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NON-VECTOR SPACE GENERALIZATIONS OF VON NEUMANN'S MINIMAX THEOREM

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Recently I. Joó [1] published a very short and elementary proof for Ky Fan's generalization of the von Neumann minimax theorem. A careful examination of his method leads us to a conclusion that may be a bit surprising: By carrying out some simple changes, we can arrive at a proof where the only property arising from convexity which we need is the connectedness of straight line segments (in real vector spaces). Therefore the most suitable concept in describing the topological situation that occurs in the minimax principles is perhaps the following structure that we call interval space and which is richer but axiomatically simpler than the familiar topological vector spaces:

Definition. By an interval space we mean a topological space X equipped with a mapping $[\cdot, \cdot]: X \times X \to \{\text{connected subsets of } X\}$ such that

$$x_1, x_2 \in [x_1, x_2] = [x_2, x_1] \quad \forall x_1, x_2 \in X.$$

The set $[x_1, x_2]$ is called the interval connecting x_1 and x_2.

In interval spaces, it makes sense to speak of convexity of subsets and quasiconvexity of $X \to \mathbf{R}$ mappings in a natural way:

Definition. A subset K of an interval space is *convex* if $[x_1, x_2] \subset \subset K \quad \forall x_1, x_2 \in K$. A function $f\colon X \to \mathbf{R}$ is *quasiconvex* if $f(z) \leqslant \leqslant \max \{f(x_1), f(x_2)\}$ whenever $z \in [x_1, x_2]$.

In this context, we have the following minimax principle:

Theorem 1. *Let X, Y be compact interval spaces and let $f\colon X \times Y \to \to \mathbf{R}$ be a continuous function such that*

(1) *the function $x \to f(x, y)$ is quasiconcave for any (fixed) $y \in Y$,*

(1′) *$y \to f(x, y)$ is quasiconvex for any $x \in X$.*

Then $\max_x \min_y f(x, y) = \min_y \max_x f(x, y)$.

The key step of the proof of Theorem 1 is the proposition below that, in some sense, plays an analogous role as Brouwer's fixed point theorem in the classical proofs:

Proposition. *Let Y be an interval space, X a topological space and $K\colon X \to$ {non-empty compact X-subsets} a mapping with the properties*

(2) *$K(z) \subset K(y_1) \cup K(y_2)$ whenever $z \in [y_1, y_2]$ and $y_1, y_2 \in Y$,*

(3) *$\bigcap_{i=1}^{n} K(y_i)$ is connected (possibly empty) for every $y_1, \ldots, y_n \in \in Y$ $(n = 1, 2, \ldots)$,*

(4) *$x \in K(y)$ whenever $y = \lim_{i \in I} y_i$, $x = \lim_{i \in I} x_i$ and $x_i \in K(y_i)$ for all $i \in I$.*

Then we have $\bigcap_{y \in Y} K(y) \neq \phi$.

Question. Is there a tricky choice of X, Y and $K(\cdot)$ in this proposition such that the conclusion $\bigcap_{y \in Y} K(y) \neq \phi$ be a known equivalent of Brouwer's fixed point theorem?

By shifting the emphasis from the topology on the order structure of the underlying spaces, we can prove a topological version of the strongest known minimax principle due to Brezis – Nirenberg – Stampacchia [2]:

Definition. We say that an interval space Y is *Dedekind complete* if for every pair of points $y_1, y_2 \in Y$ and convex subsets $H_1, H_2 \subset Y$ with $y_1 \in H_1$, $y_2 \in H_2$ and $[y_1, y_2] \subset H_1 \cup H_2$ there exists $z \in H_1 \cup H_2$ such that $[y_j, z] \setminus \{z\} \in H_j$ $(j = 1, 2)$.

Theorem 2. *Suppose that X is an interval space, Y a Dedekind complete Hausdorff interval space and let $f\colon X \times Y \to \mathbf{R}$ be a function with the properties*

(5) *$x \to f(x, y)$ is upper semicontinuous and quasiconcave on X (for all $y \in Y$),*

(6) *$y \to f(x, y)$ is quasiconvex and lower semicontinuous when restricted to any interval in Y (for all $x \in X$),*

(7) *for some $\gamma < \inf_y \sup_x f(x, y)$ and $y \in Y$, the set $\{x\colon f(x, y) \geqslant \gamma\}$ is compact.*

Then we have $\max_x \inf_y f(x, y) = \inf_y \sup_x f(x, y)$.

In the light of the proof of Theorem 2 we can answer (negatively) the question raised by L. Nirenberg [3] whether the condition (5) in Theorem 2 can be replaced by the weaker assumption

(5) *$x \to f(x, y)$ is quasiconcave and upper semicontinuous when restricted to any interval in X (for all $y \in Y$) if X and Y denote compact convex subsets in some topological vector spaces.*

Counterexample. Set $Y \equiv \{\mathbf{N} \to [0, 1]$ functions$\}$, $F \equiv \{\mathbf{N} \to \mathbf{R}$ functions$\}$, $E \equiv \{Y \to \mathbf{R}$ functions$\}$, $X \equiv \{Y \to [0, 1]$ functions$\}$ and let the vector spaces E, F be endowed with the pointwise convergence topology. Introduce the sets $H_n \equiv \operatorname{co}\{1_{\{i\}}\colon i > n\}$ $(n = 1, 2, \ldots)$ and $K(y) \equiv$ $\equiv \operatorname{co}\{1_{H_n}\colon n \geqslant m(y)\}$ $(y \in Y)$ where $m(y) \equiv \min\{n \in \mathbf{N}\colon y \notin H_n\}$ (here co stands for the convex hull operation and 1_{H_n} is the characteristic function of the set H_n). Now we define the function f: $X \times Y \to \{0, 1\}$ by $f(x, y) \equiv 1_{K(y)}(x)$. Then the sets X and Y are compact in E and F, respectively. The function f satisfies (5′), (6) and (7). However, we have

$$0 = \max_x \min_y f(x, y) \quad \text{and} \quad 1 = \min_y \max_x f(x, y).$$

(Detailed proofs had already appeared in [4].)

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