ON THE ALGEBRAIC CLASSIFICATION OF BOUNDED CIRCULAR DOMAINS

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ABSTRACT

The category of the partial Jordan structures associated to bounded balanced domains by Kaup's canonical construction is characterised in terms of topological algebra. It is shown that this category coincides with the category of weakly commutative partial JB*-triples. The converse question is also studied: which domains lead by Kaup's construction to a given weakly commutative partial JB*-triple?

1. Introduction

By a celebrated theorem of Kaup [7] there is a bijective functor between the categories of bounded symmetric domains in complex Banach spaces and Jordan triple product *-algebras with positive spectrum, the so-called JB*-triples. In finite dimensions this result contains the holomorphic classification of bounded symmetric domains due to Cartan [3] using Lie theoretical methods and to the school of Koecher [10; 11] via a Jordan theoretical approach respectively. The main point of both approaches is the observation that a bounded balanced symmetric domain is the integral manifold through zero of its complete holomorphic vector fields. It is well known that the complete holomorphic vector fields of a bounded balanced domain determine canonically a *partial* Hermitian Jordan triple star algebra (partial J*-triple for short) structure on the space [2; 9].

The purpose of this paper is to examine the converse question. Given a partial J*-triple structure $(E, E_0, \{*\})$ on a complex Banach space E (for def. see [13; 2] or Section 2 below), when does there exist a neighbourhood B of zero such that the integral manifold of the family of vector fields $\{(a - \{za^*z\})\, \partial/\partial z \colon a \in E_0\}$ through B is a bounded balanced domain? That is, in the terminology of [13], under which algebraic conditions are partial J*-triples geometric?

We solve this problem completely. Geometric partial J*-triples are exactly, up to linear isomorphisms, the partial JB*-triples satisfying the identity of weak commutativity $\{\{xa^*x\}b^*x\} = \{xa^*\{xb^*x\}\}$ $(a, b \in E_0)$.

We begin the paper with some considerations that may have independent interest. We prove that in a weakly commutative partial JB*-triple $(E, E_0, \{*\})$

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the germs around the unit ball of E_0 of the mappings $\exp[(a - \{za^*z\})\, \partial/\partial z]$ with $a \in E_0$ generate a Banach–Lie group **G**. This implies a Cartan-type uniqueness theorem and hence a semidirect product decomposition into a linear and an exponential quadratic part for **G**—a germ analogue of a result of Vigué and Isidro [16]. With the aid of this decomposition we show that, given a connected circular neighbourhood B of the origin that is invariant under the Jordan automorphisms of E with the property that the integral curves through B are complete in E for all the vector fields $(a - \{za^*z\})\, \partial/\partial z$, the orbit $\cup_{a \in E_0} \exp[(a - \{xa^*x\})\, \partial/\partial x](B)$ is the integral manifold through B of the Banach–Lie algebra of vector fields $\{(a + iL - \{xa^*x\})\, \partial/\partial x\colon a \in E_0, L \in \mathrm{Der}(E)\}$. This shape estimate is a circular generalisation of a similar theorem of Panou [12] for bicircular partial JB*-triples. This is our main geometrical tool in proving the equivalence of geometric partial J*-triples and weakly commutative partial JB*-triples.

2. The group of automorphism germs

Throughout this section let $(E, E_0, \{*\})$ denote a partial JB*-triple, i.e. E is a complex Banach space, E_0 a closed complex subspace of E and

$$\{*\}\colon (x, a, y) \mapsto \{xa^*y\}$$

is a continuous real-trilinear operation $E \times E_0 \times E \to E$ symmetric bilinear in x, y and conjugate linear in the variable a, such that

$$\{E_0E_0^*E_0\} \subset E_0 \quad \text{and} \quad \text{for all } a, b, c \in E_0, \quad x, y \in E$$

$$\{ab^*\{xc^*y\}\} = \{\{ab^*x\}c^*y\} - \{x\{ba^*c\}^*y\} + \{xc^*\{ab^*y\}\} \tag{J1}$$

$$\|\{aa^*a\}\| = \|a\|^3 \quad \text{for all } a \in E_0 \tag{J2}$$

$$a \,\square\, a^* \in \mathrm{Her}_+(E) \quad \text{for all } a \in E_0. \tag{J3}$$

Here $v \,\square\, a^*$ is the notation for the linear operator $x \mapsto \{va^*x\}$ and

$$\mathrm{Her}_+(E) := \{\alpha \in L(E)\colon \|\exp(\zeta\alpha)\| \leq 1 \quad \text{if } \mathrm{Re}(\zeta) \leq 0\}$$

is the family of E-hermitian operators with non-negative spectrum. (As usual, $L(E)$ denotes the Banach algebra of bounded linear operators in E with the operator norm.)

We shall denote by D_0 the unit ball of E_0. It is well known [8] that D_0 is the integral manifold through 0 of the family of vector fields

$$\mathscr{P} := \{Z_a\colon a \in E_0\} \quad \text{where } Z_a := (a - \{za^*z\})\frac{\partial}{\partial z}.$$

We regard $\mathscr{P}$ as a real-linear submanifold of the topological Lie algebra $\mathscr{H}$ of all holomorphic vector fields on E endowed with the topology of local uniform convergence and the Poisson product

$$\left[h_1(z)\frac{\partial}{\partial z}, h_2(z)\frac{\partial}{\partial z}\right] := (\mathrm{d}h_1(z)h_2(z) - \mathrm{d}h_2(z)h_1(z))\frac{\partial}{\partial z}.$$

In particular (see e.g. [5, prop. 10.4, 10.10]), by setting

$$L_b := i \cdot b \,\Box\, b^*(z)\frac{\partial}{\partial z} \qquad (b \in E_0),$$

for any $a, b \in E_0$

$$\begin{aligned}
[Z_a, Z_b] &= L_{a+ib} - L_{a-ib} + (\{\{za^*z\}b^*z\} - \{\{zb^*z\}a^*z\})\frac{\partial}{\partial z} \\
[L_a, Z_b] &= Z_{L_a b} \\
[L_a, L_b] &= L_{\{bb^*a\}+ia} - \lambda_{\{bb^*a\}-ia}.
\end{aligned} \tag{2.1}$$

Henceforth let $\mathscr{L}$ denote the closed real-linear hull

$$\mathscr{L} := \mathrm{Span}_{\mathbb{R}}\{L_a : a \in E_0\} \cup \left\{i \cdot z\frac{\partial}{\partial z}\right\}$$

in $\mathscr{H}$. It is easy to see that the sum $\mathscr{P} + \mathscr{L}$ is topologically direct in $\mathscr{H}$. Moreover,

$$\left\|Z_a + l(z)\frac{\partial}{\partial z}\right\| := \|a\| + \|l\| \qquad \left(a \in E_0, l(z)\frac{\partial}{\partial z} \in \mathscr{L}\right)$$

is a Banach norm on $\mathscr{P} + \mathscr{L}$, determining its relative topology from $\mathscr{H}$. Therefore we have the following lemma.

Lemma 2.2. *$\mathscr{P} + \mathscr{L}$ is a Banach–Lie subalgebra of $\mathscr{H}$ if and only if*

$$\{\{za^*z\}b^*z\} = \{\{zb^*z\}a^*z\} \qquad (a, b \in E_0, z \in E). \tag{J4}$$

Definition 2.3. We call a partial JB*-triple *weakly commutative* if it satisfies postulate (J4).

Remark 2.4. It is well known [6; 12; 5, prop. 7.10.(c)] that geometric partial JB*-triples are weakly commutative. Furthermore, if we have $E = E_0$ then (J4) is an algebraic consequence of the main identity (J1) (see [1]).

Recall that given a vector field $h(z)\,\partial/\partial z \in \mathscr{H}$, its exponential can be defined in terms of the maximal solutions $z_{h,x}(\,\cdot\,)$ of the initial value problems

$$\frac{d}{dt}z(t) = h(z(t)) \qquad z(0) = x$$

as follows

$$\exp\left(t \cdot h(z)\frac{\partial}{\partial z}\right)(x) := z_{h,x}(t)$$

whenever $z_{h,x}$ is defined on some neighbourhood of [0, t] in $\mathbb{R}$.

We know that every vector field $Z \in \mathcal{P} + \mathcal{L}$ is complete in D_0, i.e. for any $t \in \mathbb{R}$, $\exp(tZ)$ is defined on D_0 and $\exp(tZ)(D_0) = D_0$. Since the elements of $\mathcal{P} + \mathcal{L}$ are of polynomial type, it follows from the Picard–Lindelöf theorem that for each $Z \in \mathcal{P} + \mathcal{L}$, $\exp(Z)$ is a holomorphic mapping defined on some neighbourhood of D_0.

Henceforth let us denote by $\mathcal{F}$ the family of all holomorphic mappings with stable subset D_0 defined on some neighbourhood of D_0 and having values in E. Notice that $\mathcal{F}$ is a semigroup with respect to composition. Furthermore let $\mathcal{G}$ be the subsemigroup of $\mathcal{F}$ generated by the family $\{\exp(Z) : Z \in \mathcal{P} + \mathcal{L}\}$. Given $\phi \in \mathcal{F}$, we shall write $\phi^{\sim}$ for the *germ* of ϕ *around* D_0, i.e.

$$\phi^{\sim} := \{\psi \in \mathcal{F}: \phi | U = \psi | U \quad \text{for some neighbourhood } U \text{ of } D_0\}.$$

Observe that $\phi_1^{\sim} = \psi_1^{\sim}$ and $\phi_2^{\sim} = \psi_2^{\sim}$ imply $(\phi_1 \circ \phi_2)^{\sim} = (\psi_1 \circ \psi_2)^{\sim}$ in $\mathcal{F}$. Thus the family $\mathbf{F} := \{\phi^{\sim} : \phi \in \mathcal{F}\}$ carries a natural composition semigroup structure. Since the range of $\exp(Z)$ coincides with the domain of $\exp(-Z)$ and $\exp(-Z) \circ \exp(Z) = \mathrm{id}$ on dom $\exp(Z)$ for $Z \in \mathcal{P} + \mathcal{L}$, the germ image

$$\mathbf{G} := \{\phi^{\sim} : \phi \in \mathcal{G}\} \quad \text{with operation } \phi_1^{\sim}\phi_2^{\sim} := (\phi_1 \circ \phi_2)^{\sim} \quad (\phi_1, \phi_2 \in \mathcal{G})$$

is a subgroup of $\mathbf{F}$.

Definition 2.5. We call $\mathbf{G}$ the *group of automorphism germs* in $(E, E_0, \{*\})$. This terminology is motivated by our purpose of looking for a bounded circular neighbourhood D of D_0 where any element of $\mathbf{G}$ is the germ of some holomorphic automorphism of D.

Theorem 2.6. *If $(E, E_0, \{*\})$ is a weakly commutative partial JB*-triple then* $\mathbf{G}$ *can be equipped with a connected Banach–Lie group structure whose Lie algebra is isomorphic to* $\mathcal{P} + \mathcal{L}$.

Proof. We modify slightly Upmeier's arguments [14] as follows.

Identify the vector field $Z := h(z)\,\partial/\partial z$ with the differential operator $f \mapsto [df(z)h(z): z \in \mathrm{dom}(f)]$ as usual. Then

$$f \circ \exp(Z)|U_{2e/M} = \sum_{n=0}^{\infty} \frac{1}{n!} Z^n(f|U_{2e/M}) \quad \text{where } M := \sup_{z\in U} \|h(z)\| \tag{2.7}$$

whenever $U \subset E$ is an open set, $f\colon U \to E$ is a bounded holomorphic function and U_r is the inner parallel set $U_r := \{z \in U\colon \inf_{x\notin U} \|z-x\| > r\}$ (cf. [5, (4.7), lemma 6.45]). By writing C for the Campbell–Hausdorff series in the Banach–Lie algebra $\mathcal{P} + \mathcal{L}$, it follows that

$$\begin{aligned}\exp(Z_1)\exp(Z_2)|B &= \sum_{k=0}^{\infty} \frac{1}{k!} Z_1^k \sum_{l=0}^{\infty} \frac{1}{l!} Z_1^l\,(\mathrm{id}_B) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} C(Z_1, Z_2)^n(\mathrm{id}_B) = \exp(C(Z_1, Z_2))|B\end{aligned} \tag{2.8}$$

whenever B is the unit ball of E and $Z_k = h_k(z)\,\partial/\partial z$ $(k = 1, 2)$ have such small norms that $(Z_1, Z_2) \in \mathrm{dom}(C)$ and $\sup_{\|z\|\leq 2e+1} \|h_k(z)\| < 1$ $(k = 1, 2)$.

Finally notice that $Z \mapsto \exp(Z)|D_0$ is injective on some neighbourhood of 0 in $\mathcal{P} + \mathcal{L}$ (see e.g. [5, lemma 6.47]).

Hence the theorem is a consequence of the following proposition.

Proposition 2.9. *Let L be a Banach–Lie algebra, G a group. Suppose Φ is a bijective mapping of some neighbourhood U of 0 in L into G such that*

(i) *$\Phi(U)$ generates G and*

(ii) *$\Phi^{-1}(\Phi(Z_1), \Phi(Z_2)) = C(Z_1, Z_2)$ for Z_1, Z_2 in some neighbourhood V of 0 in L.*

Then there is a (unique) Hausdorff group topology $\mathcal{T}$ on G such that for any $g \in G$, $\{g\Phi(\epsilon U)\colon \epsilon > 0\}$ is a filter base for the $\mathcal{T}$-neighbourhoods around g. Moreover, there is a complex manifold structure $\mathcal{M}$ on G compatible with the topology $\mathcal{T}$ such that for a suitable neighbourhood W of 0 the mappings $\Phi_g := g\Phi|W$ $(g \in G)$ form a complete system of local charts for $\mathcal{M}$.

PROOF. We may assume without loss of generality that U is the unit ball of L and $V = \delta U$ for some $\delta > 0$. Then $\{\Phi(\epsilon U)\colon \epsilon > 0\}$ is a filter base shrinking to the unit e of G. (Indeed, $\bigcap_{\epsilon>0} \Phi(\epsilon U) = \{\Phi(0)\}$ and $\Phi(0)\Phi(Z) = \Phi(C(0, Z)) = \Phi(Z)$ $(Z \in L)$ whence $\Phi(0) = e$.) By (ii), $\Phi(-Z) = \Phi(Z)^{-1}$ for $Z \in V$. Thus $\Phi(\epsilon U) = \Phi(-\epsilon U) = \Phi(U)^{-1}$ whenever $0 < \epsilon \leq \delta$. To establish the existence of the topology $\mathcal{T}$ with the required properties, by the arguments of the proof of [5, theorem 6.52] we have to show that given $\epsilon > 0$ and $g \in G$

$$\Phi(\lambda U)\Phi(\lambda U) \subset g\Phi(\epsilon U)g^{-1} \qquad \text{for some } \lambda > 0. \tag{2.10}$$

By assumption (i), we may write $g = \Phi(Z_n)\,\Phi(Z_{n-1})\ldots\Phi(Z_1)$ with a suitable sequence $Z_1, \ldots, Z_n \in V$. It is well known* that, for fixed k,

$$C(Z_k, C(Z, -Z_k)) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}(Z_k))^n Z \qquad (Z \in L)$$

where $\mathrm{ad}(Z_k)$ is the adjoint $Z \mapsto [Z_k, Z]$. Hence $Z \mapsto \Phi^{-1}(\Phi(Z_k)\Phi(Z) \times \Phi(Z_k)^{-1})$ is well defined on some neighbourhood of 0. Moreover, it is the restriction of some invertible linear operator. Therefore $Z \mapsto \Phi^{-1}(g\Phi(Z)g^{-1})$ is also well defined on some neighbourhood of 0 as the restriction of an invertible linear operator. Hence $(Y_1, Y_2) \mapsto \Phi^{-1}(g^{-1}\Phi(Y_1)\Phi(Y_2)g)$ is well defined and continuous in some neighbourhood of $0 \in L \times L$, proving (2.10).

Let us choose the neighbourhood W of 0 so that $C(W, -W) \subset V$. From the construction of $\mathcal{T}$ it is immediate that Φ_g is a local homeomorphism for any $g \in G$. It only remains to verify that $\Phi_{g_1}^{-1} \circ \Phi_{g_2}$ is holomorphic for any $g_1, g_2 \in G$. Let $g \in \Phi_{g_1}(W) \cap \Phi_{g_2}(W)$; i.e., $g = g_1\Phi(Z_1) = g_2\Phi(Z_2)$ for some $Z_1, Z_2 \in W$. Thus $g_1^{-1}g_2 = \Phi(Z_2)\Phi(Z_1)^{-1} = \Phi(C(Z_2, -Z_1))$. Hence for any $Z \in V$ we have

$$\begin{aligned}\Phi_{g_1}^{-1}(\Phi_{g_2}(Z)) &= \Phi^{-1}(g_1^{-1}g_2\Phi(Z)) = \Phi^{-1}(\Phi(C(Z_2), -Z_1))\Phi(Z)) \\ &= C(C(Z_2, -Z_1), Z)\end{aligned}$$

which is a holomorphic mapping of Z.

3. Canonical decomposition of automorphism germs

Throughout this section let $(E, E_0, \{*\})$ be a weakly commutative partial JB*-triple and use the notations of the previous section.

According to the construction 2.9 of the Banach–Lie group $\mathbf{G}$ of automorphism germs, the continuous one-parameter subgroups of $\mathbf{G}$ have the form

$$t \mapsto \exp(tZ)^{\sim} \qquad (Z \in \mathcal{P} + \mathcal{L}).$$

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*For example, this is an easy consequence of the formulation of C in terms of non-commutative formal power series. The series of $\log(e^A e^Z e^{-A})$ is $\Sigma_k\, 1/k!\, A^k Z \Sigma_m (-1)^m/m!\, A^m$ (because $[d/dt\, e^A e^{tZ} e^{-A}] = (e^A Z e^{-A})[e^A e^{tZ} e^{-A}]$ whence $e^A e^Z e^{-A} = \exp[e^A Z e^{-A}]$, cf. [15]). According to [4, section 3], by writing $[X_0, \ldots, X_n] := [\cdots[[X_0, X_1], X_2], \ldots, X_n] = (-1)^n(\mathrm{ad}\, X_n)\ldots(\mathrm{ad}\, X_1)X_0$, we have

$$\begin{aligned}C(C(A, Z), -A) &= C(A, C(Z, -A)) \\ &= \sum_{k,m} \frac{(-1)^m}{k!m!} \frac{1}{k+m+1} [\underbrace{A, \ldots, A}_{k}, Z, \underbrace{A, \ldots, A}_{m}] \\ &= \sum_m \frac{(-1)^n}{m!} \left\{ \frac{1}{m+1} [Z, \underbrace{A, \ldots, A}_{m}] + \frac{1}{m+2} [A, Z, \underbrace{A, \ldots, A}_{m}] + \right\} = \sum_m \frac{1}{m!} (\mathrm{ad}\, A)^m Z.\end{aligned}$$

Thus given any $g \in \mathbf{G}$, the tangent space $T_g(\mathbf{G})$ consists of the vectors

$$v_g(Z)\colon \varphi \mapsto \frac{d}{dt}\Big|_0 \varphi(\exp(tZ)^{\sim} g) \qquad (Z \in \mathscr{P} + \mathscr{L}). \tag{3.1}$$

Remark, furthermore, that we may define the derivatives of the automorphism germs at the points of D_0 (the unit ball of E_0) by the formula

$$(\phi^{\sim})_x^{(n)} := \mathrm{d}_x^n \phi \qquad (\phi \in \mathscr{G}, x \in D_0)$$

because $\phi^{\sim} = \psi^{\sim}$ implies the coincidence of ϕ with ψ in some neighbourhood of D_0 for any couple $\phi, \psi \in \mathscr{G}$. (The symbol d_x^n means the nth Fréchet derivative at the point x.) Since $\mathbf{G}$ is generated by the family $\{\exp(Z)^{\sim}\colon Z \in \mathscr{P} + \mathscr{L}\}$, it readily follows from (2.7) that the mappings $(x, g) \mapsto g_x^{(n)}$ $(n = 0, 1, \ldots)$ are real analytic from $D_0 \times \mathbf{G}$ into $L_s^n(E)$, the space of all symmetric n-linear operations $E^n \to E$.

Definition 3.2. Henceforth we denote by $\mathbf{L}$ the subgroup of $\mathbf{G}$ generated by the family $\{\exp(L)^{\sim}\colon L \in \mathscr{L}\}$.

Since $\mathscr{L} \subset \{l(z)\, \partial/\partial z\colon il \in \mathrm{Her}(E)\}$, every element of $\mathbf{L}$ is the germ around D_0 of some linear E-unitary mapping.

Theorem 3.3. *For any $g \in \mathbf{G}$ there exists a unique couple $a \in E_0$, $u \in \mathbf{L}$ such that $g = \exp(Z_a)^{\sim} u$.*

PROOF. Let $\Phi\colon U \to \mathbf{G}$ be the local chart around 0 of $\mathbf{G}$ constructed in Theorem 2.6. Thus U is some neighbourhood of 0 in $\mathscr{P} + \mathscr{L}$ and $\Phi\colon Z \mapsto \exp(Z)^{\sim}$. Let P_0 denote the projection of $\mathscr{P} + \mathscr{L}$ onto $\mathscr{L}$ along $\mathscr{P}$ and let P_1 be its complement. By Proposition 2.9 (ii) we have

$$\exp(P_1 Z)^{\sim} \exp(P_0 Z)^{\sim} = \exp(C(P_1 Z, P_0 Z))^{\sim} \qquad (Z \in V) \tag{3.4}$$

in some neighbourhood $V \subset U$ of 0 where C denotes the Campbell–Hausdorff series in $\mathscr{P} + \mathscr{L}$. Observe that

$$\mathrm{d}_0^1[Z \mapsto C(P_1 Z, P_0 Z)] = P_1 + P_0 = \mathrm{id}_{\mathscr{P}+\mathscr{L}}. \tag{3.5}$$

Therefore, by the inverse mapping theorem, $V_1 := Z \cap \{C(P_1 Z, P_0 Z)\colon Z \in V\}$ is a neighbourhood of 0 in $\mathscr{P} + \mathscr{L}$. Thus, by (3.5), every $g \in \Phi(V)$ admits the decomposition $g = \exp(P_1 Z)^{\sim} \exp(P_0 Z)^{\sim}$ for a unique $Z \in V$. By the definition of P_0 and P_1 this means

$$g \in \{\exp(Z_a)^{\sim} \exp(L)^{\sim}\colon a \in E_0, L \in \mathscr{L}\} \qquad (g \in V_1). \tag{3.6}$$

Recall [7; 16; 5, cor. 10.37] that

$$\Gamma: b \mapsto \sum_{n=0}^{\infty} \frac{1}{2n+1} (b \square b^*)^n b \qquad (b \in E_0)$$

is a real-bianalytic mapping of D_0 onto E_0 whose inverse is $a \mapsto \exp(Z_a)(0)$. Since $0 = u(0)$ ($:= u_0^{(0)}$) for all $u \in \mathbf{L}$, it follows that

$$a = \Gamma(g(0)), u = \exp(-Z_{\Gamma(g(0))})^{\sim} g \quad \text{whenever } g = \exp(Z_a)^{\sim} u, \quad u \in \mathbf{L}. \tag{3.7}$$

(We write again simply $g(0)$ for the value $g_0^{(0)}$ and note that $g(0) \in D_0$ for all $g \in \mathbf{G}$.)

For each $g \in \mathbf{G}$ define

$$a_g := \Gamma(g(0)) \quad \text{and} \quad u_g := \exp(-Z_{\Gamma(g(0))})^{\sim} g.$$

Since $h = \exp(Z_{a_h})^{\sim} u_h$ for any $h \in \mathbf{G}$, by (3.7) it only remains to verify that

$$u_h \in \mathbf{L} \qquad (h \in \mathbf{G}). \tag{3.8}$$

Proof of (3.8). It is clear that $g \mapsto u_g$ is a real-analytic mapping $\mathbf{G} \to \mathbf{G}$. Let us first study the behaviour of its derivative on the tangent manifold $T(\mathbf{G})$ of $\mathbf{G}$. We check that, with the charts (3.1) of $T(\mathbf{G})$,

$$d[g \mapsto u_g] v_h(Z) \in v_{u_h}(\mathscr{L}) \qquad (h \in \mathbf{G}). \tag{3.9}$$

Indeed, the mapping $(Z, g) \mapsto u_{\exp(Z)_g^{\sim}} u_g^{-1}$ is real-analytic on $(\mathscr{P} + \mathscr{L}) \times \mathbf{G}$. Hence $(Z, g) \mapsto \Phi^{-1}(u_{\exp(Z)_g^{\sim}} u_g^{-1})$ is well defined and real-analytic on some neighbourhood of $\{0\} \times \mathbf{G}$. Therefore

$$\Lambda: (Z, g) \mapsto \left.\frac{d}{dt}\right|_0 \Phi^{-1}(u_{\exp(tZ)_g^{\sim}} u_g^{-1})$$

is also well defined and real-analytic on the whole $(\mathscr{P} + \mathscr{L}) \times \mathbf{G}$. Observe that by (3.1)

$$d[g \mapsto u_g] v_h(Z) = v_{u_h}(\Lambda(Z, g)) \qquad (Z \in \mathscr{P} + \mathscr{L}, h \in \mathbf{G}). \tag{3.10}$$

Consider the mapping

$$L(Z) := \Phi^{-1}(u_{\exp(Z)^{\sim}}) \qquad (Z \in V_1).$$

By Proposition 2.9 (ii), for some neighbourhood $V_2 \subset V_1$ of 0 in $\mathscr{P} + \mathscr{L}$ we have

$$\begin{aligned}\Phi^{-1}(u_{\exp(tZ)^{\sim} \exp(Y)^{\sim}} u_{\exp(Y)^{\sim}}^{-1}) &= \Phi^{-1}(\Phi(L(C(tZ, Y)))\Phi(L(Y))) \\ &= C(L(tZ, Y), L(Y)) \quad \text{if } tZ, Y \in V_2.\end{aligned}$$

Since L has values in the Banach–Lie algebra $\mathcal{L}$, $\Lambda((\mathcal{P}+\mathcal{L})\times V_2)\subset\mathcal{L}$. Thus, by real-analyticity, $\operatorname{ran}\Lambda\subset\mathcal{L}$, which proves (3.9).

Now we return to the proof of (3.8). The key observation is that the construction of Proposition 2.9 applies also to $\mathbf{L}$. Thus $\mathbf{L}$ is a Banach–Lie group when endowed with the group topology where $\{\Phi(\mathcal{L}\cap\epsilon U):\epsilon>0\}$ is a basis for the filter of e-neighbourhoods and when we equip $\mathbf{L}$ with the manifold structure of local charts $\Phi_u|(\mathcal{L}\cap U)$ $(u\in\mathbf{L})$. Now we may view the tangent manifold $T(\mathbf{L})$ as submanifold of $T(\mathbf{G})$ in the usual way. For $g\in\mathbf{L}$ and $Z\in\mathcal{L}$ we identify the vector

$$w_g(L):=\left[C^\infty(\Phi_g(\mathcal{L}\cap U))\ni\chi\mapsto\frac{d}{dt}\bigg|_0\chi(\exp(tZ)^\sim g)\right]$$

with $v_g(L)$ defined by (3.1) for functions $\varphi\in C^\infty(\Phi_g(U))$.

Let us fix $h\in\mathbf{G}$ arbitrarily. Since $\mathbf{G}$ is a connected Banach–Lie group, we may choose a $\mathscr{C}^\infty$-smooth curve $t\mapsto g(t)$ parametrised on some open interval $I\supset[0,1]$ and taking values in $\mathbf{G}$ such that $g(0)=e$ and $g(1)=h$. By (3.10) there exists a $\mathscr{C}^\infty$-smooth curve $L: I\to\mathcal{L}$ such that

$$\frac{d}{dt}u_{g(t)}=\mathrm{d}[g\mapsto u_g]\frac{d}{dt}g(t)=v_{u_{g(t)}}(L(t)).$$

Since $\mathbf{L}$ is a Banach–Lie group whose tangent manifold is $\cup_{g\in\mathbf{L}}v_g(\mathcal{L})$, the initial value problem in $\mathbf{L}$

$$\frac{d}{dt}y(t)=v_{y(t)}(L(t))\qquad y(0)=e\tag{3.11}$$

admits a solution $k: I\to\mathbf{L}$. Thus both $k(\cdot)$ and $u_{g(\cdot)}$ solve (3.11) on the interval I. By uniqueness, they must coincide. In particular $u_h=u_{g(1)}=k(1)\in\mathbf{L}$ which proves (3.8).

Corollary 3.12. *Given $Z_1,Z_2,\ldots,Z_n\in\mathcal{P}+\mathcal{L}$, there exists (a unique) $a\in E_0$ and a linear E-unitary operator W such that* $\exp(Z_1)\circ\cdots\circ\exp(Z_n)$ *coincides with* $\exp(Z_a)\circ W$ *on some neighbourhood of D_0. The operator W belongs to the subgroup of $GL(E)$ generated by* $\exp(\mathcal{L})$.

Corollary 3.13. *Every $g\in\mathbf{G}$ is determined uniquely by $g(0)$ and $g_0^{(1)}$. Indeed,* $g=\exp(Z_{\Gamma(g(0))})^\sim[\mathrm{d}_{g(0)}\exp(-Z_{\Gamma(g(0))})]^\sim g_0^{(1)\sim}$. *In particular* $\mathbf{L}=\{g\in\mathbf{G}: g(0)=0\}$.

PROOF. From (3.6) we know that $u_g=\exp(-Z_{\Gamma(g(0))})^\sim g$ is the germ around D_0 of some linear operator. Thus $u_g=u_{g0}^{(1)\sim}=\exp(-Z_{\Gamma(g(0))})_{g(0)}^{\sim(1)}g_0^{(1)\sim}$.

Corollary 3.14. $\mathbf{L}$ *is a closed topological subgroup of* $\mathbf{G}$.

PROOF. Since $\Phi(\mathcal{L}\cap\epsilon U)=\Phi(\epsilon U)\cap\mathbf{L}$ if $\epsilon\in(0,1)$, the topology of $\mathbf{L}$ is finer than the topology of $\mathbf{G}$ restricted to $\mathbf{L}$.

On the other hand, from (3.5) we see that for some neighbourhood $V_2 \subset U$ of 0 in $\mathcal{P} + \mathcal{L}$, the mapping $\Psi: Z \mapsto \exp(P_1 Z)^{\sim} \exp(P_0 Z)^{\sim}$ is real-bianalytic. In particular, $\Psi(V_2)$ is a neighbourhood of e in $\mathbf{G}$ and, by Corollary 3.13,

$$\begin{aligned}\Psi(V_2) \cap \mathbf{L} &= \{\exp(P_1 Z)^{\sim} \exp(P_0 Z)^{\sim} : Z \in V_2, \exp(P_1 Z)(0) = 0\} \\ &= \{\exp(P_0 Z)^{\sim} : Z \in V_2, P_1 Z = 0\} \\ &= \{\exp(Z)^{\sim} : Z \in V_2 \cap \mathcal{L}\}.\end{aligned}$$

Thus if U_1 is any open subset of $\Psi(V_2)$ in $\mathbf{G}$ then $U_1 \cap \mathbf{L} = \Phi(\Psi^{-1}(\mathrm{U}_1) \cap \mathcal{L})$, which proves that the topology of $\mathbf{G}$ restricted to $\mathbf{L}$ is finer than that of $\mathbf{L}$.

Since the topology of a Banach–Lie group has countable bases for the neighbourhoods of the unit, $\mathbf{G}$ is metrisable with a complete left-invariant metric (cf. [5, 6.22]). Since complete subspaces of metric spaces are closed, $\mathbf{L}$ is necessarily a closed subset of $\mathbf{G}$.

4. The structure of complete orbits

Our aim in this section is to prove the following theorem.

Theorem 4.1. *Let $(E, E_0, \{*\})$ be a weakly commutative partial JB*-triple. Assume there exists a neighbourhood B of 0 in E such that $B \subset \operatorname{dom} \exp(Z_a)$ for all $a \in E_0$ and the figure*

$$\bigcup_{a \in E_0} \exp(Z_a)(B)$$

is bounded. Then $(E, E_0, \{*\})$ is a geometric partial JB-triple, i.e. there exists a bounded balanced domain $D \subset E$ such that the vector fields $Z_a := (a - \{za^*z\})\, \partial/\partial z$ $(a \in E_0)$ are complete in D.*

Proof. With the notations of Section 3, define

$$D_B := \{u^{\wedge} \exp(Z_a)(x) : a \in E_0, u \in \mathbf{L}, x \in B\} \tag{4.2}$$

where we denote by $u^{\wedge}$ the linear operator whose germ around D_0 is $u \in \mathbf{L}$. Notice that $\mathbf{L}^{\wedge} (:= \{u^{\wedge} : u \in \mathbf{L}\})$ is the subgroup of $\mathrm{GL}(E)$ generated by the family $\{\exp(l) : l \in \mathcal{L}\}$. Since $\mathcal{L} \subset i\mathrm{Her}(E)$, the group $\mathbf{L}^{\wedge}$ consists of E-unitary operators. Hence $D_B = \mathbf{L}^{\wedge} \bigcup_{a \in E_0} \exp(Z_a)(B)$ is a bounded connected $\mathbf{L}^{\wedge}$-invariant open neighbourhood of 0. In particular, since $\{e^{it}\mathrm{id} : t \in \mathbb{R}\} \subset \mathbf{L}^{\wedge}$, the domain D_B is circular. Thus the holomorphic envelope D of D_B is a bounded balanced domain (see [2]). Therefore the theorem is an immediate consequence of the proposition below.

Proposition 4.3. *Let $x \in E$ be a point such that the integral curve of any of the vector fields Z_a $(a \in E_0)$ through x is complete in E, i.e. $x \in \operatorname{dom} \exp(Z_a)$ for all $a \in E_0$. Then the figure*

$$M := \{U \exp(Z_a)(x) \colon U \in \mathbf{L}^\wedge, a \in E_0\}$$

is the integral manifold through x of the Lie algebra of vector fields $\mathscr{P} + \mathscr{L}$ *and* $\mathscr{P} + \mathscr{L}$ *is complete in M.* (Here we write again $\mathscr{P} := \{Z_a \colon a \in E_0\}$, cf. Lemma 2.2.)

PROOF. From (2.1) it readily follows that

$$\begin{aligned}\exp(L)\exp(Z_a)\exp(-L) &= \exp(\exp(\operatorname{ad} L) Z_a) \\ &= \exp(Z_{\exp(L)(a)})\end{aligned}$$

for any $a \in E_0$ and $L \in \mathscr{L}$. Thus

$$U(\exp(Z_a)) = \exp(Z_{u(a)}) \circ U \qquad (U \in \mathbf{L}^\wedge, a \in E_0).$$

Since $U(E_0) = E_0$ for any $U \in \mathbf{L}^\wedge$, this means that

$$M^0 := \{Ux \colon U \in \mathbf{L}^\wedge\} \subset \operatorname{dom} \exp(Z_b) \qquad (b \in E_0).$$

Thus the integral curves of the fields Z_b through the points of M^0 are complete in E and

$$M = \bigcup_{a \in E_0} \exp(Z_a)(M^0). \tag{4.4}$$

Let us now fix $Z \in \mathscr{P} + \mathscr{L}$ arbitrarily and define

$$c(t, b) := \exp(tZ)\exp(Z_b)(0) \qquad (t \in \mathbb{R}, b \in E_0).$$

From Theorem 3.3 we know that

$$\exp(tZ)^\sim \exp(Z_b)^\sim = [\exp(Z_{c(t,b)}) \circ U_{t,b}]^\sim \qquad (t \in \mathbb{R}, b \in E_0) \tag{4.5}$$

for some (uniquely determined) $U_{t,b} \in \mathbf{L}^\wedge$. By (4.4) the mappings

$$\phi_{t,b} := \exp(Z_{c(t,b)}) \circ U_{t,b} \qquad (t \in \mathbb{R}, b \in E_0)$$

are well defined on M^0 and for any fixed $z \in M^0$ the map $(t, b) \mapsto \phi_{t,b}(z)$ is real-analytic $\mathbb{R} \times E_0 \to E$ and ranges in M. To complete the proof of the proposition it suffices to verify that if $Z = (a + l(z) - \{za^*z\})\, \partial/\partial z$ then

$$\frac{\partial}{\partial t}\phi_{t,b}(z) = a + l(\phi_{t,b}(z)) - \{\phi_{t,b}(z) a^* \phi_{t,b}(z)\} \qquad (z \in M^0). \tag{4.6}$$

Indeed, once we have established (4.6) we can argue as follows. Consider any $y_0 \in M$. By (4.3), for some $z_0 \in M^0$ and $b \in E_0$ we can write $y = \exp(Z_b)(z_0)$.

Define

$$y(t) := \phi_{t,b}(z_0) \qquad (t \in \mathbb{R}).$$

We have $y(0) = y_0$, $y(t) \in M$ $(t \in \mathbb{R})$ and, assuming (4.6),

$$\frac{d}{dt} y(t) = \frac{\partial}{\partial t} \phi_{t,b}(z_0) = a + l(y(t)) - \{y(t)a^* y(t)\}.$$

This shows that $y(t) = \exp(tZ)(y_0) \in M$ $(t \in \mathbb{R})$. Thus by the arbitrariness of g_0, the vector field Z is complete in M.

Proof of (4.6). Let $W_m := \{z \in E: \|z\| < \|x\| + m + 1\}$ $(m = 0, 1, 2)$ and choose $\delta > 0$ such that $\sup\{\|d + k(z) - \{zd^*z\}\|: \|d\| + \|k\| < \delta, z \in W_2\} < 1$. It is elementary that the initial value problem

$$\frac{d}{dt} z(t) = d + k(z) - \{z(t)d^* z(t)\} \qquad z(0) = w_0$$

has a solution defined in a neighbourhood of [0, 1] and ranging in W_m whenever $w_0 \in W_{m-1}$ and $\|Z_d + k(z)\,\partial/\partial z\| := \|d\| + \|k\| < \delta$ $(m = 1, 2)$. Hence

$$\operatorname{dom}[\exp(tZ)\exp(Z_b)] \supset W_0 \quad \text{if} \quad \|tZ\|, \|b\| < \delta.$$

By (4.5) it follows that

$$\phi_{t,b} | W_0 = \exp(tZ)\exp(Z_b) | W_0 \quad \text{if} \quad \|c(t, b)\|, \|tZ\|, \|b\| < \delta.$$

From the definition of the exponential of vector fields we see that (4.6) holds whenever $\|c(t, b)\|, \|tZ\|, \|b\| < \delta$. Hence, by the real analyticity of both sides of (4.6) we obtain (4.5) for all $t \in \mathbb{R}$, $b \subset E_0$.

Corollary 4.7. *If $B \subset \bigcap_{a \in E_0} \operatorname{dom} \exp(Z_a)$, then the figure D_B defined by* (4.1) *satisfies*

$$D_B \subset \bigcap_{Z \in \mathcal{P} + \mathcal{L}} \operatorname{dom} \exp(Z).$$

Moreover, D_B is the smallest $\exp(\mathcal{P} + \mathcal{L})$-invariant subset of E containing B. If B is $\exp(\mathcal{L})$-invariant then $D_B = \bigcup_{a \in E_0} \exp(Z_a)(B)$.

5. On the integration of the vector fields Z_a

Throughout this section let $(E, E_0, \{*\})$ be a partial JB*-triple. Fix a real-analytic curve $a: I \to E_0$ on some open interval $I \subset \mathbb{R}$ containing 0. Let us first study the power series of the automorphism germs g^t $(t \in I)$ defined by the solution of the initial value problem

$$\frac{d}{dt}g^t := Z_{a(t)}(g^t), \qquad g^0 = \mathrm{id}^{\sim}.$$

Here we write again $Z_a := (a - \{za^*z\})\,\partial/\partial z$. From the Picard–Lindelöf theorem concerning the existence and continuity properties of solutions of ordinary differential equations it easily follows that

$$g^t = (F^t)^{\sim} \qquad (t \in I)$$

where $(t, x) \mapsto F^t(x)$ is the maximal solution of $\partial/\partial t\, y(t, x) = Z_{a(t)}(y(t, x))$ with boundary condition $y(0, x) = x$. Let us write

$$p_k^t(x) := \frac{1}{k!}\,\mathrm{d}_0^k F^t(x, \ldots, x) \qquad (k = 0, 1, \ldots)$$

where $\mathrm{d}_0^k F^t$ is the Fréchet derivative $(h_1, \ldots, h_k) \mapsto \partial^k/\partial\tau_1 \ldots \partial\tau_k|_0 \times F^t(\Sigma_{j=1}^k \tau_j h_j)$. Since each F^t is holomorphic

$$F^t(x) = \sum_{k=0}^{\infty} p_k^t(x) \qquad \text{for } (t, x) \in \mathscr{D} \tag{5.1}$$

where $\mathscr{D} := \{(t, x) : t \in I, x \in \operatorname{dom} F^t\}$. Notice that $\mathscr{D}$ is a neighbourhood of $I \times \{0\}$ in $\mathbb{R} \times E$. By the definition of the curve $t \mapsto g^t$ (see Section 2)

$$\frac{d}{dt}F^t(x) = a(t) - \{F^t(x)a(t)^*F^t(x)\} \qquad ((t, x) \in \mathscr{D}). \tag{5.2}$$

Since the k-homogeneous polynomials p_k^t can also be obtained by means of a suitable contour integral operator applied to F^t, it follows that for each k the operation $t \mapsto p_k^t$ is real-analytic on the whole of I. Hence (5.1) and (5.2) imply

$$\frac{d}{dt}p_0^t = a(t) - \{p_0^t a(t)^* p_0^t\} \qquad p_0^0 = 0 \tag{5.3}$$

$$\frac{d}{dt}p_1^t = -2(p_0^t \,\square\, a(t)^*)p_1^t \qquad p_1^0 = \mathrm{id} \tag{5.4}$$

$$\frac{d}{dt}p_k^t(x) = -2(p_0^t \,\square\, a(t)^*)p_k^t(x) - \sum_{l=1}^{k-1} \{p_l^t(x)a(t)^*p_{k-l}^t(x)\} \qquad p_k^0 = 0 \qquad (k > 1, x \in E). \tag{5.5}$$

Henceforth we focus our attention on the case when the curve $t \mapsto a(t)$ has values in a subtriple generated by a single element $c \in E_0$. Recall that, given $c \in E_0$, by [13, lemma 2.1] there exists a topological Jordan-homomorphism T: $\mathscr{C}_0(\Omega) \to E_0$, where Ω is some bounded relatively closed subset of $(0, \infty)$ and there also exists a continuous linear operator $L: \mathscr{C}_0(\Omega) \to \mathrm{L}(E)$ such that, by writing $\xi := \mathrm{id}_\Omega$ we have $T(\xi) = c$ and

$$L(\varphi\bar{\psi}) = T(\varphi) \,\square\, T(\psi)^* \qquad (\varphi, \psi \in \mathscr{C}_0(\Omega)).$$

We assume in the sequel that

$$a(t) \in T(\mathrm{Re}\, \mathscr{C}_0(\Omega)) \qquad (t \in I).$$

It is immediate from (5.3) that also $p_0^t \in T(\mathrm{Re}\, \mathscr{C}_0(\Omega))$ for all $t \in I$. By setting

$$\alpha_t := T^{-1}(a(t)), \qquad \pi_t := T^{-1}(p_0^t) \qquad (t \in I)$$

we have

$$\frac{d}{dt}\pi_t = \alpha_t - \alpha_t \pi_t^2 \qquad \pi_0 = 0. \tag{5.6}$$

It follows that the solution of (5.4) can be written as

$$p_1^t = \exp\left[-2L\left(\int_0^t \pi_\tau \alpha_\tau \, d\tau\right)\right] \qquad (t \in I). \tag{5.7}$$

In particular $p_1^t \in \mathrm{GL}(E)$ for all $t \in I$. Therefore we may define

$$C_k^t(x) := (p_1^t)^{-1} p_k^t(x) \qquad (k \geq 1, t \in \mathbb{R}, x \in E).$$

Applying this definition to the left-hand side of (5.5) we obtain

$$p_1^t \frac{d}{dt} C_k^t(x) = -\sum_{l=1}^{k-1} \{p_l^t(x) a(t)^* p_{k-l}^t(x)\} \qquad (k > 1, t \in \mathbb{R}, x \in E). \tag{5.8}$$

Lemma 5.9. *Let A be a real flat subspace of E_0 (i.e. $b \,\square\, c^* = c \,\square\, b^*$ for all $b, c \in A$) and let $\mathscr{A}$ denote the closed real linear hull of the family of operators $A \,\square\, A^*$ in $\mathrm{L}(E)$. Then for all $\alpha \in \mathscr{A}$, $x, y \in E$ and $c \in E_0$*

$$\alpha\{xc^*y\} = \{(\alpha x)c^*y\} - \{x(\alpha c)^*y\} + \{xc^*(\alpha y)\} \tag{5.10}$$

$$\exp(\alpha)\{xc^*y\} = \{[\exp(\alpha)x][\exp(-\alpha)c]^*[\exp(\alpha)y]\}. \tag{5.11}$$

Proof. If $b_1, b_2 \in A$ then by axiom (J1)

$$b_1 \,\square\, b_2^*\{xc^*y\} = \{(b_1 \,\square\, b_2^* x)c^*y\} - \{x(b_2 \,\square\, b_1^* c)^*y\} + \{xc^*(b_1 \,\square\, b_2^* y)\}.$$

Thus, since $b_1 \,\square\, b_2^* = b_2 \,\square\, b_1^*$, (5.10) holds for $\alpha \in A \,\square\, A^*$. By passing to real linear combinations and then to limits in $\mathrm{L}(E)$, we obtain (5.10) for all $a \in \mathscr{A}$. To prove (5.11), fix α, x, y and c arbitrarily. Define $\epsilon^t := \exp(t\alpha)$ $(t \in \mathbb{R})$. Then using Leibniz's rule and then (5.10),

$$\frac{d}{dt}\{(\epsilon^t x)(\epsilon^{-t} c)^*(\epsilon^t y)\}$$

$$= \{(\alpha\epsilon^t x)(\epsilon^{-t} c)^*(\epsilon^t y)\} - \{(\epsilon^t x)(\alpha\epsilon^{-t} c)^*(\epsilon^t y)\} + \{(\epsilon^t x)(\epsilon^{-t} c)^*(\alpha\epsilon^t y)\}$$
$$= \alpha\{(\epsilon^t x)(\epsilon^{-t} c)^*(\epsilon^t y)\} \qquad (t \in \mathbb{R}).$$

Since $\epsilon^0 = \mathrm{id}$, hence (5.11) is immediate.

In order to express the right-hand side of (5.8) in terms of the homogeneous polynomials C_l^t, we apply Lemma 5.9 to

$$A := \mathrm{Span}_{\mathbb{R}}\, L(\mathrm{Re}\, \mathscr{C}_0(\Omega)), \qquad \alpha := -2L\left(\int_0^t \pi_\tau\, \alpha_\tau\, d\tau\right).$$

Since $p_1^t = \exp(\alpha)$, from (5.11) we see that

$$\{p_l^t(x) a(t)^* p_m^t(x)\} = \{[p_1^t C_l^t(x)][(p_1^t)^{-1}(p_1^t a(t))]^*[p_1^t C_m^t(x)]\}$$
$$= p_1^t\{C_l^t(x)(p_1^t a(t))^* C_m^t(x)\} \qquad (l, m \geqslant 1, t \in \mathbb{R}).$$

Thus we can write (5.8) in the form

$$\frac{d}{dt} C_k^t(x) = -\sum_{l=1}^{k-1} \{C_l^t(x)(p_1^t a(t))^* C_{k-l}^t(x)\}. \tag{5.12}$$

Remark 5.13. We may apply the previous considerations to the constant curve $t \mapsto c$ in order to get formulas for the Taylor coefficients of $\exp(tZ_c)$. However, in this trivial way, one only obtains rather complicated recursive expressions for the C_k^t that are hard to simplify directly with the identity of weak commutativity.

The first two coefficients can be calculated in this manner. Solving (5.6) and (5.7) for $\alpha_t := \xi$ $(t \in \mathbb{R})$ we get

$$\exp(tZ_c)(0) = T(\pi_t) = T(\tanh(t\xi))$$
$$= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\left(\frac{d^{2n+1}}{d\tau^{2n+1}}\bigg|_0 \tanh \tau\right)(c \,\square\, c^*)^n c \tag{5.14}$$
$$\mathrm{d}_0^1 \exp(tZ_c) = \exp[-2L(\log\cosh(t\xi))]. \tag{5.15}$$

Definition 5.16. Let $(E, E_0, \{*\})$ be a partial JB*-triple and $b \in E_0$. In the sequel we shall write

$$e(b) := \exp(Z_b)(0), \qquad l(b) := \mathrm{d}_0^1 \exp(Z_b)$$

and for $k = 1, 2, \ldots$ we define recursively

$$B_1(b, x) := x, \quad B_k(b, x) := -\frac{1}{k-1} \sum_{l=1}^{k-1} \{B_l(b, x) b^* B_{k-l}(b, x)\} \quad (x \in E).$$

Remark that $\exp(-\{zb^*z\}\,\partial/\partial z) = \Sigma_{k=1}^{\infty} B_k(b, z)$ in some neighbourhood of 0 in E (see proof of (6.5)).

Proposition 5.17. *Let $(E, E_0, \{*\})$ be a weakly commutative partial JB*-triple and $c \in E_0$. Then*

$$\mathrm{d}_0^k \exp(Z_c)(x, \ldots, x) = k!\,l(c) B_k(e(c), x) \qquad (k = 1, 2, \ldots).$$

Proof. Let us apply the previous considerations to the curve $t \mapsto T(\alpha_t)$ where

$$\alpha_t := \frac{\tanh(\xi)}{1 - [t \cdot \tanh(\xi)]^2}$$

on some open interval I containing $[0, 1]$. By (5.6)

$$\frac{d}{dt} \operatorname{areath}(\pi_t) = \alpha_t = \frac{d}{dt} \operatorname{areath}(t \cdot \tanh(\xi))$$

whence

$$p_0^t = T(\pi_t) = T(t \cdot \tanh(\xi)) = t \cdot e(c) \qquad (t \in I). \tag{5.18}$$

Substituting into (5.7), for all $t \in I$

$$p_1^t = \exp\left[-2L\left(\log \frac{1}{\sqrt{1 - [t \cdot \tanh(\xi)]^2}}\right)\right]. \tag{5.19}$$

Thus $p_0^1 = \exp(Z_c)(0)$ and $p_1^1 = \exp[-2L(\log\cosh(\xi)] = \mathrm{d}_0^1 \exp(Z_c)$ by (5.14) and (5.15). From Corollary 3.13 it follows that

$$g^1 = \exp(Z_c)^{\sim}.$$

Therefore it suffices to check that

$$C_k^t(x) = t^{k-1} B_k(e(c), x) \qquad (t \in I, x \in E, k = 1, 2, \ldots). \tag{5.20}$$

This is trivial for $k = 1$. To perform the induction step, calculate $p_1^t a(t)$ in terms of the representation T. It is easily seen that

$$L(\phi) T(\psi) = T(\phi\psi) \qquad \phi, \psi \in \mathscr{C}_0(\Omega).$$

Hence, by (5.19)

$$p_1^t a(t) = T\left(\exp\left[-2\log\frac{1}{\sqrt{1-[t\cdot\tanh(\xi)]^2}}\right]\frac{\tanh(\xi)}{1-[t\cdot\tanh(\xi)]^2}\right)$$
$$= T(\tanh(\xi)) = e(c)$$

independently of $t \in I$. Thus, by (5.12), if we assume (5.20) for all indices $\leqslant k$,

$$C_{k+1}^t = -\sum_{l=1}^{k}\int_0^t \tau^{k-1}d\tau\{B_l(e(c),x)e(c)^*B_{k-l+1}(e(c),x)\}$$
$$= t^k B_{k+1}(e(c),x).$$

Remark 5.21. In [8] Kaup proved that

$$\exp(Z_c)(b) = e(c) + l(c)[\mathrm{id} + b\,\Box\,e(c)^*]^{-1}b \qquad \text{for } b \in D_0$$

where D_0 is the unit ball of E_0. Comparing this result with Proposition 5.17, we obtain that

$$B_k(e(c),b) = (-b\,\Box\,e(c)^*)^{k-1} \qquad \text{if } b \in E_0 \tag{5.22}$$

for $k = 2, 3, \ldots$. It is a natural question, whether we have $B_k(e(c),x) = (-x\,\Box\,e(c)^*)^{k-1}x$ for all x if $(E, E_0, \{*\})$ is a weakly commutative partial JB*-triple. The answer is negative even in very simple cases.

Example 5.23. Let $E = \mathbb{C}^3$, $\{e_1, e_2, e_3\}$ be the canonical bases in E, and let $E_0 := \mathbb{C}e_1$. We define the triple product $\{*\}$ on $E \times E_0 \times E$ by sesquilinear extension from the following relations:

$$\{e_1e_1^*e_j\} = \{e_je_1^*e_1\} = e_j \quad (j = 1, 2, 3)$$
$$\{e_2e_1^*e_2\} = e_3$$
$$\{e_2e_1^*e_3\} = \{e_3e_1^*e_2\} = e_2$$
$$\{e_3e_1^*e_3\} = 0.$$

Then $(E, E_0, \{*\})$ is a weakly commutative partial JB*-triple. However,

$$(-e_2\,\Box\,\exp(tZ_{e_1}(0))^*)^3e_2 = -(\tanh t)^3e_3$$
$$\neq B_4\exp(tZ_{e_1}(0)), e_2) = -\tfrac{2}{3}(\tanh t)^3e_3.$$

6. Weakly commutative partial JB*-triples are geometric

Henceforth $(E, E_0, \{*\})$ denotes an arbitrarily fixed partial JB*-triple and we shall also use the notations established in Definition 5.16.

Lemma 6.1. *The linear operator $l(c)$ is non-expansive for any $c \in E_0$.*

Proof. Fix $c \in E_0$, set $\Omega := \{\omega > 0: \omega^2 \in \mathrm{Sp}(c \square c^*)\}$, $\xi := \mathrm{id}_\Omega$ and apply the representations $T: \mathscr{C}_0(\Omega) \to E_0$, $L: \mathscr{C}_0(\Omega) \to \mathrm{L}(E)$ used in (5.14) and (5.15). By (5.15) we have

$$\begin{aligned} l(c) &= \exp[-2L(\log\cosh(\xi))] \\ &= \exp[-2b \square b^*] \end{aligned}$$

with

$$b := T(\sqrt{\log\cosh(\xi)}).$$

Hence the statement follows by axiom (J3).

Theorem 6.2. *A partial JB*-triple is geometric if and only if it is weakly commutative.*

Proof. It is well known that geometric partial JB*-triples are weakly commutative (see [5, prop. 7.9.(c)]).

To prove the converse, let $(E, E_0, \{*\})$ be a weakly commutative partial JB*-triple. According to Theorem 4.1, we have to see the existence of a bounded neighbourhood B of 0 such that $\exp(Z_c)$ is well defined on B for any $c \in E_0$ and

$$\sup_{c \in E_0, x \in B} \|\exp(Z_c)(x)\| < \infty.$$

For this it suffices to establish that for some $\rho > 0$,

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sup_{c \in E_0} \sup_{\|x\| \leqslant \rho} \|\mathrm{d}_0^k(Z_c)(x, \ldots, x)\| < \infty. \tag{6.3}$$

Notice that $\mathrm{d}_0^0 \exp(Z_c) = \exp(Z_c)(0) = e(c)$ lies in D_0, the open unit ball of E_0, independently of the chôice of $c \in E_0$.

Let us write

$$M := \sup\{\|\{xc^*y\}\|: x, y \in E, c \in E_0, \|x\|, \|y\|, \|c\| \leqslant 1\}.$$

We prove by induction on k that

$$\|B_k(e(c), x)\| \leqslant M^{k-1} \qquad (c \in E_0, \|x\| \leqslant 1, k = 1, 2, \ldots). \tag{6.4}$$

Indeed, $B_1(e(c), x) = x$ $(c \in E_0, x \in E)$ and by the recursive Definition 5.16,

$$\|B_{n+1}(e(c), x)\| \leq \frac{1}{n} \sum_{l=1}^{n} \|\{B_l(e(c), x)e(c)^*B_{n-l+1}(e(c), x)\}\|$$
$$\leq \frac{1}{n} \sum_{l=1}^{n} M \cdot M^{l-1} \cdot M^{n-l} = M^n$$

for all $c \in E_0$, $x \in E$ with $\|x\| \leq 1$ whenever (6.4) holds for $k = 1, \ldots, n$.

In view of Proposition 5.17 and Lemma 6.1,

$$\|d_0^k \exp(Z_c)(x, \ldots, x)\| \leq k! M^{k-1} \|x\|^k \qquad (c \in E_0, x \in E)$$

for $k = 1, 2, \ldots$. Hence (6.3) is fulfilled for any $\rho < M^{-1}$.

Corollary 6.5. *In a weakly commutative partial JB*-triple we have*

$$\exp(Z_c)(x) = e(c) + l(c) \exp\left(\{ze(c)^*z\} \frac{\partial}{\partial z}\right)(x) \qquad (\|x\| \leq M^{-1}).$$

PROOF. According to Definition 5.16 and (6.4), the expression $F(t, e, x) := \Sigma_{k=0}^{\infty} t^k B_{k+1}(e, x)$ satisfies

$$\frac{\partial}{\partial t} F(t, e, x) = \{F(t, e, x)e^*F(t, e, x)\} \qquad (\|e\| < 1, \|x\| < M^{-1}).$$

Therefore $\Sigma_{k=1}^{\infty} B_k(e, x)$ is the power series of $\exp(\{ze^*z\}\, \partial/\partial z)$ around 0 and this series converges uniformly on the domains $\{(e, x) \colon \|e\| < 1, \|x\| < (1 - \epsilon)M^{-1}\}$ for any $\epsilon > 0$. Thus the statement is immediate from Proposition 5.17.

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