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A Simple Affine-Invariant Spline Interpolation over Triangular Meshes

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Abstract: Given a triangular mesh, we obtain an orthogonality-free analogue of the classical local Zlámal–Ženišek spline procedure with simple explicit affine-invariant formulas in terms of the normalized barycentric coordinates of the mesh triangles. Our input involves first-order data at mesh points, and instead of adjusting normal derivatives at the side middle points, we constructed the elementary splines by adjusting the Fréchet derivatives at three given directions along the edges with the result of bivariate polynomials of degree five. By replacing the real line $\mathbb{R}$ with a generic field $\mathbb{K}$, our results admit a natural interpretation with possible independent interest, and the proofs are short enough for graduate courses.

Keywords: polynomial $\mathcal{C}^1$-spline; triangular mesh; first-order data; affine invariance over fields

MSC: 65D07; 41A15; 65D15

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1. Introduction

With the rapid increase in computing capacity, spline interpolation over triangular meshes became a popular issue in numerical mathematics: given the data of the coordinates of points from some 2D surface, triangularization techniques and then $\mathcal{C}^1$-spline constructions are widely used for approximating the underlying surface with high accuracy. The related literature with large computational demands and a spectacular outcome is enormous. Beautiful examples relatively close to our context are Hahman (2000) [1] and Cao (2019) [2] and the references therein.

Our aim in this short note is somewhat in the opposite direction. We investigate "minimalist" approaches: given a triangular mesh on the plane, find a method producing a $\mathcal{C}^1$-spline with *polynomials of low degree* on the mesh triangles, which is "local" in the sense that the coefficients for any mesh triangle can be calculated with an explicit formula depending only on the location and the given data (as function values, differential requirements, etc.) associated with the vertices of two adjacent triangles. Our presented results originate from computer algebraic studies of the classical method by Zlámal et al. (1971) [3] based on the fact that the requirement of adjusting fifth-degree polynomials for function, gradient, and Hessian values along with normal derivatives at edge middle points of a single mesh triangle gives rise to a $\mathcal{C}^1$-spline. Originally, they only proved that the linear system of 21 equations for calculating the 21 coefficients for the adjustment admits a unique solution. Recently, Sergienko et al. (2014) [4] published the rather sophisticated related explicit formulas, which motivated us to develop an axiomatic approach to locally generated polynomial spline methods Stachó (2019) [5] The recent work is a non-straightforward application of the results there, although it is self-contained formally. We only used the principal shape functions Φ and Θ below provided by Theorem 2.3 in Stachó (2019) [5] in the simplest form without the need for any hint of their provenience.

We describe a family of local $\mathcal{C}^1$-spline procedures with really simple explicit affine-invariant five-degree polynomials in terms of barycentric coordinates by adjusting first-order data at the vertices. Though the result seems to be a variant of the procedure by

Zlámal–Ženišek (ZZ), it cannot be deduced as a special case as it is free of the concept of orthogonality. The proof, which may have independent interest, is basically different from that of ZZ.

2. Main Results

Throughout this work, let:

$$\Phi(t) := t^3(10 - 15t + 6t^2), \qquad \Theta(t) := t^3(4 - 3t).$$

Fix also any non-degenerate triangle T with vertices p_1, p_2, p_3 on the plane $\mathbb{R}^2$ along with three affine functions $x \mapsto f_i + A_i(x - p_i)$ $\big($that is, $f_i \in \mathbb{R},\ A_i \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})\big)$, and define:

$$F_0(x) := \sum_{i=1}^{3} \Big[\Phi\big(\lambda_i(x)\big) f_i + \Theta\big(\lambda_i(x)\big) A_i(x - p_i)\Big] \tag{1}$$

where $\lambda_1, \lambda_2, \lambda_3 : \mathbb{R}^2 \to \mathbb{R}$ are the *barycentric weights* determined unambiguously by the relations:

$$\sum_{i=1}^{3} \lambda_i(x) = 1, \quad x = \sum_{i=1}^{3} \lambda_i(x) p_i \quad (x \in \mathbb{R}^2).$$

Theorem 1. *Let $u_1, u_2, u_3 \in \mathbb{R}^2$ be arbitrary vectors such that $u_k \not\parallel (p_j - p_k)$ whenever $\{i, j, k\} = \{1, 2, 3\}$. Then, there exist constants $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}$ that can be formulated explicitly in terms of $\lambda_1, \lambda_2, \lambda_3$ (see (18) later) such that the function:*

$$F(x) := F_0(x) + \sum_{\ell=1}^{3} \zeta_\ell \lambda_\ell(x)^{-1} \prod_{m=1}^{3} \lambda_m(x)^2 \tag{2}$$

along with its Fréchet derivatives $F'(x)v := \frac{d}{d\tau}\big|_{\tau=0} F\big(x + \tau v\big)$ behave on the edges of T for any triple (i, j, k) of different indices as follows:

$$F(p_i) = f_i, \quad F'_v(p_i) = A_i v \quad (v \in \mathbb{R}^2), \tag{3}$$

$$F\big(tp_i + (1-t)p_j\big) = \Phi(t) f_i + [1 - \Phi(t)] f_j + [(1-t)\Theta(t) A_i - t\Theta(1-t) A_j](p_j - p_i), \tag{4}$$

$$F'\big(tp_i + (1-t)p_j\big) u_k = \big[\Theta(t) A_i + \Theta(1-t) A_j\big] u_k. \tag{5}$$

As a consequence, given a triangular mesh, we can obtain modifications of the celebrated Zlámal–Ženišek (ZZ) spline procedure Zlámal et al. (1971) [3], Sergienko et al. (2014) [4] regardless of second-order data, but with simple explicit scalar-product-free formulas in terms of affine functions. Notice that, due to affine invariance, our results cannot be deduced from ZZ, e.g., by setting the input second derivatives at the vertices to zero.

Recall that by a *triangular mesh*, we mean a family $\mathcal{T} = \{T_1, \ldots T_N\}$ of closed triangles in $\mathbb{R}^2$ such that the intersection $T_m \cap T_n$ is either a common edge or a common vertex or empty for different indices m, n. Given any triangle $T \subset \mathbb{R}^2$, $\mathrm{Vert}(T)$, resp. $\mathrm{Edge}(T)$, will denote the set of its vertices, resp. edges, and we write $\mathrm{Vert}(\mathcal{T}) := \bigcup_{n=1}^{N} \mathrm{Vert}(T_n)$, resp. $\mathrm{Edge}(\mathcal{T}) := \bigcup_{n=1}^{N} \mathrm{Edge}(T_n)$. By a *dataset of first order* for the mesh $\mathcal{T}$, we mean a family:

$$\mathcal{F} = \big\{(p, f_p, A_p) : p \in \mathrm{Vert}(\mathcal{T})\big\} \quad \text{with } f_p \in \mathbb{R},\ A_p \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}). \tag{6}$$

We call a $\mathcal{C}^1$-smooth function $f : \cup\mathcal{T} := \bigcup_{n=1}^{N} T_n \to \mathbb{R}$ a *polynomial $\mathcal{C}^1$-spline* for the data $\mathcal{F}$ over $\mathcal{T}$ if the restrictions $f|T_n$ are polynomials $\mathbb{R}^2 \to \mathbb{R}$ with Taylor expansion $f_p + A_p(x - p)$ around the points $p \in \mathrm{Vert}(T_n)$ (i.e., f is continuously differentiable on $\mathrm{Interior}\big(\cup\mathcal{T}\big)$; furthermore, there are polynomials $P_1, \ldots, P_N$ in two-variables with $f([\xi, \eta]) = P_n(\xi, \eta)$ whenever $[\xi, \eta] \in T_n$ $(n = 1, \ldots, N)$ satisfying $f(p) = f_p$, $\partial P_n / \partial \xi\big|_{[\xi,\eta]=p} = A_p[1, 0]$, $\partial P_n / \partial \eta\big|_{[\xi,\eta]=p} = A_p[0, 1]$ at the points $p \in \mathrm{Vert}(T_n)$).

For our later considerations, $\mathcal{T} = \{T_1, \ldots, T_N\}$ will stand as a fixed triangular mesh. Given any mesh triangle T_n, we write $\lambda_{n,p}$ $\big(p \in \mathrm{Vert}(T_n)\big)$ for its barycentric weights (i.e., $x = \sum_{p\in\mathrm{Vert}(T_n)}(x)p$ for any $p \in \mathbb{R}^2$), and $E_{n,p}$ denotes the edge opposite the vertex p in T_n.

Theorem 2. *Let (6) be a first-order dataset for $\mathcal{T}$, and let $\{u_E : E \in \mathrm{Edge}(\mathcal{T})\} \subset \mathbb{R}^2$ be a family of vectors with $u_E \nparallel E$. Then, we can find constants:*

$$\big\{\zeta_{p,E} :\ E \in \mathrm{Edge}(T),\ p \in \mathrm{Vert}(T)\setminus E \text{ for some } T \in \mathcal{T}\big\} \subset \mathbb{R}$$

such that the union $F : \cup\mathcal{T} \to \mathbb{R}$ of the polynomial functions $F_n : T_n \to \mathbb{R}$ obtained by replacing the terms λ_ℓ $(\ell = 1,2,3)$ in (1), (2) with $\lambda_{n,p}$ $(p \in \mathrm{Vert}(T_n))$ as:

$$F_n(x) := \sum_{p\in\mathrm{Vert}(T_n)} \Big[\Phi\big(\lambda_{n,p}(x)\big) f_p + \Theta\big(\lambda_{n,p}(x)\big) A_p(x-p) + \zeta_{p,E_{n,p}} \lambda_{n,p}(x)^{-1} \prod_{q\in\mathrm{Vert}(T_n)} \lambda_{n,p}(x)^2\Big]$$

is a polynomial $\mathcal{C}^1$-spline for the data $\mathcal{F}$ over $\mathcal{T}$ such that:

$$F'\big(tp + (1-t)q\big)u_E = \big(\Theta(t)A_p + \Theta(1-t)A_q\big)u_E \quad \text{whenever } E = [p,q] \in \mathrm{Vert}(\mathcal{T}),\ 0 < t < 1.$$

Remark 1. *In the course of the proof, with a straightforward adaptation of Theorem 1, we obtain an explicit expression for $\zeta_{p,E}$ in terms of the barycentric weights of the triangle $T := \big[\text{Convex hull of } \{p\} \cup E\big] \in \mathcal{T}$.*

3. Preliminaries and Straightforward Observations

Our basic polynomials Φ, Θ have the Hermitian interpolation properties:

$$\Phi(0) = \Phi'(0) = \Phi'(1) = 0, \quad \Phi(1) = 1, \quad \Phi'(t) = 30\,t^2(1-t)^2; \tag{7}$$
$$\Theta(0) = \Theta'(0) = \Theta(1) = 0, \quad \Theta(1) = 1, \quad \Theta'(t) = 12\,t^2(1-t). \tag{8}$$

Given any indices i, j, k with $\{i,j,k\} = \{1,2,3\}$,

$$\lambda_i(p_i) = 1, \quad \lambda_i(x) = 0 \ \text{ for } \ x \in [p_j, p_k] = \big\{(1-t)p_j + tp_k :\ t \in [0,1]\big\}; \tag{9}$$
$$\lambda_i'(z)v \equiv \lambda_i(z+p_j) = \lambda_i(z+p_k) = \lambda_i\big(z + (1-t)p_j + tp_k\big) \quad \text{independently of } z, t. \tag{10}$$

Remark 2. *It is customary to express the weight λ_i in terms of the natural inner product $\big\langle([\xi_1,\xi_2]\big|[\eta_1,\eta_2]\big\rangle = \sum_{\ell=1}^2 \xi_\ell\eta_\ell$ of $\mathbb{R}^2$ as $\lambda_i(x) = \big\langle x - p_j\big|\langle m_i|m_i\rangle^{-1}m_i\big\rangle$ where $m_i = p_i - r_i$ is the height vector of the triangle T with the closest point r_i to p_i on the line connecting p_j with p_k. The formulas obtained by means of this inner product (as the explicit form of the ZZ basic functions published recently Sergienko et al. (2014) [4]) are only invariant with respect to the isometries of $\mathbb{R}^2$, while our approach is free of metric considerations and can be generalized to purely algebraic settings by replacing $\mathbb{R}$ with an arbitrary field $\mathbb{K}$. In the sequel, we write:*

$$G_i := \big[v \mapsto \lambda_\ell'(p_i)v\big] \tag{11}$$

for the (constant) Fréchet derivative of λ_i, regarded as a linear functional $\mathbb{R} \to \mathbb{R}$, but avoiding identifying it with the gradient vector $\langle m_i|m_i\rangle^{-1}m_i$.

Notice that, as it is formulated in terms of polynomials $\mathbb{R}^2 \to \mathbb{R}$, the functions $\lambda_1, \lambda_2, \lambda_3$ and F in Theorem 1 extend to to $\mathbb{R}^2$ by means of the same algebraic expressions; furthermore, the identities (4) and (5) hold on the whole line $\big\{tp_i + (1-t)p_j : t \in \mathbb{R}\big\}$. By (9), we have $\lambda_m(p_n) = \delta_{mn}$ in terms of the Kronecker symbol $\delta_{mn} = \big[1 \text{ if } m = n,\ 0 \text{ else}\big]$; furthermore, the monomials:

$$C_{\nu_1,\nu_2,\nu_3}(x) = \lambda_1(x)^{\nu_1}\lambda_2(x)^{\nu_2}\lambda_3(x)^{\nu_3}$$

satisfy:

$$C_{\nu_1,\nu_2,\nu_3}(x) = 0 \quad \text{for } x \in \partial T = \bigcup (\text{edges of } T) \quad \text{if } \min\{\nu_1,\nu_2,\nu_3\} \geq 1,$$
$$\big[C_{\nu_1,\nu_2,\nu_3}\big]'(x)v = \sum_{\ell=1}^{3} \nu_\ell \lambda_\ell(x)^{-1} C_{\nu_1,\nu_2,\nu_3}(x) G_\ell v.$$

In particular, independently of the choice of the coefficients ζ_ℓ,

$$F(x) = F_0(x) \quad \text{for } x \in \partial T, \tag{12}$$
$$\Big[\sum_{\ell=1}^{3} \zeta_\ell \lambda_\ell^{-1} \prod_{m=1} \lambda_m^2\Big]'(x)v = \zeta_k \lambda_i(x)^2 \lambda_j(x)^2 G_k v \quad \text{for } x \in [p_i,p_j] \text{ if } \{i,j,k\} = \{1,2,3\}. \tag{13}$$

Proof of Theorem 1. Fix the indices i,j,k arbitrarily such that $\{i,j,k\} = \{1,2,3\}$. Consider a generic point:

$$x_t := tp_i + (1-t)p_j \tag{14}$$

on the edge $[p_i, p_j]$ of the triangle T. Since the weights λ_ℓ are affine functions,

$$\lambda_i(x_t) = t, \ \lambda_j(x_t) = 1-t, \ \lambda_k(x_t) = 0 \quad (0 \leq t \leq 1). \tag{15}$$

Since $\Phi(1-t) = 1 - \Phi(t)$, in view of (15), we obtain:

$$F_0(x_t) = \sum_{\ell=i,j,k} \Big[\Phi\big(\lambda_\ell(x_t)\big) f_\ell + \Theta\big(\lambda_\ell(x_t)\big) A_\ell(x_t - p_\ell)\Big] =$$
$$= \Big[\Phi(t) f_i + (1-t)\Theta(t) A_i(p_j - p_i)\Big] + \Big[\big(1-\Phi(t)\big) f_j + t\Theta(1-t) A_j(p_i - p_j)\Big] +$$
$$+ \Big[\Phi(0) f_k + \Theta(0) A_k\big(tp_i + (1-t)p_j - p_k\big)\Big].$$

That is, by (7) and (8),

$$\begin{aligned} F(x_t) =& t^3(10 - 15t + 6t^2) f_i + (1-t)^3(1 + 3t + 6t^2) f_j + \\ &+ t^3(1-t)(4-3t) A_i(p_j - p_i) + (1-t)^3 t(1+3t) A_j(p_i - p_j). \end{aligned} \tag{16}$$

As for the Fréchet derivatives along the edge $[p_i, p_j]$, in view of (12) and (13), we obtain:

$$F'(x_t)v = F_0'\big(tp_i + (1-t)p_j\big)v + \zeta_k t^2 (1-t)^2 G_k v.$$

Notice that in general, we have:

$$F_0'(x)v = \sum_{\ell=i,j,k} \Big[\Phi'\big(\lambda_\ell(x)\big)[G_\ell v] f_\ell + \Theta'\big(\lambda_\ell(x)\big)[G_\ell v] A_\ell(x - p_\ell) + \Theta\big(\lambda_\ell(x)\big) A_\ell v\Big].$$

In particular, since at the generic point (14) on $[p_i, p_j]$, we have $x_t - p_i = (1-t)(p_j - p_i)$, resp. $x_t - p_j = t(p_i - p_j)$,

$$F_0'(x_t)v = F_0'\big(tp_i + (1-t)p_j\big)v =$$
$$= \Big[\Phi'(t)[G_i v] f_i + \Theta'(t)[G_i v](1-t) A_i(p_j - p_i) + \Theta(t) A_i v\Big] +$$
$$+ \Big[\Phi'(1-t)[G_j v] f_j + \Theta'(1-t)[G_j v] t\, A_j(p_i - p_j) + \Theta(1-t) A_j v\Big] +$$
$$+ \Big[\Phi'(0)[G_k v] f_k + \Theta'(0)[G_k v] A_k(x_t - p_k) + \Theta(0) A_k v\Big].$$

Thus, in view of (7) and (8), it follows that:

$$F_0'(x_t)v = \Big[30\,t^2(1-t)^2[G_i v]f_i + 12\,t^2(1-t)^2[G_i v]A_i(p_j-p_i) + t^3(4-3t)A_i v\Big] + \\ + \Big[30\,t^2(1-t)^2[G_j v]f_j + 12\,t^2(1-t)^2[G_j v]A_j(p_i-p_j) + (1-t)^3(1+3t)A_j v\Big].$$

Hence, in view of (13), we conclude that:

$$\begin{aligned} &F'(x_t)v = t^2(1-t)^2\big[\zeta_k[G_k v] + M_{i,j}v\big] + t^3(4-3t)A_i v + (1-t)^3(1+3t)A_j v \\ &\text{where}\quad M_{i,j}v := 30\big([G_i v]f_i + [G_j v]f_j\big) + 12\big([G_i v]A_i - [G_j v]A_j\big)(p_j - p_i). \end{aligned} \tag{17}$$

At this point (3), (4) and (5) are immediate. Namely (3) follows from (16) and (17) by setting $t := 1$. Equation (4) is an equivalent form of (17). To verify (5), consider (17) with $v := u_k$. Observe that $G_k u_k \neq 0$ since $u_k \not\parallel (p_j - p_i)$. Thus, the coefficient:

$$\zeta_k := -\frac{M_{i,j}u_k}{G_k u_k} = -\frac{1}{G_k u_k}\Big[30\big([G_i u_k]f_i + [G_j u_k]f_j\big) + 12\big([G_i u_k]A_i - [G_j u_k]A_j\big)(p_j - p_i)\Big] \tag{18}$$

is well defined. Applying it, for the generic point (14) on the edge $[p_i, p_j]$, we obtain:

$$F'(x_t)u_k = t^3(4-3t)A_i u_k + (1-t)^3(1+3t)A_j u_k$$

independently of the location of the third vertex p_k of the triangle T. □

Corollary 1. *By writing $v = \alpha u_k + \beta(p_i - p_j)$, we have:*

$$\begin{aligned} F'(x_t)v &= \alpha F'(x_t)u_k + \beta\big[F'(x_t)\big](p_i - p_j) = \\ &= \alpha\Big[\Phi(t)A_i + [1-\Phi(t)]A_j\Big]u_k + \beta\frac{d}{dt}F(x_t). \end{aligned} \tag{19}$$

Proof of Theorem 2. It suffices to verify the following two statements:

(i) Given $p \in \mathrm{Vert}(\mathcal{T})$ and $v \in \mathbb{R}^2$, we have $F(p) = f_p$ and $F'(p)v = A_p v$;
(ii) Given two adjacent mesh triangles $T_m, T_n \in \mathcal{T}$, with common edge $[p, q]$, for the points $x_t = tp + (1-t)q$ on the line connecting p, q, we have $F(x_t) = F_m(x_t) = F_n(x_t)$ and $F_m'(x_t)v = F_n'(x_t)v$ for any $v \in \mathbb{R}^2$.

As for (i): Choose any mesh triangle $T_n \in \mathcal{T}$ with $p \in \mathrm{Vert}(T_n)$. By writing p_1, p_2, p_3 with $p_1 = p$ for the vertices of T_n, an application of (3) in Theorem 1 with $F := F_n$ shows that $F_n(p) = F_n(p_1) = f_{p_1} = f_p$ and $F_n'(p)v = F_n'(p_1)v = A_1 v = A_p v$ independent of which mesh triangle T_n with vertex p is considered.

As for (ii): Let $T_m, T_n \in \mathcal{T}$ be two adjacent triangles with common edge $[p, q]$. Necessarily, $\mathrm{Vert}(T_m) = \{p, q, r\}$ and $\mathrm{Vert}(T_n) = \{p, q, \bar{r}\}$ with suitable mesh points $r, \bar{r} \in \mathrm{Vert}(\mathcal{T})$. An application of (4) in Theorem 1 with $p_1 := p$, $p_2 := q$, $p_3 := r$, and $F := F_m$ shows that:

$$\begin{aligned} F_m(x_t) &= \Phi(t)f_1 + [1-\Phi(t)]f_2 + \big[\Theta(t)A_1 + \Theta(1-t)A_2\big](p_j - p_i) = \\ &= \Phi(t)f_p + [1-\Phi(t)]f_q + \big[\Theta(t)A_p + \Theta(1-t)A_q\big](q - p). \end{aligned}$$

The same conclusion holds when replacing (r, F_m) with $(\bar{r}, F_n)$. Thus, we have $F_m(x_t) = F_n(x_t)$ along the edge $[p, q]$ (moreover, along the whole straight line connecting p and q) independently of the location of the third vertices r resp. $\bar{r}$. Since the functions F_n ($n = 1, \dots, N$) are $\mathbb{R}^2 \to \mathbb{R}$ polynomials, their union $F : \bigcup_{n=1}^N T_n \to \mathbb{R}$ with $F(x) = F_n(x)$ whenever $x \in T_n$ is a well-defined continuous function.

From a similar application of (5) in Theorem 1 and (19) finishing its proof applied with $F := F_m$, $p_1 := p$, $p_2 := q$, $p_3 := r$, $u_3 := u_{[p,q]}$, $v := \alpha u_3 + \beta(p_2 - p_1) = \alpha u_{[p,q]} + \beta(q-p)$, and $\zeta_3 :=$ [obtained from (18)] we conclude that:

$$F'_m(x_t)v = \alpha\Big[\Phi(t)A_p + [1-\Phi(t)]A_q\Big]u_{[p,q]} + \beta\frac{d}{dt}F_m(x_t).$$

We obtain the same when the index m is replaced with n, though the values ζ_3 may differ in (18) when calculating with $p_3 := r$ for m and $p_3 := \overline{r}$ for n, respectively. We know already that the functions F_m, F_n coincide along the common edge $[p,q]$. Therefore, with the combined function $F\ \big(F(x) = F_n(x)$ for $x \in T_n;\ 1 \le n \le N\big)$ we indeed have $F'(x_t)v = F'_m(x_t)v = F'_n(x_t)v = \alpha\Big[\Phi(t)A_p + [1-\Phi(t)]A_q\Big]u_{[p,q]} + \beta\frac{d}{dt}F(x_t)$. $\square$

4. Version in the Pure Algebraic Setting

We consider the possibility of replacing the real line $\mathbb{R}$ with an arbitrary (possibly finite) field $\mathbb{K}$. Though ordering is no longer available, in particular, we cannot speak of edges $[p,q] = \{tp+(1-t)q : 0 \le t \le 1\}$ or triangles $T = \{tp+sq+(1-s-t)r : 0 \le s,t,s+t \le 1\}$ in $\mathbb{K}$ any longer, the concept of *lines* $\mathrm{Line}(p,q) := \{tp + (1-t)q : t \in \mathbf{K}\}$ connecting distinct points $p,q \in \mathbb{K}$ makes sense and is widely used in algebraic geometry. From classical geometry, we can also save the concept of *non-degenerate point triples* $\{p_1,p_2,p_3\} \subset \mathbb{K}^2$ by requiring that the expression $p_1 \wedge p_2 + p_2 \wedge p_3 + p_3 \wedge p_1$ (which corresponds to a non-zero multiple of the area of the triangle with vertices p_j in the case $\mathbb{K} = \mathbb{R}$) should not vanish. The parallelity of two vectors $u,v \in \mathbb{K}^2$ can also be well defined with the property $u \wedge v \ne 0$.

On the other hand, it is also well known that the formal derivation $\frac{d}{d\tau}\sum_{k=0}^n \alpha_k \tau^k := \sum_{k=1}^n k\alpha_k\tau^{k-1}$ $(\alpha_0,\dots,\alpha_n \in \mathbb{K})$ gives rise to a calculus with multivariate polynomials with coefficients in $\mathbb{K}$ preserving the familiar identities as the linearity, Leibniz rule, and derivation formula of composite maps. Thus, since our computations in Algorithm 1 involve only polynomial functions, we can conclude that the following theorem holds.

Theorem 3. *Let $[p_1,\dots,p_K]$ be a sequence of distinct points in $\mathbb{K}^2$, and let $\big[[i_{n,1},i_{n,2},i_{n,3}] : n = 1,\dots,N\big]$ be a sequence of distinct triples of indices $1 \le i_{n,1} < i_{n,2} < i_{n,3} \le K$ such that the triples:*

$$\mathrm{T}_n := \{p_{i_{n,1}}, p_{i_{n,2}}, p_{i_{n,3}}\} \quad (n = 1,\dots,N)$$

of points are non-degenerate. Then, given any sequence $\big[f_n : n = 1,\dots K\big]$, constants in $\mathbb{K}$ along with a sequence $\big[A_n : n = 1,\dots K\big]$ of linear forms $\mathbb{K} \to \mathbb{K}$ and any family $\big[u_{m,n} : 1 \le m < n \le K\big]$ of vectors in $\mathbb{K}^2$ such that $u_{m,n} \wedge (p_m - p_n) \ne 0$ $(1 \le m < n \le K)$, the sequence $[F_1,\dots,F_N]$ of polynomial functions $\mathbb{K} \to \mathbb{K}$ obtained with the calculations in Algorithm 1 has the following properties:

(i) $F_n(p_k) = f_k$, $F'_n(p_k)v = A_k v$ $(v \in \mathbb{K}^2)$ *whenever* $p_k \in \mathrm{T}_n$ *for some* n;
(ii) $F_m|\mathrm{Line}(p_i,p_j) = F_n|\mathrm{Line}(p_i,p_j)$ *whenever* $i \ne j$ *and* $\{p_i,p_j\} = \mathrm{T}_m \cap \mathrm{T}_n$;
(iii) $F'_n\big(tp_i + (1-t)p_j\big)v = \big[\Theta(t)A_i + \Theta(1-t)A_j\big)u_k$ $(t \in \mathbb{K}$ *whenever* $\{p_i,p_j,p_k\} = \mathrm{T}_n$.

Algorithm 1 . Triangular $\mathcal{C}^1$-spline with first order data

Require: $K \in \mathbf{N} = \{1, 2, \ldots\}$ for the number of mesh points;
List $\mathbf{v}_k = [v_k^x, v_k^y] \in \mathbb{R}^2$ $(k = 1, \ldots, K)$ of mesh points;
List $\mathbf{f}_k \in \mathbb{R}$ $(k = 1, \ldots, K)$ for function data at mesh points;
List $\mathbf{A}_k(\xi, \eta) = \mathbf{A}_k^x \xi + \mathbf{A}_k^y \eta$, $\mathbf{A}_k^x, \mathbf{A}_k^y \in \mathbb{R}$ $(k = 1, \ldots, K)$
$N \in \mathbf{N}$ for the number of mesh triangles;
of linear forms for prescribed derivatives at mesh points;
List $[i_{n,1}, i_{n,2}, i_{n,3}] \in \mathbf{N}^3$ $(n = 1, \ldots, N)$ of indices with $1 \leq i_{n,1} < i_{n,2} < i_{n,3} \leq N$
such that $\mathrm{Vert}(T_n) = \{\mathbf{v}_{i_{n,1}}, \mathbf{v}_{i_{n,2}}, \mathbf{v}_{i_{n,3}}\}$;
List $\mathbf{u}_{m,n} = [u_{m,n}^x, u_{m,n}^y] \in \mathbb{R}^2$ $(0 \leq m, n \leq N,\ m \neq n)$ of vectors
such that $\mathbf{u}_{m,n} = \mathbf{u}_{n,m} \not\parallel (\mathbf{v}_m - \mathbf{v}_n)$;
Ensure: List $\mathbf{F}_n(\xi, \eta)$ $(n = 1, \ldots, N)$ of polynomials with coefficients in $\mathbb{R}$.
Calculation: Consecutively, for each index $n = 1, 2, \ldots, N$, we compute the polynomial $F_n(\xi, \eta)$ by applying Theorem 1 and (18) as follows:

For $\ell = 1, 2, 3$, let:
$p_\ell := \mathbf{v}_{i_{n,\ell}}$, $f_\ell := \mathbf{f}_{i_{n,\ell}}$, $A_\ell(\xi, \eta) := \mathbf{A}_{i_{n,\ell}}(\xi, \eta)$,
$u_1 := \mathbf{u}_{i_{n,2}, i_{n,3}}$, $u_2 := \mathbf{u}_{i_{n,3}, i_{n,1}}$, $u_3 := \mathbf{u}_{i_{n,1}, i_{n,2}}$;
For technical reasons, for $m = 1, 2, 3$, we set also
$p_{m+3} := p_m$, $f_{m+3} := f_m$, $A_{m+3}(\xi, \eta) := A_m(\xi, \eta)$, $u_{ell+3} := u_m$;
After setting the actual values for using the formulas in the theorem, for $\ell = 1, 2, 3$, establish the barycentric weights and their derivatives as affine, resp. linear, forms, in terms of the outer product $[\alpha, \beta] \wedge [\gamma, \delta] := \det\left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right] = \alpha\delta - \beta\gamma$ (see [6] Berger(1987)):

$$
\begin{aligned}
D &:= p_1 \wedge p_2 + p_2 \wedge p_3 + p_3 \wedge p_1, \\
\lambda_\ell(\xi, \eta) &:= \Big[[\xi, \eta] \wedge (p_{\ell+1} - p_{\ell+2}) + p_{\ell+1} \wedge p_{\ell+2} \Big] / D, \\
G_\ell(\xi, \eta) &:= \Big[[\xi, \eta] \wedge (p_{\ell+1} - p_{\ell+2}) \Big] / D;
\end{aligned}
$$

For cyclic indexing, we set also:
$\lambda_{m+3}(\xi, \eta) := \lambda_m(\xi, \eta)$, $G_{m+3}(\xi, \eta) := G_m(\xi, \eta)$ $(m = 1, 2, 3)$.
Then, for $k = 1, 2, 3$, we compute the correction coefficients by means of (18):

$$
\zeta_k := -\frac{1}{G_k u_k} \sum_{d=1}^{2} \Big[30 [G_{k+d} u_k] f_{k+d} - (-1)^d 12 [G_{k+d} u_k]\, A_{k+d}(p_{k+2} - p_{k+1}) \Big];
$$

Finally, we let:

$$
\begin{aligned}
\mathbf{F}_n(\xi, \eta) := \sum_{\ell=1}^{3} \Big[& \Phi(\lambda_\ell(\xi, \eta)) f_\ell + \Theta(\lambda_\ell(\xi, \eta)) A_\ell([\xi, \eta] - p_\ell) + \\
& + \zeta_\ell \lambda_\ell(\xi, \eta) \lambda_{\ell+1}(\xi, \eta)^2 \lambda_{\ell+2} ell(\xi, \eta)^2 \Big].
\end{aligned}
$$

5. Conclusions

Our spline interpolation described above is a ZZ-type procedure providing well-articulated explicit formulas of independent theoretical interest working even in abstract algebraic settings. From practical view points, for classical plane splines, the method is completely parallelizable, and it is clearly easy to optimize with respect to its free ζ-parameters. Applications on 3D triangular complexes even with a non-trivial topology can also be expected, though this seems to be no longer a straightforward task.

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References

1. Hahmann, S.; Bonneau, G.-P. Triangular G^1 interpolation by 4-splitting domain triangles. *Comput. Aided Geom. Des.* **2000**, *17*, 731–757. [CrossRef]
2. Cao, J.; Chen, Z.; Wei, X.; Zhang, Y.J. A finite element framework based on bivariate simplex splines on triagle configurations. *Comput. Methods Appl. Mech. Eng.* **2019**, *357*, 112598. [CrossRef]
3. Zlámal, M.; Zěnišek, A. Mathematical aspects of the FEM. In *Technical Physical and Mathematical Priciples of the FEM*; Kolaž, V., Kratoci-Ivil, F., Zlamal, M., Zenisek, A., Eds.; Publising House: Praha, Czech Republic, 1971; pp. 15–39.
4. Sergienko, I.V.; Lytvyn, O.M.; Lytvyn, O.O.; Denisova, O.I. Explicit formulas for interpolating splines of degree 5 on the triangle. *Cybern. Syst. Anal.* **2014**, *50*, 670–678. [CrossRef]
5. Stachó, L.L. Locally Generated Polynomial C1-Splines over Triangular Meshes. *arXiv* **2019**, arXiv:1903.11336.
6. Berger, M. *Geometry I*; Springer: Berlin, Germany, 1987.