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Analisi matematica. — *On the existence of fixed points of holomorphic automorphisms.* Nota [(*)] di LÁSZLÓ L. STACHÓ, presentata dal Corrisp. G. CIMMINO.

RIASSUNTO. — Dato uno spazio di Banach complesso E, si considerano gli automorfismi olomorfi del disco unità B di E, e si esamina il problema dell'esistenza di punti uniti nei casi in cui E sia lo spazio delle funzioni continue su un compatto od un reticolo di tipo M.

By a result of Kaup-Upmeier [1], if E is any (complex) Banach space then every member of Aut B (E) (= {the Fréchet holomorphic automorphisms of the unit ball of E}) can be considered as the restriction to B (E) of a holomorphic map of some neighbourhood of $\overline{B}$ (E) (= the closed unit ball) into E. Several well-known positive examples suggested the following conjecture: These natural extensions always have a fixed point in $\overline{B}$ (E) (e.g. if E is a Hilbert space [2], if $E = L^p(X, m)$ for some measure space and $1 \leq p < \infty$ [3], if E is finite dimensional, if $E = l^\infty$). Our main purpose will be to investigate the validity of this conjecture and to prove that the answer is negative. The next observation immediately provides a large class of counter-examples. We shall denote by Aut $\overline{B}$ (E) the set formed by the continuous extensions to $\overline{B}$ (E) of the elements of Aut B (E). Let Δ be the unit disc in **C**; the group Aut $\overline{\Delta}$ is the group of Moebius transformations.

THEOREM 1. *If* $E = C(\Omega)$ *for some compact space* Ω *then every* $F \in \mathrm{Aut}\,\overline{B}(E)$ *admits a fixed point only if the following topological condition on* Ω *holds*

(☆) *for all open* $\mathscr{F}_\sigma$*-subsets* G *of* Ω, *any* $g \in C_{\text{bounded}}(G)$ *can be continuously extended to the whole* Ω.

Now it is naturally raised the question, for which kind of $F \in \mathrm{Aut}\,\overline{B}(C(\Omega))$ condition (☆) ensures the existence of fixed points (in case of compact Ω). In this direction the following theorem holds:

THEOREM 2. *If* Ω *is a compact space then* (☆) *is equivalent to*

(☆☆) *every* $F \in \mathrm{Aut}\,\overline{B}(C(\Omega))$ *of the form* $F(f) = [x \to M(x) f(Tx)]$ *where* T *is any pointwise periodic homeomorphism of* Ω *onto itself (pointwise periodic meaning* $\forall x \in \Omega\ \exists n \in \mathbf{N}\ T^n x = x$) *and* $M(\cdot)$ *denotes any continuous map of* Ω *into* Aut $\overline{\Delta}$, *has a fixed point.*

The key lemma in the proof of Theorem 2 is the following one (which may have some interest also in itself).

(*) Pervenuta all'Accademia il 28 giugno 1979.

LEMMA 1. *If* Ω *is a compact space having the property* (☆) *and if* $T : \Omega \leftrightarrow \Omega$ *is any pointwise periodic homeomorphism, then, for every* $f \in C(\Omega)$, *there exists some* $n \in \mathbf{N}$ *such that* $f = f \circ T^n$.

Since, by a theorem of E. Vesentini [4], if Ω is any compact space and $F \in \mathrm{Aut}\,\overline{B}\,(C(\Omega))$, then there exists a unique pair of continuous maps $M_F : \Omega \rightarrow \mathrm{Aut}\,\overline{\Delta}$ and $T_F : \Omega \leftrightarrow \Omega$ such that $F(f) = [x \rightarrow M_F(x) \cdot f(T_F x)]$ $\forall f \in \overline{B}\,(C(\Omega))$, one can conjecture that (☆) implies the existence of fixed points also for all $F \in \mathrm{Aut}\,\overline{B}\,(C(\Omega))$. However, as it is seen from the next theorem, this expectation is false.

THEOREM 3. *One can construct a* $\Phi \in \mathrm{Aut}\,\overline{B}\,(L^\infty(0, 1))$ *which has no fixed point.*

(The proof of this Theorem is essentially combinatorial, but requires also some ergodic theory).

COROLLARY. *Even the hyperstonianity (which is clearly a stronger property than* (☆)) *of a compact* Ω *does not ensure, in general, the existence of fixed points for every* $F \in \mathrm{Aut}\,\overline{B}\,(C(\Omega))$. *(For the spectrum of* $L^\infty(0, 1)$ *is a compact hyperstonian space).*

On the other hand, Theorem 3 is suitable in proving the next positive result.

THEOREM 4. *If* E *is a complex* M-*space (for definition see* [5]) *with separable predual then every* $F \in \mathrm{Aut}\,\overline{B}\,(E)$ *has fixed point iff* $E \cong l^\infty$ *or if* E *is finite dimensional.*

The proof of Theorem 4 is based on a characterization of M-spaces due to Rieffel-Kukutani [5], on Halmos's Isomorphism Theorem [6] and on

LEMMA 2. *If* Ω *is a discrete (not necessarily compact) space and* $E = C_{\text{bounded}}(\Omega)$ *then every* $F \in \mathrm{Aut}\,\overline{B}\,(E)$ *can be written in the form* $F(f) = [x \rightarrow M(x)\, f(Tx)]$ *for suitable maps* $M : \Omega \rightarrow \mathrm{Aut}\,\overline{\Delta}$ *and* $T : \Omega \leftrightarrow \Omega$.

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