{d}\lambda = \lambda(S \cap I_\alpha) \cdot \int f \, \mathrm{d}\lambda$$

and hence

$$\lim_\alpha \int T_\alpha(f)\, \mathbf{1}_S \,\mathrm{d}\lambda = \lambda(S) \cdot f \,\mathrm{d}\lambda$$

for every $f \in E$. Taking linear combinations we get

$$\lim_\alpha \int T_\alpha(f)\, g \,\mathrm{d}\lambda = \int g \,\mathrm{d}\lambda \cdot \int f \,\mathrm{d}\lambda$$

for every step function g on I and hence also for every $g \in L^1(I)$. This proves (4.3) for the special case where $\mu = \nu = g\,\mathrm{d}\lambda$ is absolutely continuous with respect to λ.

For the rest of the proof we assume that $\mu \geq 0$ is singular with respect to λ (i.e., $\nu = 0$) and satisfies $\mu(I) = 1$. Fix $\varepsilon > 0$ and a function $f \in E$ with $\|f\| = 1$. Then there is an index $\beta = (L, s)$ with $\mu(L) > 1 - \varepsilon$. For every $\alpha \geq \beta$ we therefore have

$$\left|\int T_\alpha(f)\,\mathrm{d}\mu\right| = \left|\int_{I \setminus L} T_\alpha(f)\,\mathrm{d}\mu\right| \leq \mu(I \setminus L) < \varepsilon$$

since $T_\alpha(f)$ vanishes on L, i.e., $\lim_\alpha \int T_\alpha(f)\,\mathrm{d}\mu = 0$ in this case. □

Corollary 4.4. *For every function $f \in \mathscr{C}_0(I)$ with $\int f\,\mathrm{d}\lambda = 0$, the net $(T_\alpha(f))$ constructed in (4.2) converges weakly to 0 in $\mathscr{C}_0(I)$, i.e., $w - \lim_\alpha T_\alpha(f) = 0$ where $w - \lim$ stands for weak limit.*

Proof. By [7] the dual E^* of $E = \mathscr{C}_0(I)$ consists precisely of all linear forms $h \mapsto \int h\,\mathrm{d}\mu$ with μ a regular complex Borel measure of bounded variation on I.

Lemma 4.5. *We have* $\mathrm{Cont}_w(E) = 0$ *for* $E := \mathscr{C}_0(I)$.

Proof. Consider an arbitrary $h \in \mathrm{Cont}_w(E)$ and fix a function $f \in E$ with $\int f\,\mathrm{d}\lambda = 0$ and $\int f^2\,\mathrm{d}\lambda = 1$. Then $w - \lim_\alpha T_\alpha(f) = 0$ holds by (4.4). This implies by the definition of $\mathrm{Cont}_w(E)$ that also $w - \lim_\alpha h_\alpha = 0$ holds for $h_\alpha := \{T_\alpha(f)\, h T_\alpha(f)\} = T_\alpha(f^2)\,\bar{h}$. For every $g \in L^1(I)$ we derive

$$0 = w - \lim_\alpha \int h_\alpha g\,\mathrm{d}\lambda = w - \lim_\alpha \int T_\alpha(f^2)\,\bar{h} g\,\mathrm{d}\lambda$$
$$= \int \bar{h} g\,\mathrm{d}\lambda \cdot \int f^2\,\mathrm{d}\lambda = \int \bar{h} g\,\mathrm{d}\lambda$$

and hence $h = 0$. □

Let us now proceed to the general case: Recall that a topological space Λ is said to be perfect if it does not contain any isolated point, and that Λ is said to be scattered if no subspace of Λ is perfect. Clearly, the union Π of all perfect subspaces of Λ is a closed perfect subspace of Λ with $\Lambda \setminus \Pi$ scattered.

Lemma 4.6. *For every locally compact topological space Ω, the* JB*-*triple $E := \mathscr{C}_0(\Omega)$ is weakly continuous if, and only if, Ω is scattered.*

Proof. Suppose the Ω is not scattered. Then, by [14] and [15] 8.5.4, there is a function $f \in E$ with $f(\Omega) = \bar{I}$, where I is the open unit interval and $\bar{I}$ is its closure in $\mathbb{R}$. $H := \{\tilde{h} \circ f : h \in \mathscr{C}_0(I)\}$ is a closed subtriple of $\mathscr{C}_0(\Omega)$ isomorphic to $\mathscr{C}_0(I)$, where every $\tilde{h}$ denotes the continuous extension of h to $\bar{I}$. By (4.5), H and hence also E is not weakly continuous.

Conversely, suppose now that Ω is scattered. Obviously it is enough to show that weak covergence coincides with pointwise convergence on bounded subsets of E: Indeed, let (f_α) be a bounded net in E converging pointwise on Ω to $f \in E$. Consider any bounded linear functional $\varphi \in E^*$. By [15], the dual space E^* can be identified in a natural way with $l^1(\Omega)$. Therefore there is a function $c \in l^1(\Omega)$ such that

$$\langle \varphi, g \rangle = \sum_{\omega \in \Omega} c(\omega)\, g(\omega)$$

holds for all $g \in E$. Given any $\varepsilon > 0$, we find a finite subset $F \subset \Omega$ with

$$\sum_{\Omega \setminus F} |c(\omega)| \cdot \sup_{\alpha} \|f_\alpha - f\| < \varepsilon/2 \quad \text{for all} \quad \alpha$$

and then also an index β with

$$\left|\sum_{F} c(\omega)\,(f_\alpha(\omega) - f(\omega))\right| < \varepsilon/2 \quad \text{for all} \quad \alpha \geq \beta\,.$$

That is, $|\langle \varphi, f_\alpha - f\rangle| < \varepsilon$ whenever $\alpha \geq \beta$. But this means $\lim_{\alpha} \langle \varphi, f_\alpha \rangle = \langle \varphi, f \rangle$. □

Theorem 4.7. *Let E be a commutative* JB*-*triple and $\pi : S \to \Omega$ the corresponding principal $\mathbb{T}$-fibre bundle realization of E as in* (4.1). *Then*

$$\mathrm{Cont}_w(E) = \{f \in E : f \mid \pi^{-1}(\Pi) = 0\}$$

holds whith Π the maximal perfect subset of the spectrum Ω of E.

Proof. Fix $f \in E$ and denote by U the closed ideal generated by f in E. An elementary argument shows that $U = \{g \in E : g \mid Z = 0\}$ holds for $Z := f^{-1}(0)$. In particular U is isometrically isomorphic to $\mathscr{C}_0^{\mathbb{T}}(S \setminus Z)$. Let $(\Omega_i)_{i \in I}$ be an open covering of $\Omega \setminus \pi(Z)$ such that the bundle $\pi : S \to \Omega$ is trivial over every Ω_i. Put $S_i : \pi^{-1}(\Omega_i)$, $E_i := \mathscr{C}_0^{\mathbb{T}}(S_i)$ and $F_i := \mathscr{C}_0(S_i)$. Then E_i and F_i can be considered in a natural way as subtriples of E and $F := \mathscr{C}_0(S)$ respectively. Furthermore

$$Pg(s) = \int_{\mathbb{T}} t^{-1} g(ts)\, \mu(\mathrm{d}t)$$

defines a contractive projection P from F onto E with μ the normalized Haar measure on the group $\mathbb{T}$. Since $\sum_{i \in I} F_i$ is dense in F by Weierstrass approximation theorem, also $\left[P \sum_{i \in I} F_i\right] = \sum_{i \in I} E_i$ is dense in E. Every E_i is a closed ideal in E isomorphic to $\mathscr{C}_0(\Omega_i)$ since $S_i \to \Omega_i$ is a trivial bundle. Therefore by (4.6) E_i is weakly continuous if, and only if, Ω_i is scattered. Now the statement follows easily using (3.3) and the fact that Ω is scattered if, and only if, every Ω_i is scattered. □

Corollary 4.8. *For a commutative* JB*-*triple* E *we have* $\mathrm{Cont}_w(E) = 0$ *if, and only if, the spectrum of* E *is perfect.*

This together with [9] gives:

Corollary 4.9. *Let* E *be a commutative* JB*-*triple with perfect spectrum (f.i.* $E = \mathscr{C}_0(S)$ *for a connected locally compact topological space* S*). Then every weakly continuous biholomorphic automorphism of the open unit ball of* E *is linear.*

5. Factor structure of weakly continuous JB*-triples

In the following let E be an arbitrary JB*-triple. For every $a \in E$ the closed subtriple E_a generated by a is commutative. Furthermore, the spectrum $S_a \subset \mathbb{R}$ of E_a is scattered if, and only if, it is countable. We call E *locally weakly continuous* if for every $a \in E$ the subtriple E_a is weakly continuous. Therefore (4.7) immediately implies: E is locally weakly continuous if, and only if, every spectrum S_a, $a \in E$, is countable. This, by a result in [4], just means that the dual of E has Radon-Nikodým property (for short, E has dual RNP), proving:

Proposition 5.1. *A* JB*-*triple has dual* RNP *if, and only if, it is locally weakly continuous. In particular every weakly continuous* JB*-*triple has dual* RNP.

It is easily verified that a Cartan factor is locally weakly continuous if, and only if, it has finite rank. On the other hand, every spin factor of infinite dimension is locally weakly continuous but not weakly continuous (compare f.i. [9]).

Keeping our convention from section 1 we denote by $\mathscr{F}$ the set of all elementary ideals of $\boldsymbol{E}$ and by $\mathscr{W}$ the subset of all weakly continuous $F \in \mathscr{F}$, that is, of all elementary ideals in $\boldsymbol{E}$ which are not a spin factor of infinite dimension. For every $F \in \mathscr{F}$ the space $\boldsymbol{F}$ is a minimal w^*-closed ideal in $\boldsymbol{E}$ isomorphic to a Cartan factor. Furthermore,

$$\boldsymbol{F}^{\perp} := \{a \in \boldsymbol{E} : (\boldsymbol{F} \,\square\, \boldsymbol{F})\, a = 0\}$$

is a maximal w^*-closed ideal of $\boldsymbol{E}$ and satisfies

$$\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{F}^{\perp}.$$

Denote by π_F the canonical projection of $\boldsymbol{E}$ onto $\boldsymbol{F}$ along $\boldsymbol{F}^{\perp}$. For convenience we also put

$$\boldsymbol{\mathscr{F}} := \{\boldsymbol{F} : F \in \mathscr{F}\} \quad \text{and} \quad \boldsymbol{\mathscr{W}} := \{\boldsymbol{F} : F \in \mathscr{W}\}.$$

A result of Chu-Iochum [5] states.

Proposition 5.2. *The* JB*-*triple* E *has dual* RNP *if, and only if,* E *is the* l^∞*-sum of Cartan factors, that is, if* $\boldsymbol{E} = \oplus^\infty \boldsymbol{\mathscr{F}}$ *holds.*

Lemma 5.3. *Assume that* E *has dual* RNP*. Then for every* $F \in \mathscr{F}$ *we have either* $E \cap \boldsymbol{F} = \{0\}$ *or* $E \cap \boldsymbol{F} = F$*. In any case* $\pi_F(E)$ *is* w^**-dense in* $\boldsymbol{F}$.

Proof. π_F is w^*-continuous on $\boldsymbol{E}$. Therefore

$$\boldsymbol{F} = \overline{\pi_F(E)} \subset \boldsymbol{R} \subset \boldsymbol{F} \quad \text{implies} \quad \boldsymbol{R} = \boldsymbol{F} \quad \text{for} \quad R := \pi_F(E).$$

The intersection $U := E \cap \boldsymbol{F}$ is an ideal in E since $\boldsymbol{F}$ is an ideal in $\boldsymbol{E}$. Therefore $\boldsymbol{U}$ is a w^*-closed ideal in $\boldsymbol{F}$. Suppose $U \neq \{0\}$. Since $\boldsymbol{F}$ is a minimal w^*-closed ideal in $\boldsymbol{E}$, necessarily $\boldsymbol{U} = \boldsymbol{F}$ holds. On the other hand $\boldsymbol{U}$ is isometrically isomorphic to the bidual of U. By Lemma 3.2 of [4] a JB-triple is elementary if, and only if, its bidual is a Cartan factor. Hence U is an elementary JB*-triple. In particular the linear span of $\mathrm{at}(U)$ is norm-dense in U. It follows $U \subset F$. Indeed, by Lemma (2.3),

$$\mathrm{at}(U) = \mathrm{at}(\boldsymbol{E}) \cap U = \mathrm{at}(\boldsymbol{F}) \cap U \subset F \cap U .$$

The w^*-topology coincides with $\sigma(\boldsymbol{F}, F^*)$ on F. This implies $U = F \cap U = F \cap \boldsymbol{F} = F$. □

Every triple morphism $\lambda: E \to F$ between JB*-triples is also called a representation on E. We call λ an elementary representation if the image $\lambda(E)$ is an elementary JB*-triple and a spin representation if $\lambda(E)$ is a spin factor.

Lemma 5.4. *Assume that E has dual* RNP. *Then the following conditions are equivalent:*

1. *Every w^*-dense representation from E to a Cartan factor is elementary;*
2. *$\pi_{\boldsymbol{F}}(E)$ is elementary for all $\boldsymbol{F} \in \mathscr{F}$;*
3. *$\pi_{\boldsymbol{F}}(E) = F$ for all $\boldsymbol{F} \in \mathscr{F}$;*
4. *$E \oplus^\infty \mathscr{F}$.*

Proof. $1 \Rightarrow 2 \Rightarrow 3$ follows from (5.3) and the fact that $F \subset \boldsymbol{F}$ is the unique w^*-dense elementary subtriple. $3 \Rightarrow 4$ is trivial, therefore let us assume that 4 holds. Fix a Cartan factor C and a w^*-dense representation $\lambda: E \to C$. By Remark 1.1 in [3] the representation λ has an extension to a w^*-continuous representation $\boldsymbol{\lambda}: \boldsymbol{E} \to C$. Then $\boldsymbol{F} := (\boldsymbol{\lambda}^{-1}(0))^\perp$ is a w^*-closed ideal in $\boldsymbol{E}$ isomorphic to C, i.e., $\boldsymbol{F} \in \mathscr{F}$ and for a suitable isomorphism $\tau: C \to \boldsymbol{F}$ we have $\pi_{\boldsymbol{F}} = \tau \circ \boldsymbol{\lambda}$. Our assumption implies $\pi_{\boldsymbol{F}}(E) \subset F$ and hence $\pi_{\boldsymbol{F}}(E) = F$ by (5.3), i.e., $\lambda(E) = \tau^{-1}(F)$ is elementary. □

Example 5.5. Let H be an infinite dimensional Hilbert space and K the elementary JB*-triple of all compact operators on H. Then $A := K + \mathbb{C}e$ is a closed JB*-subtriple of $\mathscr{L}(H)$, where e is the identity operator on H. Denote by $\lambda: A \to A/K = \mathbb{C}$ the canonical projection, i.e., $\lambda(x + te) = t$ for all $x \in K$ and $t \in \mathbb{C}$. The JB*-triple A is isomorphic to the graph

$$E := \{(a, \lambda a): a \in A\} \subset A \oplus \mathbb{C} \quad \text{of} \quad \lambda .$$

Therefore $\boldsymbol{E} = \mathscr{L}(H) \oplus \mathbb{C}$ holds, i.e., $\mathscr{F} = \{\boldsymbol{F}, \boldsymbol{K}\}$ with $F = \boldsymbol{F} = \mathbb{C}$ in this case. This implies $E \cap \boldsymbol{K} = K$ and $E \cap \boldsymbol{F} = 0$. Notice that $A = K + \mathbb{C}e$ has dual RNP but is not weakly continuous (compare the proof of Lemma 2.11 in [9]) although K and A/K are weakly

$\phi = \psi + \sum_{\mathscr{W}} \phi_F$ with $\langle W, \psi \rangle = 0$ and $\phi_F := \phi \circ \pi_F$ for all $F \in \mathscr{W}$. Thus for any index $i \in I$ we have

$$\langle \{z_i w z_i\}, \phi \rangle = \sum_{F \in \mathscr{W}} \langle \{\pi_F(z_i)\, \pi_F(w)\, \pi_F(z_i)\}, \phi_F \rangle .$$

By assumption, for fixed $F \in \mathscr{W}$,

$$\lim_i \langle \{\pi_F(z_i)\, \pi_F(w)\, \pi_F(z_i)\}, \phi_F \rangle = 0 .$$

Since

$$\sum_{\mathscr{W}} \|\phi_F\| \leq \|\phi\| < \infty$$

and

$$|\langle \{\pi_F(z_i)\, \pi_F(w)\, \pi_F(z_i)\}, \phi_F \rangle| \leq \|\phi_F\| \cdot \|w\| \quad \text{for any} \quad i \in I ,$$

a similar argument as that in the proof of (4.6) gives

$$\limsup_i |\langle \{z_i w z_i\}, \phi \rangle| \leq \limsup_i \sum_{F \in \mathscr{W}} |\langle \{\pi_F(z_i)\, \pi_F(w)\, \pi_F(z_i)\}, \phi_F \rangle| = 0 .$$

Thus $w^* - \lim_i \{z_i w z_i\} = 0$. □

Theorem 5.7. *A* JB*-*triple E is weakly continuous if, and only if, the following three conditions are satisfied:*

1. *E has dual* RNP;
2. *Every w*-dense representation from E to a Cartan factor is elementary;*
3. *Every spin representation of E is finite dimensional.*

Proof. Notice that by (3.1) and (5.6) we have $\mathrm{Cont}_w(E) = E \cap \bigoplus^\infty \mathscr{W}$ for every JB*-triple E with dual RNP. With this the statement is any easy consequence of (5.4). □

References

[1] Barton, T. J., Dang, T., Horn, G.: Normal Representations of Banach Jordan Triple Systems. Proc. Amer. Math. Soc. **102** (1988), 551–555
[2] Barton, T. J., Timoney, R. W.: Weak* continuity of Jordan Triple Products and Applications. Math. Scand. **59** (1986), 177–191
[3] Bunce, L. J., Chu, C.-H.: Compact Operations, Multipliers and Radon-Nikodym Property in JB*-triples. Pacific J. of Math. **153** (1992), 249–265
[4] Bunce, L. J., Chu, C.-H.: Dual Spaces of JB*-triples and the Radon-Nikodym Property. To appear in Math. Z.

[9] ISIDRO, J. M., KAUP, W.: Weak Continuity of Holomorphic Automorphisms in JB*-triples. Math. Z. **210** (1992), 277–288

[10] KAUP, W.: Über die Klassifikation der symmetrischen Hermiteschen Mannigfaltigkeiten unendlicher Dimension I, II. Math. Ann. **257** (1981), 463–483; **262** (1983), 503–529

[11] KAUP, W.: A Riemann Mapping Theorem for Bounded Symmetric Domains in Complex Banach spaces. Math. Z. **183** (1983), 503–529

[12] KAUP, W., UPMEIER, H.: Banach Spaces with Biholomorphically Equivalent Unit Balls are Isomorphic. Proc. Amer. Math. Soc. **58** (1976), 129–133

[13] LOOS, O.: Jordan Pairs. Lecture Notes in Mathematics Vol. 460. Berlin–Heidelberg–New York: Springer 1975

[14] PELCZYNSKI, A., SEMADENI, Z.: Spaces of Continuous Functions III. Studia Math. **18** (1959), 211–222

[15] SEMADENI, Z.: Banach Spaces of Continuous Functions. Vol. I. P.W.N. Warsaw 1971

[16] STACHÓ, L. L., ISIDRO, J. M.: Algebraically Compact Elements in JBW*-triple Systems. Acta Sci. Math. (Szeged) **54** (1990), 171–190

[17] STEGALL, C.: The Radon-Nikodým Property in Conjugate Banach Spaces. Trans. AMS **206** (1975), 213–223

Mathematisches Institut der Universität
Auf der Morgenstelle 10
D-72076 Tübingen
Germany

Bolyai Institute
Aradi Vértanuk tere 1
H-6720 Szeged
Hungary

Werner Heisenberg in Leipzig 1927–1942

Herausgegeben von Christian Kleint und Gerald Wiemers

Abhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, mathematisch-naturwissenschaftliche Klasse, Band 58, Heft 2

1993. 261 Seiten – 139 Abb. – 210 mm x 295 mm
Broschur mit Schutzumschlag DM 98,– / öS 764,– / sFr 94,–
ISBN 3-05-501585-1

Mit diesem Sammelband wird erstmalig eine Dokumentation zu den intensiven und fruchtbaren Arbeitsjahren des Physik-Nobelpreisträgers Werner Heisenberg in Leipzig vorgelegt. Besonderes Augenmerk wurde darauf gerichtet, nur bisher noch nicht publizierte Materialien – und damit Ergänzungen zur bekannten Heisenberg-Gesamtausgabe – aufzunehmen. Das trifft insbesondere für die Berichte über die Leipziger Uranmaschinenversuche zu. So wird Robert Döpels Unfallbericht von 1942 vorgelegt; die Darlegungen werden ergänzt durch Konstruktionsskizzen, Meßkurven und andere Details aus Döpels Nachlaß. Ebenfalls aufgenommen wurde ein Geheimbericht Erich Bagges.
Im zweiten Teil der Dokumentation berichten bedeutende ehemalige Leipziger Mitarbeiter und Studenten über ihr Zusammentreffen mit Heisenberg. Es sind persönliche, unverwechselbare Erinnerungen, die für weitere Forschungen als Quelle dienen können. Auch das internationale Flair in Heisenbergs Umgebung spiegelt sich in dem Sammelband wider: Neben Sir Rudolf Peierls (England) erinnern sich u. a. der Chinese Wang Foh-san, der Pole Edwin Gora, der Österreicher Victor Weisskopf und der Ungar Edward Teller an ihre Leipziger Jahre. In einem ausführlichen Interview beleuchtet C.F.v.Weizsäcker die damalige Physik in Leipzig. Statistische Angaben zu Heisenbergs Schülerkreis und zu seinen Lehrveranstaltungen sowie zwei Aufsätze zu Heisenbergs Leipziger Wirkungsstätten (Universität und Sächsische Akademie) beschließen diesen Teil.
Im Anhang des Bandes wird Werner Heisenbergs Leipziger Akademievortrag von 1967 „Philosophische Probleme in der Theorie der Elementarteilchen" erstmalig in der Redefassung wiedergegeben, die vor kurzem auf der Grundlage neuen Quellenmaterials rekonstruiert werden konnte.