Hermite interpolation sequences over fields

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Abstract

By a Hermite interpolation sequence we mean a sequence of Hermite interpolation polynomials of degree $0, 1, \ldots$ such that consecutive terms satisfy the differentiation conditions of the previous ones. We extend this concept to arbitrary fields from the reals by purely algebraic means based on the possibility of formal Taylor expansions of rational fractions around any point of the underlying field. As an application we obtain recursion-free explicit formulas for the entries of triangular decompositions of generalized Hermite-Vandermonde matrices.

Key words: Hermite interpolation, Taylor expansion, Vandermonde-Hermite matrix, Triangular factorization
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1 Introduction

Lagrange and Hermite interpolations are taught, using recursion methods, to a wide range of students as a chapter of complex numerical analysis. Recently interest has arisen on recursion free closed formulas concerning them in the setting of triangular decompositions of Vandermonde matrices over generic fields [4,5]. The aim of this note is to show that the basic ideas of Spitzbart's paper [1] which provide a natural generalization of the Lagrangian approach to Hermite interpolation with higher derivatives (actually the generalized basic polynomials are A_{jk} there) can be realized by purely algebraic means based on the possibility of formal Taylor expansions of rational fractions over fields. We continue these arguments to achieve a generalization of the Newtonian construction as well in a more flexible formulation which may be of interest

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in education. We conclude the paper with the application of the results to get a recursion-free explicit triangular decomposition of generalized Hermite-Vandermonde matrices.

2 Basic concepts

Throughout the paper we work in the setting of the polynomial ring $\mathbb{K}[x]$ of all formal expressions $p(x) = \sum_{k=0}^{n} c_k x^k$ $(c_0, c_1, \dots, c_n \in \mathbb{K};\ n \in \mathbb{Z}_+)$ where $\mathbb{K}$ is an arbitrary field. Let us emphasize that $\sum_{k=0}^{n} c_k x^k$ is not identified with its functional representation $\mathbb{K} \ni \xi \mapsto \sum_{k=0}^{n} c_k \xi^k$ as it is usual in the classical real case. Though the formal derivatives $p^{(m)}(x) = \sum_{k=m}^{n} k(k-1)\cdots(k-m+1)c_k x^{k-m}$ are well defined, in the case of $\chi := \mathrm{char}(\mathbb{K}) \neq 0$, we have inconveniently $\left[x^k\right]^{(m)} = 0$ for $m \geq \chi$ since $k(k-1)\cdots(k-m+1) = \mathrm{mod}_\chi\Big(k(k-1)\cdots(k-m+1)\Big)$ in $\mathbb{K}$. Instead, we reformulate the defining constraints of Hermite interpolation in terms of the Taylor coefficients

$$p\big|_a^k := \Big[\text{coefficient of } x^k \text{ for } p(x+a) := \sum_{\ell=0}^{n} c_\ell (x+a)^\ell\Big] = \sum_{\ell=k}^{n} \binom{\ell}{k} c_\ell\, a^{\ell-k}$$

corresponding to the terms $p^{(k)}(a)/k!$ in the classical case. Indeed we have the Taylor expansion $p = p(x) = \sum_{k=0}^{n} p\big|_a^k (x-a)^k$ for any point $a \in \mathbb{K}$.

Let $X := (x_0, x_1, x_2, \dots)$ be an arbitrary sequence in $\mathbb{K}$ indexed over $\mathbb{Z}_+$. Define

$$\omega_n^X := \prod_{j:\ j<n} (x - x_j), \qquad \nu^X(n,i) := \#\Big\{j:\ j<n,\ x_j = x_i\Big\} \tag{2.1}$$

with the convention $\omega_0^X := \prod_\emptyset = 1 = x^0$ and with $\#$ standing for cardinality. Observe that for any $n, i = 0, 1, \dots$ we have[1]

$$\begin{aligned} \omega_n^X &= (x - x_i)^{\nu^X(n,i)} \prod_{\substack{j:j<n,\\ x_j \neq x_i}} \Big(\underbrace{x - x_j}_{(x-x_i)+(x_i-x_j)}\Big) = \\ &= \Big(x - x_i\Big)^{\nu^X(n,i)} \Big[\prod_{\substack{j:j<n,\\ x_j \neq x_i}} (x_i - x_j)\Big]\Big[1 + (x - x_i)\,\mathrm{pol}(x)\Big]. \end{aligned}$$

[1] In the formula below and later on, the symbol $\mathrm{pol}(x)$ stands for a suitable polynomial from $\mathbb{K}[x]$ which we do not intend to specify any further.

In particular, x_i is a root of ω_n^X with multiplicity $\nu^X(n,i)$ and therefore ω_n^X is the unique polynomial of degree $\leq n$ such that

$$\omega_n^X\Big|_{x_i}^{\nu^X(i,i)} = 0 \quad (i<n), \qquad \omega_n^X\Big|_{x_n}^{\nu^X(n,n)} = \prod_{\substack{j:j<n,\\ x_j\neq x_n}} (x_n - x_j) \tag{2.2}$$

Definition 2.3 Given two sequences $X := (x_0, x_1, x_2, \ldots)$, $Y := (y_0, y_1, y_2, \ldots)$ in $\mathbb{K}$, let

$$\mathcal{H}_n^{X,Y} := \Big\{ p \in \mathbb{K}[x] : \ p|_{x_i}^{\nu^X(i,i)} = y_i \quad (i = 0, \ldots, n) \Big\}$$

be the set of all polynomials satisfying the *Hermite interpolation condition of order n with coefficients from* (X,Y). By the *Hermite interpolation sequence associated with* (X,Y) we mean the sequence $h_n^{X,Y}$ $(n = 0, 1, \ldots)$ of polynomials in $\mathbb{K}[x]$ defined recursively as

$$h_0^{X,Y} := y_0 x^0, \quad h_n^{X,Y} := h_{n-1}^{X,Y} + \gamma_n^{X,Y}\omega_n^X \quad \text{where} \quad \gamma_n^{X,Y} := \frac{y_n - h_{n-1}^{X,Y}\Big|_{x_n}^{\nu^X(n,n)}}{\prod_{\substack{j:j<n,\\ x_j\neq x_n}} (x_n - x_j)} .$$

Proposition 2.4 *For any n, $h_n^{X,Y}$ is the unique polynomial in $\mathcal{H}_n^{X,Y}$ with degree $\leq n$.*

Clearly $\{h_0^{X,Y}\} = \{p \in \mathcal{H}_0^{X,Y} : \deg(p) = 0\}$. Provided $\mathcal{H}_n^{X,Y} \neq \emptyset$, the difference of two polynomials from $\mathcal{H}_n^{X,Y}$ must have root of multiplicity $\nu^X(n,i)$ at any point x_i with $i < n+1$, it immediately follows that

$$\mathcal{H}_n^{X,Y} = h_n^{X,Y} + \omega_{n+1}^X \mathbb{K}[x].$$

We see by induction that $\mathcal{H}_n^{X,Y} \neq \emptyset$ for $n \in \mathbb{Z}_+$. Trivially $y_0 x^0 + (x - x_0)\mathbb{K}[x] = \mathcal{H}_0^{X,Y}$. Suppose $\mathcal{H}_{n-1}^{X,Y} \neq \emptyset$. Then a polynomial p belongs to $\mathcal{H}_n^{X,Y}$ if and only if $p = h_{n-1}^{X,Y} + \omega_n^X q$ for some $q \in \mathbb{K}[x]$ and $p|_{x_n}^{\nu^X(n,n)} = y_n$. By choosing q in the form $q := \gamma_n^{X,Y} = \gamma_n^{X,Y} x^0$ we get $p = h_{n-1}^{X,Y} + \gamma_n^{X,Y}\omega_n^X$ with $p|_{x_n}^{\nu(n,n)} = h_{n-1}^{X,Y}\Big|_{x_n}^{\nu(n,n)} + \gamma_n^{X,Y}\,\omega_n^X\Big|_{x_n}^{\nu(n,n)} = h_{n-1}^{X,Y}\Big|_{x_n}^{\nu(n,n)} + \gamma_n^{X,Y} \prod_{\substack{j:j<n,\\ x_j\neq x_n}} (x_n - x_j) = y_n$.

Remark 2.5 In the classical case $\mathbb{K} = \mathbb{R}$, the interpolation polynomials are of the form

$$f\begin{pmatrix} b_1^{(0)},\ldots,b_1^{(m_1)} \\ a_1 \end{pmatrix}\begin{pmatrix} b_2^{(0)},\ldots,b_2^{(m_2)} \\ a_2 \end{pmatrix} \ldots \begin{pmatrix} b_r^{(0)}\ldots,b_r^{(m_r)} \\ \ldots\ a_r \end{pmatrix}$$

defined to be the unique polynomial $f \in \mathbb{R}[x]$ such that $\deg(f) \leq n$ and $f^{(d)}(a_k) = b_k^{(d)}$ $(k = 1, \ldots, r;\ d = 0, \ldots, m_k)$. In our terminology, $f\begin{pmatrix} b_1^{(0)},\ldots,b_1^{(m_1)} \\ a_1 \end{pmatrix},\ldots,\begin{pmatrix} b_r^{(0)}\ldots,b_r^{(m_r)} \\ ,\ldots,a_r \end{pmatrix} = h_n^{X,Y}$ whenever X and Y have the pattern

$$X = \Big(\underbrace{a_1, \ldots, a_1}_{m_1+1}, \ldots, \underbrace{a_r, \ldots, a_r}_{m_r+1}, \ldots\Big), \ Y = \Big(\frac{b_1^{(0)}}{0!}, \ldots, \frac{b_1^{(m_1)}}{m_1!}, \ldots, \frac{b_r^{(0)}}{0!}, \ldots, \frac{b_r^{(m_r)}}{m_r!}, \ldots\Big).$$

Definition 2.6 The *Newtonian form* of a Hermite sequence is the representation $h_n^{X,Y} = \sum_{k=0}^{n} \gamma_k^{X,Y} \omega_k^X$ $(n = 0, 1, \ldots)$. We shall write δ_i for the sequence $\left(\delta_{0,i}, \delta_{1,i}, \delta_{2,i}, \ldots\right)$ with the Kronecker symbol $\delta_{n,i} := \left[1 \text{ if } n = i,\ 0 \text{ else}\right]$. We call the members of the double sequences

$$H_{n,i}^X := h_n^{X,\delta_i}, \quad \overline{\omega}_{n,i}^X := \prod_{j:j \leq n,\, x_j \neq x_i} \left(x - x_j\right) \qquad (i, n = 0, 1 \ldots) \tag{2.7}$$

the *basic Hermite interpolation polynomials* resp. the *complementary Newton factors* over the sequence X. By the *Lagrange form* of a Hermite sequence we mean the representation $h_n^{X,Y} = \sum_{i=0}^{n} y_i H_{n,i}^X$ $(n = 0, 1, \ldots)$.

Example 2.8 Let $\mathbb{K} = \mathbb{Z}_2 = \{0, 1\}$ and consider the sequences

$$X := [0, 1, 0, 1, 0, 1, ...], \quad Y := [1, 1, 1, \ldots].$$

Then, for $1 < n = 2k + r$ with $r := \mathrm{mod}_2(n)$ and $k := \lfloor n/2 \rfloor$ we have $\omega_n^X = x^{k+r}(x - 1)^k = x^{k+r}(x + 1)^k$. From the identity $2 = 1 + 1 = 0$ we also get $\omega_{2k}^X(t + 1) = (t + 1)^k t^k = \omega_{2k}^X(t)$ and $(1 + x)^{2^k} = 1 + x^{2^k}$ for any $k = 0, 1, \ldots$. Thus *the Newtonian form of* $h_n^{X,Y}$ *is simply* $\sum_{m:2^m-2 \leq n} \omega_{2^m-2}^X$ *for any index* n. Actually also $h_n^X = \sum_{\ell=0}^{M(n)} x^\ell$ with $M(n) := \max\{2^m - 2 : 2^m - 2 \leq n,\ m \in\}$. We obtain the *classical* Newtonian form of $h_{2k+r}^{X,Y}$ by evaluating $f_{0\ \ \ 1}^{(0!,\ldots,k!)\ (0!,\ldots,(k-1+r)!)}$ over $\mathbb{R}$ with a Newton difference scheme and then taking the coefficients mod_2. The result is a rather sophisticated linear combination from the factors $1, x, \ldots, x^{k+1}, x^{k+1}(x + 1), \ldots, x^{k+1}(x + 1)^{k-1+r}$.

3 Numerical issue: modified Newton difference schemes

Classical Hermitian interpolation seems to be well understood in terms of Newtonian difference schemes as done in the nice survey [6]. Working over a field $\mathbb{K}$ of general type, can be transferred in a straightforward manner into generalized Newtonian differences schemes, if we consider the classical sequences described in Remark 2.5. However, in a non-classical case as in Example 2.8 we should be more careful when using schemes from classical rearrangements. For the sake of completeness, below we outline a self-contained approach.

In order to avoid using any sophisticated notation, throughout this section let the sequences X, Y be fixed arbitrarily and let us write $h_n, \mathcal{H}_n, \gamma_n, \nu(n, i), \omega_n$ instead of $h_n^{X,Y}, \mathcal{H}_n^{X,Y}, \gamma_n^{X,Y}$ and $\nu^X(n, i), \omega_n^X$, respectively. Observe that, for any n, we can rearrange the conditions $p|_{x_0}^{\nu^X(0,0)} = y_0, \ldots, p|_{x_n}^{\nu^X(n,n)} = y_n$ determining the space $\mathcal{H}_n$ into the form

$$p|_{a_1}^0 = b_1^{(0)}, \ldots, p|_{a_1}^{m_1^{(n)}} = b_1^{(m_1^{(n)})}, \ldots, p|_{a_{r(n)}}^0 = b_{r(n)}^{(0)}, \ldots, p|_{a_{r(n)}}^{m_{r(n)}^{(n)}} = b_{r(n)}^{(m_{r(n)}^{(n)})}.$$

This can be done in a unique way if we require that $a_1, a_2, \ldots$ are the distinct enumeration of $x_0, x_1, \ldots$ in order of first appearance, that is for $n = 0, 1, \ldots$ we have $\{x_0, \ldots, x_n\} = \{a_1, \ldots, a_{r(n)}\}$ where

$$r(n) = \#\{x_0, \ldots, x_n\}, \quad m_k^{(n)} = \#\{j \le n : \ x_j = a_k\} - 1. \tag{3.1}$$

As mentioned, to calculate the interpolating polynomials h_n we can use Newton difference schemes developed for the setting described in Remark 2.5 with the straightforward modification of replacing the terms $\frac{f^{(d)}(a_k)}{d!}$ there with the $b_k^{(d)}$ above. Namely, we can calculate each coefficient γ_n as the last bottom term of a lower triangular matrix $\Delta^{(n)}$ whose columns store the column elements of the classical Newton difference scheme. To realize this, let us store the rearranged data sequences in the $(n+1)$-vectors

$$\mathbf{a}^{(n)} = \Big[\alpha_k^{(n)}\Big]_{k=0}^{n} := \Big[\underbrace{a_1 \cdots a_1}_{1+m_1^{(n)}} \ \cdots \ \underbrace{a_{r(n)} \cdots a_{r(n)}}_{1+m_{r(n)}^{(n)}}\Big] \tag{3.2}$$

and define the columns of the matrix $\Delta^{(n)} := \mathrm{lowertr}\Big[\Delta_{k,d}^{(n)}\Big]_{k,d=0}^{n}$ recursively as follows. For the starting column let

$$\Delta_{k,0}^{(n)} := b_s^{(0)} \quad \text{if} \quad \alpha_k^{(n)} = a_s \qquad (k = 0, \ldots, n).$$

Having constructed column d, we set

$$\Delta_{k,d+1}^{(n)} := \begin{cases} \dfrac{\Delta_{k,d}^{(n)} - \Delta_{k-1,d}^{(n)}}{\alpha_k^{(n)} - \alpha_{k-d-1}^{(n)}} & \text{if} \quad \alpha_k^{(n)} \ne \alpha_{k-d-1}^{(n)} \\ b_s^{(d+1)} & \text{if} \quad \alpha_k^{(n)} = \alpha_{k-d-1}^{(n)} = a_s \end{cases} \qquad (k = d+1, \ldots, n).$$

As a final result for any $N \in \mathbb{Z}_+$, we get

$$h_N = \sum_{n=0}^{N} \gamma_n \, \omega_n = \sum_{n=0}^{N} \Delta_{n,n}^{(n)} \ (x - x_0) \cdots (x - x_{n-1}) \ . \tag{3.3}$$

To do this we need to calculate all the matrices $\Delta^{(n)}$ $(n = 0, \ldots, N)$. However, large parts in $\Delta^{(n+1)}$ and $\Delta^{(n)}$ coincide. Clearly, if $x_{n+1} \notin \{x_0, \ldots, x_n\}$ then $\Delta^{(n+1)}$ simply enlarges $\Delta^{(n)}$: in this case $\Delta_{k,d}^{(n+1)} = \Delta_{k,d}^{(n)}$ for $k, d \le n$. Consider the case $x_{n+1} = a_s \in \{x_0, \ldots, x_n\}$. Let $m^* := \max\{j \le n : \ \alpha_j^{(n)} = a_s\} + 1$ be the position after the last index where the point $x_{n+1}=a_s$ appears in $\mathbf{a}^{(n)}$:

$$\mathbf{a}^{(n+1)} = \Big[\underbrace{\cdots a_{s-1}\ \overbrace{a_s \cdots a_s}^{1+m_s^{(n)}}}_{\text{first } m^* \text{ terms of } \mathbf{a}^{(n)}}\ a_s\ \underbrace{a_{s+1}\ \cdots\ a_{r(n)}}_{\text{rest of } \mathbf{a}^{(n)}}\Big]$$

From the recursion rules it follows by induction on d that

$$\begin{aligned} &\Delta^{(n+1)}_{m^*,d} = b_s^{(d)} \quad \text{for} \quad d = 0,\ldots,m_s^{(n)}+1 = m_s^{(n+1)}, \\ &\Delta^{(n+1)}_{k,d} = \Delta^{(n)}_{k,d} \quad (d \le k < m^*), \qquad \Delta^{(n+1)}_{k,d} = \Delta^{(n)}_{k-1,d} \quad (k > \max\{d, m^*\}). \end{aligned}$$

Thus *to obtain* $\Delta^{(n+1)}$ *from* $\Delta^{(n)}$*, arithmetic operations are required only for the* $(n-m^*+2)(m^*-m_s^{(n)})$ *entries*

$$\Delta^{(n+1)}_{\overline{k},\overline{d}} \quad \text{with} \quad (\overline{k},\overline{d}) \in \Big\{(k,d):\ k \ge m^*,\ k-(m^*-m_s^{(n)}) \le d \le k\Big\}$$

displayed black in Fig. 1, which form mostly a rather narrow parallelogram.

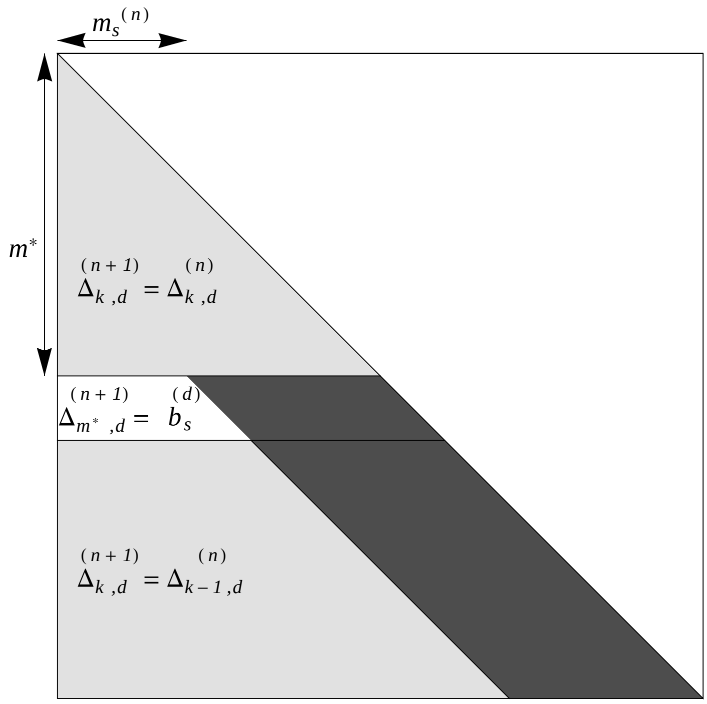

Fig1. Pieces of $\Delta^{(n)}$ in $\Delta^{(n+1)}$ (gray) and nontrivial new entries (black)

4 Basic polynomials via formal power series

As usual, let $\mathbb{K}\langle x\rangle$ denote the field of all formal fractions p/q with $p,q \in \mathbb{K}[x]$, $q \neq 0$ up to the identification $p_1/q_1 = p_2/q_2$ whenever $p_1q_2 = p_2q_1$ in $\mathbb{K}[x]$.

Given any point $a \in \mathbb{K}$ along with a fraction $\varrho(x) := p(x)/q(x) \in \mathbb{K}\langle x\rangle$ such that $q(a) \neq 0$, the *formal Taylor series*

$$\varrho(x) \sim \sum_{n=0}^{\infty} \varrho|_a^n \, (x-a)^n \tag{4.1}$$

of ρ around a is well-defined by the requirement $p \sim q \sum_{j=0}^{\infty} \varrho|_a^j \, (x-a)^j$. Actually, if $p(x) = \sum_{k=0}^{n} a_k (x-a)^k$ and $q(x) = \sum_{i=0}^{m} b_i (x-a)^i$ then we have the recursion

$$\varrho|_a^0 = \frac{p(a)}{q(a)} \; ; \qquad \varrho|_a^k = \frac{1}{q(a)} \left[p|_a^k - \sum_{j=0}^{k-1} q|_a^{k-j} \, \varrho|_a^j \right] \quad (k = 1, 2, \ldots)$$

where $p|_a^k = a_k \; (0 \leq k \leq n)$, $p|_a^k = 0$ for $k > n$ and $q|_a^i = b_i \; (0 \leq i \leq m)$, $q|_a^i = 0$ for $i > m$. We have the same Taylor series around a for $\varrho := p/q$ and $\widetilde{\varrho} := \widetilde{p}/\widetilde{q}$ with $q(a), \widetilde{q}(a) \neq 0$ whenever they represent the same object in $\mathbb{K}\langle x\rangle$ i.e. if $p\widetilde{q} = \widetilde{p}q$, because then both $T := \sum_{n=0}^{\infty} \varrho|_a^n \, (x-a)^n$ and $\widetilde{T} := \sum_{n=0}^{\infty} \widetilde{\varrho}|_a^n \, (x-a)^n$ satisfy $\widetilde{q}qT \sim p\widetilde{q}$ resp. $\widetilde{q}q\widetilde{T} \sim \widetilde{p}q = p\widetilde{q}$. In the classical case $\mathbb{K} = \mathbb{R}$, in terms of derivations again we have $\varrho|_a^n = \varrho^{(n)}(a)/n!$. In particular, if $a \neq b$ then by Newton's binomial theorem we have

$$\frac{1}{(x-b)^m} \sim \frac{1}{(a-b)^m} \left[1 + \frac{x-a}{a-b} \right]^{-m} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(a-b)^{m+k}} \binom{m+k-1}{k} (x-a)^k$$

even in the case of generic fields $\mathbb{K}$ instead of $\mathbb{R}$, since the coefficients here satisfy the same recursion as in the real case. If $\chi = \mathrm{char}(\mathbb{K}) \neq 0$ then $\binom{m+k-1}{k} = \mathrm{mod}_\chi \binom{m+k-1}{k}$.

For the initial segments of formal Taylor series, we introduce the shorthand notation

$$\varrho\|_a^N := \sum_{k=0}^{N} \varrho|_a^k \, (x-a)^k \qquad (\varrho \in \mathbb{K}\langle x\rangle). \tag{4.2}$$

Lemma 4.3 *Let $a \in \mathbb{K}$ and $\varrho(x) := p(x)/q(x)$ with $p, q \in \mathbb{K}[x]$ and $q(a) \neq 0$. Then*

$$\varrho\|_a^N \, q(x) = p(x) + (x-a)^{N+1} \mathrm{pol}(x) \qquad \big(N = \deg(p), \deg(p){+}1, \ldots\big).$$

Let $p(x) = \sum\limits_{k=0}^{N} a_k (x-a)^k$, $q(x) = \sum\limits_{i=0}^{m} b_i (x-a)^i$, $\varrho(x) = \sum\limits_{j=0}^{\infty} c_j (x-a)^j$.

Recall that, by definition a_k is the coefficient of $(x-a)^k$ in the formal product

$q(x)\varrho(x) = \sum_{i=0}^{m} b_i(x-a)^i \sum_{j=0}^{\infty} c_j(x-a)^j = \sum_{k=0}^{\infty} b_i c_j (x-a)^k$. That is

$$a_k = \sum_{(i,j):\ i+j=k} b_i c_j \quad (k=0,\dots,N), \qquad 0 = \sum_{(i,j):\ i+j=k} b_i c_j \quad (k>N).$$

Hence we conclude that

$$\begin{aligned} q(x)\sum_{j=0}^{N} c_j(x-a)^j &= \sum_{i=0}^{m} b_i(x-a)^i \sum_{j=0}^{N} c_j(x-a)^j = \\ &= \sum_{k=0}^{N}\sum_{i+j=k} b_i c_j (x-a)^k + \sum_{k=N+1}^{N+m} \sum_{\substack{i+j=k\\ i\le m}} b_i c_j (x-a)^k = \\ &= p(x) + (x-a)^{N+1} \sum_{\ell=0}^{m-1} \left[\sum_{i=0}^{\min\{m,N+1+\ell\}} c_i b_{N+1+\ell-i} \right] (x-a)^\ell . \end{aligned}$$

Let us introduce some handy notation which make the rest of the formulae more readable and the reasoning shorter.

Definition 4.4 Henceforth we write

$$\sigma^X(n,i) := \#\Big\{ j\in(i,n] : x_j = x_i \Big\}, \quad \mu^X(n,a) := \#\Big\{ j\le n :\ x_j = a \Big\}. \qquad (4.5)$$

Proposition 4.6 *We have* $H^X_{n,i} = 0$ *for* $n < i$. *Otherwise, with the terms introduced in* (2.1) *resp.* (2.7),

$$H^X_{n,i} = \frac{(x-x_i)^{\nu^X(i,i)}}{\overline{\omega}^X_{n,i}} \Bigg\|_{x_i}^{\nu^X(n+1,i)-1} \overline{\omega}^X_{n,i} \qquad (n = i, i+1, i+2, \dots). \qquad (4.7)$$

In the case $n < i$ the polynomial $H^X_{n,i}$ of degree n has roots with total multiplicity $n+1$ entailing $H_{n,i} = 0$. Assume $n \ge i$. Now consider any index $j \in \{0,\dots,n\} \setminus \{i\}$ and let $a := x_j$. Observe that the point a must be a root of the polynomial $H^X_{n,i}$ of as many algebraic multiplicity as the number of appearances of the point a in the sequence $x_0,\dots,x_n$. That is $(x-a)^{\mu^X(n,a)} \Big| H^X_{n,i}$ ($p\big|q$ standing for *p is a divisor of q*). Therefore

$$\overline{\omega}^X_{n,i} = \prod_{\substack{j:j\le n\\ x_j\ne x_i}} (x-x_j) = \prod_{a\in\{x_0,\dots x_n\}\setminus\{x_i\}} (x-a)^{\mu^X(n,a)} \Bigg| H^X_{n,i} \ .$$

The polynomial $H^X_{n,i}$ also satisfies the remaining $(n+1) - \#\{j \le n : x_j \ne x_i\}$ constraints

$$H^X_{n,i}\Big|_{x_i}^{d} = 0 \ \text{ for } \ d \in \Big\{0,\dots,n - \#\{j\le n : x_j \ne x_i\}\Big\} \setminus \{\nu^X(i,i)\} \ \text{and} \ H^X_{n,i}\Big|_{x_i}^{\nu^X(i,i)} = 1.$$

Thus $H_{n,i}^X$ must be of the form

$$H_{n,i}^X = (x - x_i)^{\nu^X(i,i)} + (x - x_i)^{n+1-\#\{j\leq n: x_j \neq x_i\} - \nu^X(i,i)} \mathrm{pol}(x).$$

Observe here that $n+1-\#\{j \leq n : x_j \neq x_i\} = \#\{j \leq n : x_j = x_i\} = \nu^X(n+1, i)$. Applying Lemma 4.3 with $p := (x - x_i)^{\nu(i,i)}$, $N := \nu^X(n+1, i) - 1$ and $q := \prod_{\substack{j: j \leq n \\ x_j \neq x_i}} (x - x_j)$ we see the polynomial $h := \dfrac{(x - x)^{\nu^X(i,i)}}{\overline{\omega}_{n,i}^X} \Big\|_{x_i}^{\nu^X(n+1,i)-1} \overline{\omega}_{n,i}^X$ has the expansion form $h = (x - x_i)^{\nu(i,i)} + (x - x_i)^{\nu(n+1,i)-\nu(i,i)} \mathrm{pol}(x)$. However, also $\overline{\omega}_{n,i}^X \big| h$ by definition. Thus h satisfies all the constraints defining $H_{n,i}^X$, which entails $h = H_{n,i}^X$.

Next we proceed to the Newtonian form of the sequences $H_{n,i}^X = h_n^{X,\delta_i}$ $(n \in \mathbb{Z}_+)$.

Corollary 4.8 *For any fixed index $i \in \mathbb{Z}_+$ and $N = i, i+1, i+2, \ldots$ we have*

$$H_{N,i}^X = \sum_{n=0}^{N} \Gamma_{n,i}^X \omega_n^X \quad \text{with} \quad \Gamma_{n,i}^X := \gamma_n^{X,\delta_i} = \prod_{j \leq n, x_j \neq x_i} \Big(x - x_j\Big)^{-1} \Big|_{x_i}^{\sigma^X(n,i)} \qquad (n \geq i).$$

Recall that $\Gamma_{n,i}^X = 0$ for $n < i$. According to Definition 2.3, $H_{N,i}^X = \sum_{n=0}^N \gamma_n^{X,\delta_i} \omega_n^X$ $(N = 0, 1, \ldots)$ implying $H_{n,i}^X - H_{n-1,i}^X = \gamma_n^{X,\delta_i} \omega_n^X$ $(n = 1, 2, \ldots)$. Thus γ_n^{X,δ_i} is the coefficient of x^n in $H_{n,i}^X - H_{n-1,i}^X$. Since $H_{n-1,i}^X$ is of degree $\leq n - 1$, it means that

$$\begin{aligned} \gamma_n^{X,\delta_i} &= H_{n,i}^X \Big|_0^n = H_{n,i}^X \Big|_{x_i}^n = \\ &= \left[\left[(x - x_i)^{\nu^X(i,i)} / \overline{\omega}_{n,i}^X \right] \Big\|_{x_i}^{\nu^X(n+1,i)-1} \overline{\omega}_{n,i}^X \right] \Big|_{x_i}^n = \\ &= \left[(x - x_i)^{\nu^X(i,i)} / \overline{\omega}_{n,i}^X \right] \Big\|_{x_i}^{\nu^X(n+1,i)-1} \Big|_{x_i}^{\nu^X(n+1,i)-1} = \\ &= \left[(x - x_i)^{\nu^X(i,i)} / \overline{\omega}_{n,i}^X \right] \Big|_{x_i}^{\nu^X(n+1,i)-1-\nu^X(i,i)}. \end{aligned}$$

Here we have $\nu^X(n+1, i) - \nu^X(i, i) - 1 = \#\Big\{j \leq n : \ x_j = x_i\Big\} - 1 - \#\Big\{j < i : x_j = x_i\Big\} = \#\Big\{j : \ i < j \leq n, x_j = x_i\Big\} = \sigma^X(n, i)$.

5 Closed formulas for the Newtonian representation

Inserting the Newtonian form of the basic polynomials into the Lagrange representation of $h_n^{X,Y}$, on the basis of the observation that Newton's binomial theorem works even in the case of formal power series, we get the following closed explicit combinatorial form for Hermite interpolation sequences.

Theorem 5.1 *For all* $N = 0, 1, \dots$ *we have* $h_N^{X,Y} = \sum_{n=0}^{N} \Big(\sum_{i=0}^{n} y_i \Gamma_{n,i}^X \Big) \omega_n^X$ *where*

$$\Gamma_{n,i}^X = (-1)^{\sigma^X(n,i)} \sum_{\kappa \in \mathcal{K}_{n,i}(X)} \prod_{a \in \mathcal{A}_{n,i}(X)} \Big(x_i - a \Big)^{-[\mu^X(n,a)+\kappa(a)]} \tag{5.2}$$

with the notations $\mathcal{A}_{n,i}(X) := \{x_0, \dots, x_n\} \setminus \{x_i\}$ *and*

$$\mathcal{K}_{n,i}(X) := \Big\{ \kappa \in \Big\{ \mathcal{A}_{n,i}(X) \to \mathbb{Z}_+ \text{ functions} \Big\} : \sum_{a \in \mathcal{A}_{n,i}(X)} \kappa(a) = \sigma^X(n,i) \Big\}.$$

We have already seen that

$$h_N^{X,Y} = \sum_{i=0}^{N} y_i H_{n,i}^X = \sum_{n=0}^{N} \Big(\sum_{i=0}^{n} y_i \Gamma_{n,i}^X \Big) \omega_n^X$$

where

$$\Gamma_{n,i}^X = \Big[1/\overline{\omega}_{n,i}^X \Big] \Big|_{x_i}^{\sigma(n,i)} = \Bigg[\prod_{\substack{j:j \le n \\ x_j \ne x_i}} (x - x_j)^{-1} \Bigg] \Bigg|_{x_i}^{\sigma(n,i)} = \Bigg[\prod_{a \in X_n \setminus \{x_i\}} (x - a)^{-\mu(n,i)} \Bigg] \Bigg|_{x_i}^{\sigma(n,i)}.$$

We know also that in general

$$(x - a)^{-m} \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(x_i - a)^{m+k}} \binom{m+k-1}{k} (x - x_i)^k.$$

Thus in the expression $\Gamma_{n,i}^X$ we get

$$\prod_{a \in X_n \setminus \{x_i\}} (x - a)^{-\mu(n,i)} \sim \prod_{a \in X_n \setminus \{x_i\}} \Bigg[\sum_{k_a=0}^{\infty} \frac{(-1)^{k_a} (x - x_i)^{k_a}}{(x_i - a)^{\mu(n,a)+k_a}} \binom{\mu(n,a) + k_a - 1}{k_a} \Bigg] =$$

$$= \sum_{k=0}^{\infty} \Bigg[\sum_{\substack{\kappa : X_n \setminus \{x_i\} \to \mathbb{Z}_+ \\ \sum_{a \in X_n \setminus \{x_i\}} \kappa(a) = k}} \prod_{a \in X_n \setminus \{x_i\}} \frac{(-1)^{\kappa(a)}}{(x_i - a)^{\mu(n,a)+\kappa(a)}} \binom{\mu(n,a) + \kappa(a) - 1}{\kappa(a)} \Bigg] (x - a)^k.$$

Hence the theorem follows immediately.

6 Triangular decomposition for Vandermonde matrices of Hermite sequences

Throughout this section, let N resp. $X = (x_0, x_1, \ldots)$ be a fixed positive integer and a sequence in $\mathbb{K}$. Consider the *Hermite-Vandermonde matrix*

$$\mathbf{V} := \Big[\, x^n\big|_{x_i}^{\nu^X(i,i)} \,\Big]_{i,n=0}^{N} = \Big[\binom{n}{\nu^X(i,i)} x_i^{n-\nu^X(i,i)} \Big]_{i,n=0}^{N} \tag{6.1}$$

corresponding to the system of equations $\sum_{n=0}^{N} c_n x^n \Big|_{x_i}^{\nu^X(i,i)} = y_i$ $(0 \le i \le N)$ established implicitly for the coefficients $c_0, \ldots, c_N \in \mathbb{K}$ of the Hermite polynomial $h_N^{X,Y}$ by Definition 2.3. We can state Rushanan's hidden idea treating the classical case in [2] in a concise formulation for the setting of Hermite-Vandermonde matrices as $\mathbf{V} = \mathbf{t}\mathbf{x}$ with the formal column resp row vectors

$$\mathbf{t} := \Big[\cdot\big|_{x_0}^{\nu^X(0,0)} \;\; \cdot\big|_{x_1}^{\nu^X(1,1)} \; \ldots \; \cdot\big|_{x_N}^{\nu^X(N,N)} \Big]^{\mathrm{T}}, \qquad \mathbf{x} := \Big[1 \; x \; x^2 \; \cdots \; x^N \Big]$$

where x is the indeterminate symbol for the polynomial ring $\mathbb{K}[x]$ and $\cdot\big|_a^k$ abbreviates the operation $\mathbb{K}[x] \to \mathbb{K}$, $p \mapsto p\big|_a^k$. Notice that if the points x_i are pairwise distinct, for $\mathbf{V}$ we get the classical Vandermonde matrix $\big[x_i^n\big]_{i,n=0}^{N}$.

Proposition 6.2 *The matrix* $\mathbf{V}$ *is invertible and we have the UL-decomposition*[2]

$$\mathbf{V}^{-1} = \Omega\Gamma \quad \text{where} \quad \Omega := \Big[\, \omega_n^X\big|_0^i \,\Big]_{i,n=0}^{N}, \qquad \Gamma := \mathrm{lowertr}\Big[\Gamma_{n,i}^X\Big]_{n,i=0}^{N}. \tag{6.3}$$

Consider any sequence $Y = (y_0, y_1, \ldots)$ in $\mathbb{K}$. We can express the defining relations $h_N^{X,Y}\Big|_{x_i}^{\nu^X(i,i)} = y_i$ $(i = 0, \ldots, N)$ in terms of the operator vector $\mathbf{t}$ as

$$\mathbf{t} h_N^{X,Y} = \mathbf{y} \quad \text{where} \quad \mathbf{y} := \Big[y_0 \; y_1 \; y_2 \; \cdots \; y_N \Big]^{\mathrm{T}}.$$

But, by Theorem 5.1 we have

$$\Big[\omega_0^X \; \omega_1^X \; \cdots \; \omega_N^X \Big] \Gamma \mathbf{y} = h_N^{X,Y}.$$

Observe as well that we can write $\mathbf{x}\Omega = \Big[\omega_0^X \; \omega_1^X \; \cdots \; \omega_N^X \Big]$. Since each Newton factor ω_n^X is a polynomial of degree n with unit leading coefficient, Ω is an *upper triangular* matrix with terms 1 in the main diagonal. It follows that

[2] The terms L and U stand for lower and upper triangular, respectively.

$\mathbf{t}\Big[\mathbf{x}\Omega\Big]\Gamma\mathbf{y} = \mathbf{y}$. Thus, from the arbitrariness of the sequence Y we conclude that

$$I_N = \Big[\delta_{in}\Big]_{i,n=0}^{N} = \mathbf{t}\Big[\mathbf{x}\Omega\Big]\Gamma = \mathbf{V}\Omega\Gamma \ . \qquad \square$$

Corollary 6.4 *For the entries with indices (i,n) resp. (n,i) with $0 \le i \le N$ of the triangular factor matrices Ω, Γ and their inverses we have*

$$\Omega_{i,n} = (-1)^{n-i} \sum_{0\le k_1<\cdots<k_{n-i}<n} x_{k_1}x_{k_2}\cdots x_{k_{n-i}} \, , \quad \Big[\Omega^{-1}\Big]_{i,n} = \sum_{\substack{d_0+\cdots+d_i=n-i\\ d_0,d_1,\ldots,d_i\ge 0}} x_0^{d_0}x_1^{d_1}\cdots x_i^{d_i} \ ;$$

$$\Gamma_{n,i} = \Gamma_{n,i}^X \text{ (defined in 4.8)}, \Big[\Gamma^{-1}\Big]_{n,i} = \sum_{0\le k_1<\cdots<k_{[i-\nu^X(n,n)]_+}<i} (x_n - x_{k_1})\cdots(x_n - x_{k_{[i-\nu^X(n,n)]_+}}).$$

Using the results of the previous sections, we can provide closed formulas for all the entries in the matrices Γ, Γ^{-1} appearing in the LU- resp. UL-decompositions $\mathbf{V} = \Gamma^{-1}\Omega^{-1}$ and $\mathbf{V}^{-1} = \Omega\Gamma$. The terms $\Gamma_{n,i}^X$ have already been calculated in several ways. Also

$$\Gamma^{-1} = \mathbf{V}\Omega = \mathbf{t}\Big[\mathbf{x}\Omega\Big] = \mathbf{t}\Big[\omega_0^X \ \omega_1^X \ \cdots \ \omega_N^X\Big] \ .$$

Since $\omega_i^X = (x-x_0)\cdots(x-x_{i-1}) = \Big[(x-x_n)+(x_n-x_0)\Big]\cdots\Big[(x-x_n)+(x_n-x_{i-1})\Big]$,

$$\Big[\Gamma^{-1}\Big]_{n,i} = \omega_i^X\Big|_{x_n}^{\nu^X(n,n)} = \sum_{0\le k_1<\cdots<k_{[i-\nu^X(n,n)]_+}} (x_n-x_{k_1})\cdots(x_n-x_{k_{[i-\nu^X(n,n)]_+}}) \quad (0\le i\le n\le N).$$

The matrices $\Omega = \Big[\varphi(x_0,\ldots,x_N)\Big]_{i,n=0}^{N}$, $\Omega^{-1} = \Big[\psi(x_0,\ldots,x_N)\Big]_{i,n=0}^{N}$ are well known [4] for the real case with $x_i \ne x_j$ $(i \ne j)$. Their entries are polynomials with integer coefficients in the variables $x_0,\ldots,x_N$. By continuity, the identities $\delta_{ij} = \sum_\ell \varphi_{i\ell}\psi_{\ell j}$ hold for all $x_0,\ldots,x_N \in \mathbb{R}$. Actually, a polynomial expression in several variables that involves only integer coefficients and vanishes for all real substitutions must have vanishing coefficients. Therefore the formulas obtained there admit a straightforward extension to our cases for the entries of Ω resp. Ω^{-1} stated.

Remark 6.5 We can express the above results in closed combinatorial formulas as follows. All the below statements follow immediately from the analogous results above, given in terms of $x_0,\ldots,x_N$ by the aid of interpreting the number of repetitions in terms of standard combinations and combinations with repetitions, respectively. The binomial coefficients have to be taken with mod_χ if $\chi := \mathrm{char}(\mathbb{K}) \ne 0$. *In terms of the rearrangement vector $\mathbf{a}^{(N)}$ and the associated multiplicities $r_n, m_s^{(n)}$ introduced in* (3.1) *resp.* (3.2), *for the non-trivial entries of the LU- resp. UL-decompositions $\mathbf{V} = \Gamma^{-1}\Omega^{-1}$ and $\mathbf{V}^{-1} = \Omega\Gamma$ with $0 \le i \le n \le N$ below we have*

$$\Gamma_{n,i} = \sum_{\substack{k_1+\cdots+k_{r(n)}=\sigma^X(n,i)\\ k_j=0,\ \text{if}\ a_{k_j}=x_i}} \prod_{s=1,a_s\neq x_i}^{r(n)} (-1)^{k_s}(x_i-a_s)^{-[m_s^{(n)}+1+k_s]}\binom{m_s^{(n)}+k_s}{k_s}$$

$$\left[\Gamma^{-1}\right]_{n,i} = \omega_i^X\Big|_{x_n}^{\nu^X(n,n)} = \sum_{0\le k_1<\cdots<k_{[i-\nu^X(n,n)]_+}<i} (x_n-x_{k_1})\cdots(x_n-x_{k_{[i-\nu^X(n,n)]_+}})\ ,$$

$$\Omega_{i,n} = (-1)^{n-i}\sum_{\substack{m_1+\cdots+m_{r(n)}=n-i\\ m_1,\ldots,m_{r(n)}\ge 0}} \binom{\mu^X(n-1,a_1)}{m_1}\cdots\binom{\mu^X(n-1,a_{r(n)})}{m_{r(n)}} a_1^{m_1}\cdots a_{r(n)}^{m_{r(n)}},$$

$$\left[\Omega^{-1}\right]_{i,n} = \sum_{\substack{\ell_1+\cdots+\ell_{r(i)}=n-i\\ \ell_1,\ldots,\ell_{r(i)}\ge 0}} \binom{\mu^X(i,a_1)+\ell_1-1}{\ell_1}\cdots\binom{\mu^X(i,a_{r(i)})+\ell_{r(i)}-1}{\ell_{r(i)}} a_1^{\ell_1}\cdots a_{r(i)}^{\ell_{r(i)}}.$$

Example 6.6 As in Example 2.8, let X again be the sequence $[0,1,0,1,\ldots]$. We calculate the explicit form of the entries $\Gamma_{n,i}$, $[\Gamma^{-1}]_{n,i}$, $\Omega_{i,n}$, $[\Omega^{-1}]_{i,n}$ for the classical case, i.e., over the reals. In terms of Remark 6.5, then we have $a_1=0$, $a_2=1$, $r(n)=\min\{2,n+1\}$, $\omega_n^X = x^{\lceil n/2\rceil}(x-1)^{\lfloor n/2\rfloor}$. Observe that the product terms in the expressions of $\Gamma_{n,i}, \Omega_{i,n}, [\Omega^{-1}]_{i,n}$ vanish for all but one case: we get $\Omega_{i,n}$ with $m_1=0, m_2=n-i$ and $\mu^X(n-1,1)=\#\{j\le n-1:\ x_j=1\}=\lfloor\frac{n}{2}\rfloor$, $[\Omega^{-1}]_{i,n}$ with $\ell_1=0, \ell_2=n-i$ and the non-vanishing term for $\Gamma_{n,i}$ appears with the choice of $s\in\{1,2\}$ such that $\{a_s\}=\{0,1\}\setminus\{x_i\}$ and $m_s^{(n)}=\#\{j\le n:\ x_j=x_i\}=\lceil\frac{n+\mathrm{mod}_2(n-i)}{2}\rceil-1$ with $k_s=\sigma^X(n,i)=\#\{j\in(i,n]:\ x_j=x_i\}=\lfloor\frac{n-i}{2}\rfloor$, respectively. It follows

$$\Omega_{i,n} = (-1)^{n-i}\binom{\lfloor n/2\rfloor}{n-i},\quad [\Omega^{-1}]_{i,n} = \binom{\lceil i/2\rceil+n-i-1}{n-i},$$

$$\Gamma_{n,i} = (-1)^{[m\ \text{if}\ 2|i,\ k\ \text{else}]}\binom{m+k}{k}\quad\text{with}\quad \begin{array}{l} k:=\lfloor(n-i)/2\rfloor,\\ m:=\lceil n+\mathrm{mod}_2(n-i))/2\rceil-1.\end{array}$$

To calculate $[\Gamma^{-1}]_{n,i}$, observe that the products $(x_n-x_{k_1})\cdots(x_n-x_{k_{[i-\nu^X(n,n)]_+}})$ do not vanish either if $x_n=0$ and $x_{k_1}=\cdots=x_{k_{[i-\nu^X(n,n)]_+}}=1$ or if $x_n=1$ and $x_{k_1}=\cdots=x_{k_{[i-\nu^X(n,n)]_+}}=0$. Such products appear $\binom{\#\{k<i:\ x_k\neq x_n\}}{i-\nu^X(n,n)} = \binom{\lceil(i-\mathrm{mod}_2(n-i))/2\rceil}{i-\lfloor n/2\rfloor}$ times. Thus

$$[\Gamma^{-1}]_{n,i} = (-1)^{[(i-\lfloor n/2\rfloor)\ \text{if}\ 2|n,\ 0\ \text{else}]}\binom{\lceil(i-\mathrm{mod}_2(n-i))/2\rceil}{i-\lfloor n/2\rfloor}.$$

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which seems to be extremely worth for future studies: *Is there a way to study Birkhoff's concept* [7] *of the Hermite type interpolation also in this general algebraic setting?*

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References

[1] A. Spitzbart, A generalization of Hermite's interpolation formula, Amer. Math. Monthly 67 (1960), 42–46.

[2] J. J. Rushanan, On the Vandermonde matrix. Amer. Math. Monthly 96 (1989), no. 10, 921–924.

[3] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, Springer, New York, 2002.

[4] H. Oruc, G. M. Phillips, Explicit factorization of the Vandermonde matrix. Linear Algebra Appl. 315 (2000), no. 1-3, 113–123.

[5] H. Oruc, LU factorization of the Vandermonde matrix and its applications. Appl. Math. Lett. 20 (2007), no. 9, 982–987.

[6] C. de Boor, Divided differences, Surv. Approx. Theory, 1 (2005), 46–69 (electronic).

[7] G. D. Birkhoff, General mean value and remainder theorems with applications to mechanical differentiation and quadrature, Trans. Amer. Math. Soc., 7 no. 1 (1906), 107–136.