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On Fixed Points of Holomorphic Automorphisms (*).

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Summary. – *Banach M-lattices are studied from the view point whether all the biholomorphic automorphisms of their unit balls admit fixed points when continuously extended to the closure of the unit ball. A characterization of compact topological F-spaces is found in terms of the fixed points of the elements of* $\mathrm{Aut}\,\bar{B}(C(\Omega))$ *which enables to establish some particular properties also of the topological automorphisms of compact F-spaces. Finally it is shown that if the M-lattice E admits a predual then each member of* $\mathrm{Aut}\,\bar{B}(E)$ *has fixed point if and only if E is isometrically isomorphic with some* l^∞*-space.*

Let $B(E)$ denote the open unit ball of a complex Banach space E and let $\mathrm{Aut}\,B(E)$ be the group of all Fréchet-holomorphic automorphisms of $B(E)$. By a result of KAUP-UPMEIER [6] every $F \in \mathrm{Aut}\,B(E)$ is the restriction to $B(E)$ of a holomorphic map of some neighborhood of $\bar{B}(E)$ into E. If, in particular, the space E is reflexive and separable, a theorem of HAYDEN-SUFFRIDGE [4] establishes the existence of fixed points of $e^{i\theta}F$ for almost every $\theta \in \mathbf{R}$ for all continuous mappings of $\bar{B}(E)$ into itself which are holomorphic on $B(E)$. However, the same article [4] remarks that by relaxing reflexivity of E such a fixed point theorem is no longer valid in general (e.g. in the case of $c_0(N)$, as it is shown there).

The following simple example indicates that even the weaker conjecture stating that each $F \in \mathrm{Aut}\,\bar{B}(E)$ (1) admits fixed point fails in most M-lattices (for def. see [9]): The mapping

$$F: f \to \left[\bar{\Delta} \in \zeta \mapsto \frac{f(\zeta) + \zeta/2}{1 + f(\zeta)/2}\right] \tag{1}$$

defined for the continuous functions $f: \bar{\Delta} \to \bar{\Delta}$ (Δ standing for the open unit disc of C) clearly belongs to $\mathrm{Aut}\,\bar{B}(C(\bar{\Delta}))$ but $Ff_0 = f_0$ would imply $f_0(\zeta)^2 = \zeta/\bar{\zeta}$, $\zeta \in \bar{\Delta}\setminus\{0\}$ contradicting the continuity of f_0 at the point 0.

In this paper we shall investigate M-lattices with order unit, a category whose behaviour seems particularly interesting from the view point whether every (holomorphic) automorphism of the closed unit ball has fixed point.

Our main results concern a characterization of compact F-spaces (def. see [3]) in

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(1) We shall denote by $\mathrm{Aut}\,\bar{B}(E)$ the set of all continuous extensions to $\bar{B}(E)$ of the elements of $\mathrm{Aut}\,B(E)$.

terms of the automorphisms of the closed unit ball of their continuous function spaces and a theorem asserting that, up to isometries the l^∞-spaces are the only M-lattices with predual where any automorphism of the closed unit ball admits a fixed point.

1. – Fixed point free elements of $\bar{B}(C_b(\Omega))$. Remarks on Aut $\bar{\Delta}$.

The construction of (1) suggests an approach promising positive results to the question: What is the necessary and sufficient topological condition on a regular topological space Ω to admit a member of Aut $\bar{B}(C(\Omega))$ without fixed points?

PROPOSITION 1. – Suppose Ω is a topological space such that every $F \in \text{Aut}\,\bar{B}(C_b(\Omega))$ has a fixed point. Then Ω is necessarily an F-space (²).

PROOF. – Let $t(\cdot)$ be any continuous function on Ω; set $G \equiv \{x \in \Omega : t(x) \neq 0\}$ and fix any $\varphi \in C_b(G)$. We may assume without loss of generality that range $(t) \subset$ $\subset [0, \pi/2]$ (thus $G = \{x \in \Omega : t(x) > 0\}$). Define the functions $k : \Omega \to \partial\Delta$ and

$$u : \Omega \to \frac{1}{2}\bar{\Delta} \quad \text{by} \quad k(y) \equiv e^{it(y)} \quad \text{and} \quad u(x) \equiv -\frac{2i\varphi(x)\, e^{-it(x)/2}}{1 + |\varphi(x)|^2} \sin\frac{t(x)}{2}$$

if $x \in G$, $u(x) \equiv 0$ for $x \in \Omega \setminus G$. Observe that the transformations

$$N(x) \equiv \left[\bar{\Delta} \in \zeta \mapsto k(x)\frac{\zeta + u(x)}{1 + \overline{u(x)}\zeta}\right]$$

are in Aut $\bar{\Delta}$ for all fixed $x \in \Omega$ since $|k(x)| = 1$ and $|u(x)| < \frac{1}{2} < 1$. Moreover the map $N : \Omega \to \text{Aut}\,\bar{\Delta}$ is continuous because so are k and u. Consider now the automorphism F of $\bar{B}(C_b(\Omega))$ defined by $F(f) \equiv [x \mapsto N(x) f(x)]$. By hypothesis, for some $f_0 \in \bar{B}(C_b(\Omega))$ we have $F(f_0) = f_0$. Thus

$$k(x)\frac{f_0(x) + u(x)}{1 + \overline{u(x)}\, f_0(x)} = f_0(x) \qquad \forall x \in \Omega$$

and therefore

$$f_0^2 \frac{2i\bar{\varphi} e^{it/2}}{1 + |\varphi|^2} \sin\frac{t}{2} + (1 - e^{it}) f_0 + \frac{2i\varphi e^{it/2}}{1 + |\varphi|^2} \sin\frac{t}{2} = 0$$

(²) I.e. given any cozero set G in Ω, each function $\varphi \in C_b(G)$ has a continuous extension to Ω (cf. [3; 14.25 Theorem (6)]). $C_b(\Omega)$ denotes the space of bounded continuous $\Omega \to \mathbf{C}$ functions.

on G. Dividing by

$$\frac{2ie^{it/2}}{1+|\varphi|^2}\sin\frac{t}{2}\left(=\frac{e^{it}-1}{1+|\varphi|^2}\neq 0 \quad \text{since } 0<t<\frac{\pi}{2} \text{ on } G\right)$$

we obtain $\overline{\varphi(x)}f_0(x)^2-\left(1+|\varphi(x)|^2\right)f_0(x)+\varphi(x)=0$ i.e. $f_0(x)\in\{\varphi(x),\,1/\overline{\varphi(x)}\}$ $\forall x\in G$. But $\|f_0\|<1$ and hence necessarily $f_0|G=\varphi$. Thus f_0 is a continuous extension of φ. □

In order to prove some converse of Proposition 1 and to generalize it, we go back to Aut $\bar{\Delta}$. Recall that any Möbius transformation M has a unique representation of the form

$$M=\left[\bar{\Delta}\ni\zeta\mapsto k_M\frac{\zeta+u_M}{1+\overline{u_M}\,\zeta}\right] \qquad \text{with } |k_M|=1 \text{ and } |u_M|<1\,,$$

and the mapping $M\mapsto(k_M,u_M)$ establishes a homeomorphism between Aut $\bar{\Delta}$ and $(\partial\Delta)\times\Delta$. We shall reserve the notation (k_M,u_M) for this mapping.

A simple computation yields

LEMMA 1. – Let $\mathrm{id}_{\bar{\Delta}}\neq M\in\operatorname{Aut}\bar{\Delta}$ and $e^{it}=k_M$. Then M has:

a) a unique fixed point which lies in Δ iff

$$|u_M|<\left|\sin\frac{t}{2}\right|\left(=\frac{k_M-1}{2}\right)$$

b) two distinct fixed points lying in $\partial\Delta$ iff $|u_M|>|\sin t/2|$;

c) a unique fixed point lying is $\partial\Delta$ iff $|u_M|=|\sin t/2|$.

LEMMA 2. – There are exactly two different continuous mappings from $(\operatorname{Aut}\bar{\Delta})\diagdown\{\mathrm{id}_{\bar{\Delta}}\}$ into $\bar{\Delta}$ which associate to any (non-identical) Möbius transformation one of its fixed points.

PROOF. – Recall that, in general, if $0<r<1$ and $F\in\operatorname{Aut}\bar{B}(E)$ where E is any complex Banach space then the mapping rF has always a unique fixed point (cf. [2]). Thus we may define the function Q: $[0,1)\times\operatorname{Aut}_{\bar{\Delta}}\to\bar{\Delta}$ by $Q(r,M)\equiv$ [the fixed point of rM]. If $r_j\to r(\in[0,1))$ and $M_j\to M$ then the net $Q(r_j,M_j)\ (=r_jM_jQ(r_j,M_j))$ tends obviously to some fixed point of rM, showing the continuity of Q. We shall prove that for every $\mathrm{id}_{\bar{\Delta}}\neq M\in\operatorname{Aut}\bar{\Delta}$, the sets

$$S_M\equiv\{\zeta\colon \exists\text{net } [(s_j,N_j)\colon j\in J] \quad (s_j,N_j)\to(1,M) \quad \text{and} \quad Q(s_j,N_j)\to\zeta\}$$

contain exactly one point. In fact, on the one hand

$$S_M=\bigcap_{n=1}^{\infty}\overline{Q\left\{(s,N)\colon 1-\frac{1}{n}<s<1 \quad \text{and} \quad |k_M-k_N|,\ |u_M-u_N|<\frac{1}{n}\right\}}$$

i.e. the intersection of a decreasing sequence of non-empty connected compact subsets of $\bar{\Delta}$, thus $S_M \neq \emptyset$ is connected and compact. On the other hand, $S_M \subset \{\zeta: M\zeta = \zeta\}$ which implies cardinality$(S_M) < 2$. But then cardinality $(S_M) = 1\ \forall M \in (\operatorname{Aut} \bar{\Delta}) \setminus \{\mathrm{id}_{\bar{\Delta}}\}$ means that the function R: $(\operatorname{Aut} \bar{\Delta}) \setminus \{\mathrm{id}_{\bar{\Delta}}\} \to \bar{\Delta}$ is well-defined by $R(M) \equiv \lim_{r \to 1} Q(r, M)$ and is continuous. Since $\{R(M)\} = S_M \subset \{\zeta: M\zeta = \zeta\}$, the mapping R is a continuous section of the multifunction ϕ: $M \mapsto \{\zeta \in \bar{\Delta}: M\zeta = \zeta\}$.

If R' denotes another continuous section of ϕ (defined on $(\operatorname{Aut} \bar{\Delta}) \setminus \{\mathrm{id}_{\bar{\Delta}}\}$) then, by Lemma 1 *a*), *c*),

$$\{M: R(M) \neq R'(M)\} \subset D \equiv \left\{M: |u_M| > \left|\frac{k_M - 1}{2}\right|\right\}.$$

Since $\phi(M) = \{\zeta \in \bar{\Delta}: \overline{u_M}\zeta^2 + (1 - k_M)\zeta - k_M u_M = 0\}\ \forall M \in D$, we have by continuity of the roots of polynomials depending on their coefficients, that the mapping $\phi|D$ is continuous from D into the space of the non-empty compact subsets of $\boldsymbol{C}$ endowed with the Hausdorff distance. Since cardinality $\phi(M) = 2\ \ \forall M \in D$, it easily follows that $\{M \in D: R'(M) = R(M)\}$ is open-closed in D. But D is connected because it is homeomorphic to

$$\left\{(k, u) \in (\partial\Delta) \times \Delta: |u| > \left|\frac{k-1}{2}\right|\right\} = \left\{(e^{it}, re^{i\delta}): t \in (-\pi, \pi),\ 1 > r > \left|\sin\frac{t}{2}\right|,\ \delta \in \boldsymbol{R}\right\}$$

which is a continuous image of the connected set $\{(t, r): t \in (-\pi, \pi),\ 1 > r > |\sin t/2\} \times \boldsymbol{R}$. Thus if $R' \neq R$ then we necessarily have that

$$\{R'(M)\} = \phi(M) \setminus \{R(M)\} \qquad \forall M \in D\,. \tag{2}$$

On the other hand, it directly follows that if we define R' by (2) on D and to coincide with R elsewhere, then R' is continuous. □

LEMMA 3. – For any $M \in \operatorname{Aut} \bar{\Delta}$ with $M \neq \mathrm{id}_{\bar{\Delta}}$ there exists a Lie homomorphism $t \mapsto M^t$ of $\boldsymbol{R}$ into $\operatorname{Aut} \bar{\Delta}$ such that $M^1 = M$ and, by setting $t_0 \equiv \inf\{t > 0: M^t = \mathrm{id}_{\bar{\Delta}}\}$ (convention: $\inf \emptyset \equiv +\infty$), we have

$$\{\zeta: M^t\zeta = \zeta\} = \{\zeta: M\zeta = \zeta\} \qquad \forall t \in (0, t_0)\,. \tag{3}$$

PROOF. – Fix M arbitrarily. According to Lemma 1, only the following cases are possible: *a*) M has a fixed point in Δ, *b*) M has two fixed point on $\partial\Delta$, *c*) the unique fixed point of M lies in $\partial\Delta$.

a) Since $\operatorname{Aut} \bar{\Delta}$ acts transitively on Δ, we can choose $N \in \operatorname{Aut} \bar{\Delta}$ which sends the fixed point of M into 0. Thus 0 is the fixed point of $K \equiv NMN^{-1}$. By the Schwarz Lemma, for some $\delta \in \boldsymbol{R}$, $K = [\zeta \mapsto e^{i\delta}\zeta]$. Set $K^t \equiv [\zeta \mapsto e^{i\delta t}\zeta]$ $(t \in \boldsymbol{R})$. Since $t \mapsto K^t$ is trivially a Lie homomorphism of $\boldsymbol{R}$ into $\operatorname{Aut} \bar{\Delta}$, we may define M^t by $M^t \equiv N^{-1}K^tN$.

b) The group Aut $\bar{\Delta}$ is doubly transitive on $\partial\Delta$. Thus we can find $N \in \mathrm{Aut}\,\bar{\Delta}$ such that one fixed point of M is sent by N into 1 and the other into -1. Now the fixed points of $K \equiv NMN^{-1}$ are -1 and 1. Observe that $k_K = 1$ and $u_K \in \boldsymbol{R}$ $\left(\text{for } k_K \dfrac{1+u_K}{1+\bar{u}_K} = 1 \text{ and } k_K \dfrac{(-1)+u_K}{1-\bar{u}_K} = -1 \quad \text{imply} \dfrac{1+u_K}{1-u_K}\Big/\left(\overline{\dfrac{1+u_K}{1-u_K}}\right) = 1\right)$. Now set $\delta \equiv \mathrm{areath}\,(u_K)$ and $K^t \equiv \left[\zeta \mapsto \dfrac{\zeta + \mathrm{th}\,(t\delta)}{1+\zeta\,\mathrm{th}\,(t\delta)}\right]$. A direct calculation shows $K^{t+s} = K^t K^s$ $\forall t, s \in \boldsymbol{R}$. Thus, also in this case, we may put $M^t \equiv N^{-1}K^t N$ $(t \in \boldsymbol{R})$.

c) Let ζ_0 denote the unique fixed point of M and fix $N' \in \mathrm{Aut}\,\bar{\Delta}$ so that $N\zeta_0 = 1$. Further let N'' be the Cayley transformation $\zeta \mapsto (1/i)\big((\zeta+1)/(\zeta-1)\big)$ mapping $\bar{\Delta}$ onto the closure of $\Pi \equiv \big(\{\infty, \zeta \in \boldsymbol{C}\colon \mathrm{Im}\,\zeta > 0\}\big)$ and set $N \equiv N''N'$. Now the mapping $K \equiv NMN^{-1}$ belongs to $\mathrm{Aut}\,\Pi$ and satisfies $K(\infty) = \infty$. Therefore K is linear, moreover $\exists \alpha, \beta \in \boldsymbol{R}$ $K = [\zeta \mapsto \alpha\zeta + \beta]$. Since the only fixed point of M is ζ_0, K must have no other fixed point than ∞. But hence K is a translation (i.e. there exists $\beta \in \boldsymbol{R}$ such that $K = [\zeta \mapsto \zeta + \beta]$). Then by letting $K^t \equiv [\zeta \mapsto \zeta + \beta t]$ and $M^t \equiv N^{-1}K^t N$ $(t \in \boldsymbol{R})$ the proof is complete. □

LEMMA 4. – Let $R(\cdot)$ denote any one of the continuous sections of $M \mapsto \{\zeta\colon M\zeta = \zeta\}$ (on $(\mathrm{Aut}\,\bar{\Delta})\setminus\{\mathrm{id}_{\bar{\Delta}}\}$). Then *a*) $MR(M) = R(M)$, *b*) $R(M^n) = R(M)$ whenever $M^n \neq \mathrm{id}_{\bar{\Delta}}$ $(n = 0, \pm 1, \pm 2, \ldots)$, *c*) $R(NMN^{-1}) = NR(M)$ for all $N \in \mathrm{Aut}\,\bar{\Delta}$.

PROOF. – *a*) Is trivial. *b*) Fix M and n and take a Lie homomorphism $t \mapsto M^t$ as in Lemma 3, set also $t_0 = \inf\{t > 0\colon M^t \neq \mathrm{id}_{\bar{\Delta}}\}$. From *a*) and (3) we deduce $R\big(\{M^t\colon t \in (0, t_0)\}\big) \subset \{\zeta\colon M\zeta = \zeta\} \in R(M)$. Hence the function $\varrho \equiv [(0, t_0) \in t \mapsto R(M^t)]$ is constant (recall, M has at most two fixed points). Thus if $M^n = \mathrm{id}_{\bar{\Delta}}$, $R(M^n) = R(M^{\mathrm{mod}_{t_0} n}) = \varrho\,(\mathrm{mod}_{t_0} n) = \varrho(\mathrm{mod}_{t_0} 1) = R(M^{\mathrm{mod}_{t_0} 1}) = R(M^1) = R(M)$ [3].

c) Let $t \mapsto N^t$ be any Lie homomorphism $\boldsymbol{R} \to \mathrm{Aut}\,\bar{\Delta}$ with $N^1 = N$. Observe that $N^{-t}R(N^tMN^{-t}) \in \{\zeta\colon M\zeta = \zeta\}$ (since $N^tMN^{-t}\eta = \eta \Leftrightarrow M(N^{-t}\eta) = N^{-t}\eta$), $\forall t \in \boldsymbol{R}$. Therefore the function $t \mapsto N^{-t}R(N^tMN^{-t})$ is constant. In particular, $N^{-1}R(NMN^{-1}) = N^0R(N^0MN^0) = R(M)$. □

DEFINITION 1. – Let δ_n denote the metric on $\boldsymbol{C}^n$ defined by $\delta_n\big((\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)\big) \equiv \max\{|a_j - \beta_j|\colon j = 1, \ldots, n\}$. For any $N^* \equiv (N_0, \ldots, N_{n-2}) \in (\mathrm{Aut}\,\bar{\Delta})^{n-1}$ and $\zeta^* \equiv (\zeta_0, \ldots, \zeta_{n-1}) \in \bar{\Delta}^n$ set

$$P_n(N^*, \zeta^*) \equiv \big[\text{that } \eta \in \bar{\Delta} \text{ for which } \delta_n\big(\zeta^*, (\eta, N_0\eta, \ldots, N_{n-2}\eta)\big) \quad \text{is minimal}\big]. \tag{4}$$

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[3] $\mathrm{mod}_\alpha \beta \equiv \inf([0, \infty) \geqslant \{\beta + n\alpha\colon n \in \boldsymbol{Z}c\})$, $\mathrm{mod}_\alpha \beta \equiv \beta$ for all $\alpha > 0$, $\beta \in \boldsymbol{R}$.

LEMMA 5. – The definition of P_n makes sense (i.e. there is a unique $\eta \in \bar{\Delta}$ with $\delta_n(\zeta^*, (\eta, N_0\eta, \dots, N_{n-2}\eta)) < \delta_n(\zeta^*, (\eta', N_0\eta', \dots, N_{n-2}\eta'))$ $\forall \eta' \in \bar{\Delta}$). Furthermore, if $M_0 \dots M_{n-1} = \mathrm{id}_{\bar{\Delta}}$ and $P_n((M_0^{-1}, M_1^{-1}M_0^{-1}, \dots, M_{n-2}^{-1} \dots M_0^{-}), \zeta^*) = \eta$ then $P_n((M_1^{-1}, M_2^{-1}M_1^{-1}, \dots, M_{n-1}^{-1} \dots M_1^{-1}), (\zeta_1, \zeta_2, \dots, \zeta_{n-1}, \zeta_0)) = M_0^{-1}\eta$ (here $\zeta^* \equiv (\zeta_1, \dots, \zeta_{n-1})$).

PROOF. – A standard compactness argument shows the existence of at least one minimizing η in (4). Set $\varepsilon \equiv \delta_n(\zeta^*, \{(\eta, N_0\eta, \dots, N_{n-2}\eta): \eta \in \bar{\Delta}\})$. Observe then that $\varepsilon = \min\{\varepsilon' > 0: (\zeta^* + \varepsilon'\bar{\Delta}^n) \cap \{(\eta, N_0\eta, \dots, N_{n-2}\eta): \eta \in \bar{\Delta}\} \neq \emptyset\}$. Thus for the set $Z \equiv \{(\eta, N_0\eta, \dots, N_{n-2}\eta): \eta \in \bar{\Delta}\}$ we have $Z \cap (\zeta^* + \varepsilon\Delta^n) = \emptyset$ and $\{\eta \in \bar{\Delta}: \delta_n(\zeta^*, (\eta, N_0\eta, \dots, N_{n-2}\eta)) = \varepsilon\} \subset Z \cap \partial(\zeta^* + \varepsilon\bar{\Delta}^n)$. Let Φ denote the map $\Phi: (\alpha_0, \dots, \alpha_{n-1}) \mapsto (\alpha_0, N_0^{-n}\alpha_1, \dots, N_{n-2}^{-1}\alpha_{n-1})$. Then $\Phi(Z) = \{(\zeta, \dots, \zeta) \in \boldsymbol{C}^n: \zeta \in \bar{\Delta}\}$ and $\Phi(\zeta^* + \varepsilon\Delta)$ is a set of the form $\{(\alpha_0, \dots, \alpha_{n-1}): |\alpha_0 - \beta_0| < \varepsilon_0, \dots, |\alpha_{n-1} - \beta_{n-1}| < \varepsilon_{n-1}\}$ for some $\beta^* \in \boldsymbol{C}^n$ and $\varepsilon^* \in [0, \infty)^n$. So it suffices to prove that if $\Delta_0, \dots, \Delta_{n-1}$ are open discs in $\boldsymbol{C}$ then the set $D \equiv \{(\lambda, \dots, \lambda) \in \boldsymbol{C}^n: \lambda \in \bar{\Delta}\}$ intersects the boundary of $C \equiv \Delta_0 \times \dots \times \Delta_{n-1}$ in at most one point whenever $D \cap C = \emptyset$. Proceed by contradiction: Assume $(\lambda_k, \dots, \lambda_k) \in D \cap \partial C$ $(k = 1, 2)$ and $D \cap C = \emptyset$ and set $\lambda \equiv \frac{1}{2}(\lambda_1 + \lambda_2)$, $\mu \equiv \frac{1}{2}(\lambda_1 - \lambda_2)$ and $F_j \equiv \bar{\Delta}_0 \times \dots \times \overline{\Delta_{j-1}} \times (\partial\Delta_j) \times \overline{\Delta_{j+1}} \times \dots \times \overline{\Delta_{n-1}}$ $(j = 0, \dots, n-1)$. Since C and D are convex, we have $(\lambda, \dots, \lambda) + [-1, 1]\,(\mu, \dots, \mu) \subset \partial C = F_0 \cup \dots \cup F_{n-1}$.
Therefore for some index j, the intersection $F_j \cap [(\lambda, \dots, \lambda) + [-1, 1]\cdot(\mu, \dots, \mu)]$ contains an inner point of the segment $(\lambda, \dots, \lambda) + [-1, 1]\cdot(\mu, \dots, \mu)$. That is, for some j and for some $\lambda' \in \boldsymbol{C}$ and $\mu' \in \boldsymbol{C} \setminus \{0\}$ we have $(\lambda', \dots, \lambda') + (-1, 1)\cdot(\mu', \dots, \mu') \subset F_j$ But this would mean that $\lambda' + \tau\mu' \in \partial\Delta_j$, $\forall \tau \in (-1, 1)$ which is impossible. Thus (4) makes sense.

To prove the second statement, observe that, by definitions of P_n and δ_n we have $\delta_n((\zeta_0, \dots, \zeta_{n-1}), (\eta, M_0^{-1}\eta, \dots, M_{n-2}^{-1} \dots M_0^{-1}\eta)) \leqslant \delta_n((\zeta_0, \dots, \zeta_{n-1}), (\eta', M_0^{-1}\eta', \dots, M_{n-2}^{-1} \dots M_0^{-1}\eta'))$ $\forall \eta' \in \bar{\Delta}$. Thus for any $\eta' \in \bar{\Delta}$, $\delta_n((\zeta_1, \dots, \zeta_{n-1}, \zeta_0), (M_0^{-1}\eta, \dots, M_{n-2}^{-1} \dots M_0^{-1}\eta, \eta)) \leqslant \delta_n((\zeta_1, \dots, \zeta_{n-1}, \zeta_0), (M_0^{-1}\eta', \dots, M_{n-2} \dots M_0\eta', \eta'))$ or which is the same, $\delta_n((\zeta_1, \dots, \zeta_{n-1}, \zeta_0), ((M_0^{-1}), M_1^{-1}(M_0^{-1}\eta), \dots, M_{n-2}^{-1} \dots M_1^{-1}(M_0^{-1}\eta), M_{n-1}^{-1} \dots M_1^{-1}(M_0^{-1}\eta))) \leqslant$ $\leqslant$[same expression with η' in place of η]. Since $\bar{\Delta} = \{M_0^{-1}\eta': \eta' \in \bar{\Delta}\}$, this means that the function $\lambda \mapsto \delta_n((\zeta_1, \dots, \zeta_{n-1}, \zeta_0), (\lambda, M_1^{-1}\lambda, \dots, M_{n-1}^{-1} \dots M_1^{-1}\lambda))$ attains its minimum over $\bar{\Delta}$ at the point $M_0^{-1}\eta$. □

LEMMA 6. – The mapping $P_n: (\mathrm{Aut}\,\bar{\Delta})^{n-1} \times \bar{\Delta}^n \to \bar{\Delta}$ is continuous.

PROOF. – Since P_n is a map of a locally compact space into a compact space, it suffices to see that its graph is closed. To do this, examine first the function: $(\mathrm{Aut}\,\bar{\Delta})^{n-1} \times \bar{\Delta}^n \to [0, \infty)$ defined by $(N^*, \zeta^*) \equiv \delta_n(\zeta^*, \{(\eta, N_0\eta, \dots, N_{n-2}\eta): \eta \in \bar{\Delta}\})$. Clearly, $\varphi = \inf\{\varphi_{\cdot}: \eta \in \bar{\Delta}\}$ where $\varphi_n \equiv [(N^*, \zeta^*) \mapsto \delta_n(\zeta^*, (\eta, N_0\eta, \dots, N_{n-2}\eta))]$. It follows from the triangle inequality that all φ_n (with $\eta \in \bar{\Delta}$) satisfy the Lipschitz condition

$$|\varphi_\eta(N^*, \zeta^*) - \varphi_\eta(N'^*, \zeta'^*)| \leqslant \sum_{j=0}^{n-1} |\zeta_j - \zeta'_j| + \sum_{j=0}^{n-2} \sup\{|N_j\xi - N'_j\xi|: \zeta \in \bar{\Delta}\}$$

$(\forall N^*, N'^* \in (\operatorname{Aut} \bar{\Delta})^{n-1}, \forall \zeta^*, \zeta'^* \in \bar{\Delta}^n)$. But then also their infimum satisfies the same Lipschitz condition. Thus φ is continuous. Now if $(N^{(i)*}, \zeta^{(i)*}) \to (N^*, \zeta^*)$ and $P_n(N^{(i)*}, \zeta^{(i)*}) \to \eta$ then

$$\begin{array}{c} (N^{(i)*}, \zeta^{(i)*}) = \delta_n\big(\zeta^{(i)*}, (P_n(N^{(i)*}, \zeta^{(i)*}), N_0^{(i)} P_n(N^{(i)*}, \zeta^{(i)*}), \ldots, N_{n-2}^{(i)} P_n(N^{(i)*}, \zeta^{(i)*}))\big) \\ \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ (N^*, \zeta^*) = \inf\{\delta_n(\zeta^*, (\eta', N_0\eta', \ldots, N_{n-2}\eta')) : \eta' \in \Delta\} = \delta_n(\zeta^*, (\eta, N_0\eta, \ldots, N_{n-2}\eta)) . \end{array}$$

But this latter inequality is the definition of the relation $P_n(N^*, \zeta^*) = \eta$. □

THEOREM 1. – Let Ω denote any topological space. Then the following statements are equivalent

a) All the automorphisms of $\bar{B}(C_b(\Omega))$ of the form $f \mapsto [x \mapsto M(x) f(x)]$ where M is any continuous $\Omega \to \operatorname{Aut} \bar{\Delta}$ mapping have fixed points.

b) All the automorphisms of $\bar{B}(C_b(\Omega))$ of the form $f \mapsto [x \mapsto M(x) f(Tx)]$ where $M(\cdot)$ is a continuous $\Omega \to \operatorname{Aut} \bar{\Delta}$ mapping and T is a periodic automorphism [4] of Ω have a fixed point.

c) Ω is an F-space.

PROOF. – *b*) $\Rightarrow$ *a*) is evident and *a*) $\Rightarrow$ *c*) is established by the proof of Proposition 1. To prove *c*) $\Rightarrow$ *b*), suppose that Ω is an F-space and let M and T be a continuous $\Omega \to \operatorname{Aut} \bar{\Delta}$ mapping and a topological automorphism of Ω, respectively. Define F by $F(f) \equiv [x \mapsto M(x) f(Tx)]$ (for all $f \in \bar{B}(C_b(\Omega))$). (Clearly $F \in \operatorname{Aut} \bar{B}(C_b(\Omega))$). Furthermore assume $T^n = \mathrm{id}_{\bar{\Delta}}$, and let R denote a continuous section defined on $(\operatorname{Aut} \bar{\Delta}) \setminus \{\mathrm{id}_{\bar{\Delta}}\}$ of the multifunction $M \mapsto \{\zeta : M\zeta = \zeta\}$ (its existence is seen in Lemma 2).

Consider the set $G \equiv \{x \in \Omega : M(x) M(Tx) \ldots M(T^{n-1}x) \neq \mathrm{id}_{\bar{\Delta}}\}$ and define the function $g : G \to \bar{\Delta}$ by $g(x) \equiv R(M(x) \ldots M(T^{n-1}x))$. Since G is the inverse image of the open subset $(\operatorname{Aut} \bar{\Delta}) \setminus \{\mathrm{id}_{\bar{\Delta}}\}$ of the metrizable space $\operatorname{Aut}_{\bar{\Delta}}$ by the continuous mapping $x \mapsto M(x) \ldots M(T^{n-1}x)$ the set G is a cozero subset of Ω (namely we have is particular $G = \{x \in \Omega : |k_{M(x) \ldots M(T^{n-1}x)}| + |u_{M(x) \ldots M(T^{n-1}x)}| \neq 0\}$). On the other hand, G is also T-invariant because in case of $M(x) \ldots M(T^{n-1}x) = \mathrm{id}_{\bar{\Delta}}$ we have $M(Tx) \ldots M(T^{n-1}x) \cdot M(x) = \mathrm{id}_{\bar{\Delta}}$ and here the last term can be written as $M(x) = M(T^n x) = M(T^{n-1}(Tx))$. About the function g we can state the following:

$$g(x) = M(x) g(Tx) \qquad \forall x \in G . \tag{5}$$

[4] The (topological) automorphisms of a topological space are its homeomorphisms onto itself.

Indeed, if $x \in G$, we have $g(Tx) = R\big(M(Tx) \ldots M(T^{n-1}(Tx))\big) = R\big(M(Tx \ldots M \cdot (T^{n-1}x) M(x)\big) = R\big(M(x)^{-1}[M(x) \ldots M(T^{n-1}x)] M(x)\big) =$ by Lemma 4 $c) = M(x)^{-1} \cdot R\big(M(x) \ldots M(T^{n-1}x)\big) = M(x)^{-1} g(x)$.

Now let h be a continuous extension of g from G to Ω. The existence of such a function h is established by [3; 14.25. Theorem (6)] since Ω is assumed to be an F-space. Since $|g| < 1$, we may assume without any loss of generality that also $|h| = 1$. Thus let $h \in \bar{B}\big(C_b(\Omega)\big)$ with $h|G = g$. Define the function $f\colon \Omega \to \bar{\Delta}$ (which will be our candidate to be a fixed point of F) by

$$f(x) \equiv P_n\Big(\big(M(x)^{-1}, M(Tx)^{-1} M(x)^{-1}, \ldots, M(T^{n-2}x)^{-1} \ldots M(x)^{-1},$$
$$\big(h(x), h(Tx), \ldots, h(T^{n-1}x)\big)\Big)\,.$$

We check now that for any $x \in \Omega$, $f(x) = M(x) f(Tx)$.

First let $x \in G$. Then $h(x) = g(x)$. But we have $g = M \cdot (g \circ T)$ which implies

$$g(Tx) = M(x)^{-1} g(x),\ g(T^2x) = M(Tx)^{-1} g(Tx) = M(Tx)^{-1} M(x)^{-1} g(x), \ldots, g(T^{n-1}x) =$$
$$= M(T^{n-2}x)^{-1} \ldots M(x)^{-1} g(x)$$

Thus

$$f(x) = P_n\Big(\big(M(x)^{-1}, \ldots, M(T^{n-2}x)^{-1} \ldots M(x)^{-1}\big),$$
$$\big(g(x), M(x)^{-1} g(x), \ldots, M(T^{n-2}x)^{-1} \ldots M(x)^{-1} g(x)\big)\Big)\,. \tag{6}$$

A direct application of Definition 1 to the right hand side of (6) yields that $f(x) = g(x)$. It follows then from (5) that $f(x) = g(x) = M(x) g(Tx) =$ applying (6) to $Tx (\in G)$ in place of $x = M(x) f(Tx)$.

Then let $x \in \Omega \setminus G$. Now $M(x) \ldots M(T^{n-1}x) = \mathrm{id}_{\bar{\Delta}}$. Thus

$$f(Tx) = P_n\Big(\big(M(Tx)^{-1}, \ldots, M(T^{n-1}x)^{-1} \ldots M(Tx)^{-1}\big), \big(h(Tx), \ldots, h(T^n x)\big)\Big) =$$
$$= P_n\Big(\big(M(Tx)^{-1}, \ldots, M(T^{n-1}x)^{-1} \ldots M(Tx)^{-1}\big), \big(h(Tx), \ldots, h(T^{n-1}x), h(x)\big)\Big)\,.$$

Therefore, by substituting $M_j \equiv M(T^j x)$, $\zeta_j \equiv h(T^j x)$ $(j = 0, \ldots, n-1)$ and $\eta \equiv f(x)$ in Lemma 8, we can verify $f(Tx) = M(x)^{-1} f(x)$.

The continuity of f is an immediate consequence of Lemma 6. □

2. – On Aut $\bar{B}\big(C(\Omega)\big)$ in case of compact F-spaces Ω.

It is well-known [11] that for a compact topological space Ω, the automorphisms of $D \equiv \bar{B}\big(C(\Omega)\big)$ are exactly those transformations $F\colon D \to C(\Omega)$ which can be represented in the form

$$F(f) = [\Omega \ni x \mapsto M_F(x) f(T_F X)] \qquad (f \in D) \tag{7}$$

where T_E and M_F are a homeomorphism of Ω onto itself and a continuous $\Omega \to \operatorname{Aut} \bar{\Delta}$ mapping, respectively. Both T_F and M_F are uniquely determined by F. In the sequel we reserve the notations T_F, M_F to indicate the topological automorphism of Ω and the $\Omega \to \operatorname{Aut} \bar{\Delta}$ mapping, defined implicitly by (7) whenever $F \in \operatorname{Aut} \bar{B}(C(\Omega))$.

Since for any F-space Ω there exists a completely regular F-space $\tilde{\Omega}$ such that $C_b(\Omega) \simeq C_b(\tilde{\Omega})$ (i.e. $C_b(\Omega)$ is isometrically isomorphic with $C_b(\tilde{\Omega})$; cf. [3; 3.9. Theorem]) and since the Stone-Čech compactification of any (completely regular) F-space is an F-space (cf. [3; 14.25. Theorem (1)]), it suffices to restrict our attention to compact F-spaces Ω (by Proposition 1) when looking for those space Ω that admit elements with fixed points for $\operatorname{Aut} \bar{B}(C(\Omega))$. Fortunately, in this case the description provided by (7) yields a precise controll of $\operatorname{Aut} \bar{B}(C(\Omega))$. However, the complete characterization of those compact space Ω for which any $F \in \operatorname{Aut} \bar{B}(C(\Omega))$ has some fixed point seems to be extremely difficult.

DEFINITION 2. – If T is a mapping of some set Ω into itself and $x \in \Omega$ then we shall call the number $\inf \{n \in \boldsymbol{N}\colon T^n x = x\}$ the rank of T at the point x and we shall denote it by $r_T(x)$. T will be said pointwise periodic if $r_T(x) < \infty$ (i.e. $\{n \in \boldsymbol{N}\colon T^n x = x\} \neq \emptyset$) for all $x \in \Omega$.

LEMMA 7. – Let Ω be a Baire space and T a pointwise periodic automorphism of Ω. For $n = 1, 2, \ldots$ set $\Omega_n \equiv \{x \in \Omega\colon r{:}(x) < n\}$ and let $G \equiv \bigcup_{n=1}^{\infty} (\Omega_n \setminus \Omega_{n-1})^0$ where $\Omega_0 \equiv \emptyset$ (^{0} denoting the interior). Then G is an open dense T-invariant subset of Ω. Furthermore we have

$$\overline{\lim_{y \to x}}\, r_T(y) = \overline{\lim_{G \ni y \to x}}\, r_T(y) \qquad \forall x \in \Omega\,.$$

PROOF. – If $(x_j\colon j \in J)$ is a net such that $x_j \to x$ (in Ω) and $T^m x_j = x_j$, $\forall j \in J$ then obviously $T^m x = x$. Thus the function $r_T\colon \Omega \to \boldsymbol{R}$ is lower semicontinuous. Therefore $\Omega_1, \Omega_2, \ldots$ are all closed. Since the pointwise periodicity of T is equivalent to $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$, this means that the set $G' \equiv \bigcup_{n=1}^{\infty} \Omega_n^0$ is dense in Ω. Consider now any open $U \subset \Omega$. By the density of G' in Ω, we can find n_0 with $\Omega_{n_0}^0 \cap U \neq 0$. Since $r_T(x) < n_0$, $\forall x \in \Omega_{n_0}$, there exists $y_0 \in \Omega_{n_0}^0 \cap U$ such that $r_T(y_0) = \max \{r_T(x)\colon x \in \Omega_{n_0}^0 \cap U\}$. But $\{x \in \Omega_{n_0} \cap U\colon r_T(x) = r_T(y_0)\} = U \cap \Omega_{n_0}^0 \cap \{x \in \Omega\colon r_T(x) > r_T(y_0) - 1\}$ is an open neighbourhood of the point y_0. Therefore the set $G'' \equiv \{y \in \Omega\colon \exists U$ neighbourhood of y $\forall x \in U \;\; r_T(x) = r_T(y)\}$ is dense in Ω. But $G'' = \bigcup_{n=1}^{\infty} \{y \in \Omega\colon \exists U$ neighbourhood of y $\forall x \in U \;\; r_T(x) = n\} = \bigcup_{n=1}^{\infty} \{x \in \Omega\colon r_T(x) = n\}^0 = \bigcup_{n=1}^{\infty} (\Omega_n \setminus \Omega_{n-1})^0$.

The T-invariance of G is immediate.

To prove the second statement, observe that by the lower semicontinuity of $r_T(\cdot)$ we have $\underset{G\ni z\to y}{\underline{\lim}}\, r_T(z) \geq r_T(y)\ \forall y \in \Omega$. Thus

$$\overline{\lim_{G\ni y\to x}} r_T(y) \leq \overline{\lim_{y\to x}} r_T(y) \leq \overline{\lim_{y\to x}}\ \overline{\lim_{G\ni z\to y}} r_T(z) = \lim_{G\ni z\to x} r_T(z)\,. \qquad \square$$

LEMMA 8. – Let Ω be a compact space, T an automorphism of Ω, $f_1, f_2, \ldots \in C(\Omega)$ and let A denote the closed C^*-subalgebra of $C(\Omega)$ (with the usual complex-conjugate involution) generated by the functions 1_Ω and $f_n \circ T^m$ $(n \in \boldsymbol{N}, m \in \boldsymbol{Z})$. Then there are a compact metric space K, a surjective continuous map $\phi\colon \Omega \to K$ and a homeomorphism $\tilde{T}$ of K onto itself such that $A = C(K)\circ\phi$ (5) and $\phi\circ T = \tilde{T}\circ\phi$.

PROOF. – The commutative Gel'fand-Neumark theorem establishes the existence of a compact Hausdorff space K and an isometric *-isomorphism ψ between $C(K)$ and A. Fix such a space K and a mapping ψ. Since A is separable, $C(K)$ is also separable and therefore K is metrizable (cf. [9]). Let us evaluate $\psi^* \delta_x$ for an arbitrary $x \in \Omega$ (ψ^* and δ_x denoting respectively the adjoint map of ψ and the Dirac-δ associated to the point x): $\psi^*\delta_x(\tilde{f}) = \langle \tilde{f}, \psi^*\delta_x\rangle = \langle \psi\tilde{f}, \delta_x\rangle\ \forall \tilde{f} \in C(K)$. Thus $\psi^*\delta_x$ is a non-vanishing multiplicative linear functional over $C(K)$. Hence there is a unique $\tilde{x} \in K$ such that $\psi^*\delta_x = \delta_{\tilde{x}}$. Let $\phi\colon \Omega \to K$ be constant map of Ω whose value is the point $\tilde{x} \in K$ that satisfies $\psi^*\delta_x = \delta_{\tilde{x}}$. Now $(\psi\tilde{f})(x) = \langle \psi\tilde{f}, \delta_x\rangle = \langle \tilde{f}, \psi^*\delta_x\rangle = \langle \tilde{f}, \delta_{\phi(x)}\rangle = \tilde{f}(\phi(x))\ \ \forall x \in \Omega$ i.e. $\psi\tilde{f} = \tilde{f}\circ\phi\ \ \forall \tilde{f} \in C(K)$. Thus $A = \psi C(K) = C(K)\circ\phi$. Furthermore we have $\|\tilde{f}\| = \|\psi\tilde{f}\| = \|\tilde{f}\circ\phi\|\ \forall \tilde{f} \in C(K)$, and this implies also that range$(\phi) = K$.

To complete the argument, consider the transformation $Q\colon C(K) \to C(K)$ defined by $Q\tilde{f} \equiv \psi^{-1}[(\psi\tilde{f})\circ T]$. Observe that Q is an order preserving surjective isometry of $C(K)$. So there is a unique homeomorphism $\tilde{T}\colon K \to K$ with $Q(\tilde{f}) = \tilde{f}\circ\tilde{T}\ \ \forall \tilde{f} \in C(K)$ (see [9]). Defining $\tilde{T}$ in this way, we have $\tilde{f}\circ\tilde{T}\circ\phi = (Q\tilde{f})\circ\phi = \big(\psi^{-1}[(\psi\tilde{f})\circ T]\big)\circ\phi = [\psi^{-1}(\tilde{f}\circ\phi\circ T)]\circ\phi = \psi[\psi^{-1}(\tilde{f}\circ\phi\circ T)] = \tilde{f}\circ\phi\circ T\ \forall \tilde{f} \in C(K)$. Therefore $\tilde{T}\circ\phi = \phi\circ T$. $\square$

COROLLARY 1. – If $f_1 = f_2 = \ldots = f(\in C(\Omega))$ then

$$\inf\{n \in \boldsymbol{N}\colon f(T^k x) = f(T^{\operatorname{mod}_n k} x) \quad \forall k \in \boldsymbol{Z}\} = r_{\tilde{T}}(\phi(x))\,.$$

PROOF. – Choose a function $\tilde{f} \in C(K)$ such that $f = \tilde{f}\circ\phi$ and set $r^*(x) \equiv \inf\{n \in \boldsymbol{N}\colon \forall k \in \boldsymbol{Z}, f(T^k x) = f(T^{\operatorname{mod}_n k} x)\}$ (for $x \in \Omega$), $\tilde{r}^*(\tilde{x}) \equiv \inf\{n \in \boldsymbol{N}\colon \forall k \in \boldsymbol{Z}, \tilde{f}(\tilde{T}^k \tilde{x}) = \tilde{f}(\tilde{T}^{\operatorname{mod}_n k}\tilde{x})\}$ (for $\tilde{x} \in K$).

First we see that $r^*(x) = \tilde{r}^*(\phi(x))\ \forall x \in \Omega$. Indeed, $r^*(x) < l$ means that for some $0 < n \leq l$, $\forall k \in \boldsymbol{Z}\ f(T^{\operatorname{mod}_n k} x) = f(T^k x)$, i.e. $\exists\, 0 < n < l\ \forall k \in \boldsymbol{Z}\ \tilde{f}(\tilde{T}^{\operatorname{mod}_n k}\phi(x)) = \tilde{f}(\phi(T^{\operatorname{mod}_n k} x)) = \tilde{f}(\phi(T^k x)) = \tilde{f}(\tilde{T}^k\phi(x))$, which can be interpreted as $\tilde{r}^*(\phi(x)) \leq l$, for any $l \in \boldsymbol{N}$.

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(5) I.e. $\forall f \in C(\Omega) \quad (f \in A \Leftrightarrow \exists \tilde{f} \in C(K) \quad f = \tilde{f}\circ\phi)$.

We prove now that $\tilde{r}^* = r_T$. Since $A = C(K)\circ\phi$, for each pair $x, y \in \Omega$ with $\phi(x) \neq \phi(y)$ there exists $g \in A$ such that $g(x) \neq g(y)$. By definition of A and by the Stone-Weierstrass theorem, there exists $k \in \mathbf{Z}$ such that $f(T^k x) \neq f(T^k y)$. Thus $\phi(x) = \phi(y)$ iff $\forall k \in \mathbf{Z}$ $f(T^k x) = f(T^k y)$. Therefore if $\tilde{x} \in K$ and $x \in \phi^{-1}(\{x\})$ then $\tilde{T}^n \tilde{x} = \tilde{x}$ iff $\tilde{T}^n \phi(x) = \phi(x)$ iff $\phi(T^n x) = \phi(x)$ iff $\forall k \in \mathbf{Z}$ $f(T^{n+k}x) = f(T^k x)$ (these equivalences hold for any $n \in \mathbf{N}$). Thus for all $n \in \mathbf{N}$, the conditions $\tilde{T}^n \tilde{x} = \tilde{x}$ and $\forall k \in \mathbf{Z}$ $f(T^k x) = f(T^{\bmod_n k} x)$ are equivalent. This implies that $\inf\{n \in \mathbf{N}: \tilde{T}^n \tilde{x} = \tilde{x}\} = \inf\{n \in \mathbf{N}: \tilde{f}(\tilde{T}^{\bmod_n k}\tilde{x}) = f(\tilde{T}^k \tilde{x}) \ \ \forall k \in \mathbf{Z}\}$. □

LEMMA 9. – Let Ω be a compact F-space, T a pointwise periodic automorphism of Ω. Then for all $f \in C(\Omega)$, there exists $n_0 \in \mathbf{N}$ such that $f = f\circ T^{n_0}$.

PROOF. – Set again $f_1 \equiv f_2 \equiv \ldots \equiv f$ and let A, K, ϕ and $\tilde{T}$ be as in Lemma 8. Suppose the contrary of the statement of Lemma 9, i.e. that, in view of Corollary 1, $\sup\{r_{\tilde{T}}(\tilde{x}): \tilde{x} \in K\} = \infty$. Since clearly $r_{\tilde{T}}(\phi(x)) \leqslant r_T(x)$, $\forall x \in \Omega$, the homeomorphism $\tilde{T}: K \leftrightarrow K$ is also pointwise periodic. Hence we can apply Lemma 7 to K and $\tilde{T}$. This shows, in view of the lower semicontinuity of the function $r_{\tilde{T}}$, that there is a sequence $\tilde{x}_1, \tilde{x}_2, \ldots$ K with $r_{\tilde{T}}(\tilde{x}_n)\uparrow\infty$ $(n \to \infty)$ such that $\tilde{x}_n$ is an inner point of $\{\tilde{x} \in K: r_{\tilde{T}}(\tilde{x}) = r_{\tilde{T}}(\tilde{x}_n)\}$ for all n. For any $n \in \mathbf{N}$, let $V_n^0, \ldots, V_n^{r_{\tilde{T}}(x_n)-1}$ $\{\tilde{x} \in K: r_{\tilde{T}}(\tilde{x}) = r_T(x_n)\}$ be pairwise disjoint neighbourhoods of the points $\tilde{x}_n, \tilde{T}\tilde{x}_n, \ldots, \tilde{T}^{r_{\tilde{T}}(x_n)-1}\tilde{x}_n$, respectively. (Remark: $\{\tilde{x} \in K: r_{\tilde{T}}(\tilde{x}) = r_{\tilde{T}}(\tilde{x}_n)\} = \{\tilde{x} \in K: r_{\tilde{T}}(\tilde{x}) = r_{\tilde{T}}(\tilde{T}^k \tilde{x}_n)\}$, $\forall k \in \mathbf{Z}$.) Set $U_n^k \equiv \bigcap_{l=0}^{r_{\tilde{T}}(x_n)-1} (\tilde{T}^{k-1} V_n^l)$ (for $n \in \mathbf{N}$, $k \in \mathbf{Z}$). Now the family $\{U_n^k: n \in \mathbf{N}, 0 \leqslant k < r_{\tilde{T}}(\tilde{x}_n)\}$ is disjoint and $U_k^n = \tilde{T}^k U_n^0$ (for all n and $0 \leqslant k < r_{\tilde{T}}(\tilde{x}_n)$). Let us fix an irrational number δ and a sequence of integers $l_1, l_2, \ldots$ with $l_n / r_{\tilde{T}}(\tilde{x}_n) \to \delta$ $(n \to \infty)$. Define the functions $\tilde{g}, \tilde{h}$ on $\bigcup_{n=1}^{\infty} \bigcup_{k=0}^{r_{\tilde{T}}(x_n)-1} U_n^k$ by $\tilde{g}(\tilde{x}) \equiv \exp(2\pi i k l_n / r_{\tilde{T}}(\tilde{x}_n))$ and $\tilde{h}(\tilde{x}) \equiv \exp(2\pi i l_n / r_{\tilde{T}}(\tilde{x}_n))$ for all $\tilde{x} \in U_n^k$ $(n \in \mathbf{N}, 0 \leqslant k < r_{\tilde{T}}(\tilde{x}_n))$. Set $g_0 \equiv \tilde{g}\circ\phi$ and $h_0 \equiv \tilde{h}\circ\phi$ with domain $G \equiv \phi^{-1}\Big(\bigcup_{n=1}^{\infty} \bigcup_{k=0}^{r_{\tilde{T}}(x_n)-1} U_n^k\Big)$. Then we have $\tilde{g}(\tilde{T}\tilde{x}) = \tilde{h}(\tilde{x})\cdot\tilde{g}(\tilde{x})$, $\forall x$. Therefore $g_0 \circ T = h_0 \cdot g_0$. Since the set G is the inverse image by a continuous mapping of an open sebset of a metric space, it is a cozero set. Thus we can find continuous extensions g, h of the functions g_0, h_0 to the whole space Ω, respectively. Since $g_0(Tx) = h_0(x) g_0(x)$ $\forall x \in G$, we have $g(Tx) = h(x) g(x)$ $\forall x \in \overline{G}(!)$. In particular, if $x_n \in \phi^{-1}(\{\tilde{x}_n\})$ $(n = 1, 2, \ldots)$ and $x (\in \Omega)$ is a cluster point of the sequence $(x_1, x_2, \ldots)$ then $1 = \lim_{n\to\infty} g(x_n) = g(x) = g(T^{r_T(x)}x) = h(T^{r_T(x)-1}x)\, g(T^{r_T(x)-1}x) = \ldots = [h(T^{r_T(x)-1}x) \ldots h(x)]\cdot g(x)$. Similarly,

$$g(T^{r_T(x)}x_n) = [h(T^{r_T(x)-1}x_n) \ldots h(x_n)]\, g(x_n) = h(T^{r_T(x)-1}x_n) \ldots h(x_n) = \exp[2\pi i r{:}(x)\, l_n / r{:}(x_n)] \qquad \forall n \in \mathbf{N}.$$

But then $1 = g(x) = g(^{r_T(x)}x) = \lim_{n\to\infty} g(T^{r_T(x)}x_n) = \exp[2\pi i r_T(x)\cdot\delta] \neq 1$, a contradiction. □

What we have shown in Lemma 9 means that the automorphism $f \mapsto f \circ T$ of $C(\Omega)$ is pointwise periodic whenever the underlying automorphism T of the compact F-space Ω is pointwise periodic. However, the following simple Banach space principle holds:

LEMMA 10. – Let E be a Banach space, les T denote a linear pointwise periodic E-isometry. Then T is periodic.

PROOF. – Assume T is not periodic. Now $\forall n \in \boldsymbol{N}$ $\exists f \in E, T^n f \neq f$. Therefore (and by linearity of T) we can define a sequence $f_1, f_2, \ldots \in E$ in the following manner. We choose f_1 so that $T^1 f_1 \neq f_1$. If $f_1, \ldots, f_j$ are already defined then we set $\delta_j \equiv \dim \{T^n f_j : n \in \boldsymbol{N}\}$ and $\varepsilon_j \equiv \min \{\|T^n f_j - f_j\| : T^n f_j \neq f_j, n \in \boldsymbol{N}\}$ and then we choose f_{j+1} to satisfy the relations $T^{j+1} f_{j+1} \neq f_{j+1}$ and $\operatorname{diam}\{T^n f_{j+1} : n \in \boldsymbol{N}\} < \varepsilon_j/3$. Thereafter consider the vector $f \equiv \sum_{j=1}^{\infty} f_j$. Let $n (\in \boldsymbol{N})$ be arbitrarily fixed and set $n_0 \equiv \min\{j : T^n f_j \neq f_j\}$. Then $T^n f - f = \sum_{j \geqslant n_0} (T^n f_j - f_j)$. Thus $\|T^n f - f\| \geqslant \|T^n f_{n_0} - f_{n_0}\| - \sum_{j \geqslant n_0} \|T^n f_j - f_j\| \geqslant \delta_{n_0} - \sum_{j > n_0} \delta_j$. But we have $\delta_j < \frac{1}{3} \varepsilon_{j-1} < \frac{1}{3} \delta_{j-1}$ $\forall j \in \boldsymbol{N}$ whence $\sum_{j > n_0} \delta_j \leqslant \delta_{nr+1} \sum_{k=0}^{\infty} 3^{-k} = \frac{3}{2} \delta_{n_0+1} \leqslant \frac{1}{2} \varepsilon_{n_0}$. Thus $\|T^n f - f\| \geqslant \varepsilon_{n_0}/2 > 0$ $\forall n \in \boldsymbol{N}$, i.e. T is not pointwise periodic. $\square$

THEOREM 3. – Let Ω be a compact F-space and T a pointwise periodic automorphism of Ω. Then T is necessarily periodic.

PROOF. – Lemma 9 and Lemma 10 directly yield that we can find n such that $f \circ T^n = f$, $\forall f \in C(\Omega)$. Hence necessarily $T^n = \mathrm{id}_\Omega$ (since $T^n x \neq x$ would imply $f T^n \neq f$ whenever $f(x) = 0 \neq f(T^n x)$, and $C(\Omega)$ separates the points of Ω by its compactness).

Hence we obtain the following refinement of Theorem 1:

THEOREM 1′. – The following two conditions are equivalent for a compact space Ω:

a) Every $F \in \operatorname{Aut} \bar{B}(C(\Omega))$ with pointwise periodic T_F has fixed point, *b*) Ω is an F-space.

3. – The case of M-lattices with predual.

Having established Theorem 1′, it is natural to ask whether the F property of a compact space Ω ensures the existence of fixed points for every $F \in \operatorname{Aut} \bar{B}(C(\Omega))$. The question can be stated equivalently in the following way: Consider any commutative C^*-algebra with unit whose maximal ideal space is an F-space. Does any biholomorphic automorphism of the unit ball have a fixed point? In the latter setting, we can expect a negative answer. In fact, as we shall see, the space $E \equiv L^\infty(0,1)$ admits an $F \in \operatorname{Aut} \bar{B}(E)$ of the form $F: f \mapsto [x \mapsto M(x)f(Tx)]$ with an ergodic transformation T of the interval $(0,1)$ and a Borel measurable function $M: (0,1) \to \operatorname{Aut} \bar{\Delta}$ without fixed point. (The maximal ideal space of $L^\infty(0,1)$ is hyperstonian (see [10]) hence obviously an F-space).

Throughout this Chapter, let M_1, M_2 denote the transformations

$$[\boldsymbol{C} \in \zeta \mapsto -\zeta] \quad \text{and} \quad \left[\boldsymbol{C} \ni \zeta \mapsto \frac{\zeta + \operatorname{th}(1)}{1 + \zeta \operatorname{th}(1)}\right],$$

respectively. (Note: $M_1|_{\bar{\Delta}}, M_2|_{\bar{\Delta}} \in \operatorname{Aut} \bar{\Delta}$.) In view of Lemma 3, the fixed point preserving Lie group homomorphism $M_1^{(\cdot)}: \boldsymbol{R} \to \operatorname{Aut} \bar{\Delta}$ defined by $M_2^1 = M_2$, is given by

$$M_2^t: \zeta \mapsto \frac{\zeta + \operatorname{th}(t)}{1 + \zeta \operatorname{th}(t)} \qquad (t \in \boldsymbol{R}).$$

Let λ be the normed Lebesgue measure on the unit circle $\partial\Delta$ of $\boldsymbol{C}$ (i.e. $\lambda \equiv 1/2\pi \operatorname{length}|_{\partial\Delta}$). Fix an irrational number $\delta \in (0,1)$ and denote by T the clockwise rotation of $\partial\Delta$ by the angle $2\pi\delta$, i.e. $T: x \mapsto \exp(-2\pi i\delta)\cdot x$. The space $L^\infty(\partial\Delta, \lambda)$ is considered, as usual as $\{\tilde{\varphi}: \varphi \text{ is a bounded Borel } \partial\Delta \to \boldsymbol{C} \text{ function}\}$ where $\tilde{\varphi} \equiv \{\psi(: \partial\Delta \to \boldsymbol{C}): \lambda\{x \in \partial\Delta: \psi(x) \neq \varphi(x)\} = 0\}$. Finally, let $M: \partial\Delta \to \operatorname{Aut} \bar{\boldsymbol{C}}$ be the function

$$\exp(2\pi i\tau) \mapsto \begin{cases} M_1 & \text{if} \quad 0 \leqslant \tau < \delta \\ M_2 & \text{if} \quad \delta \leqslant \tau < 1' \end{cases}$$

and define $F: \bar{B}(L^\infty(\partial\Delta, \lambda)) \to L^\infty(\partial\Delta, \lambda)$ by $F(\tilde{\varphi}) \equiv [x \mapsto M(x)\varphi(Tx)]$ for all Borel measurable $\varphi: \partial\Delta \to \bar{\Delta}$. Clearly, $F \in \operatorname{Aut} \bar{B}(L^\infty(\partial\Delta, \lambda))$.

THEOREM 4. – The transformation F (defined above) has no fixed point.

The proof is divided into eight steps

1) Let G be the subgroup of $\operatorname{Aut} \bar{\boldsymbol{C}}$ generated by M_1 and M_2. Since

$$M_2 M_1 = M_1 M_2^{-1} \quad (\text{and } M_1 M_2 = M_2^{-1} M_1), \tag{8}$$

we have $G = \{M_1^s M_2^t: s = 0, 1;\ t \in \boldsymbol{Z}\}$. This representation of G is unique in the sense that if $s, s' \in \{0,1\}$ and $t, t' \in \boldsymbol{Z}$ with $M_1^s M_2^t = M_1^{s'} M_2^{t'}$ then $s = s'$ and $t = t'$ $\left(\text{since } \operatorname{id}_{\boldsymbol{C}} = M_1^{s-s'} M_2^{t'-t} = \left[\zeta \mapsto (-1)^{s-s'} \dfrac{\zeta + \operatorname{th}(t'-t)}{1 + \zeta \operatorname{th}(t'-t)}\right]\right)$.

2) In the following we shall argue by contradiction assuming that Theorem 4 does not hold. Denote by f_0 a fixed point of F and let $\varphi_0: \partial\Delta \to \bar{\Delta}$ be a representant of f_0 (thus $f_0 = \tilde{\varphi}_0$). The symbol $\forall_\lambda$ will indicate « λ-almost everywhere ». Now

$\varphi_0(Tx) = M(x)^{-1}\varphi_0(x)\ \forall_\lambda x \in \partial\Delta\ \ \forall n \in \mathbf{Q}$, and therefore

$$\varphi_0(T^n x) = M(T^{n-1}x)^{-1} \dots M(x)^{-1}\varphi_0(x) \qquad \forall_\lambda x \in \partial\Delta\ \ \forall n \in \boldsymbol{N}. \tag{9}$$

Thus range $(\varphi_0 \circ T^n) \subset G \cdot (\text{range}\ \varphi_0)\ \ \forall n \in \boldsymbol{N}$.

3) It is well-known that the transformation T is ergodic (cf. [5]). Hence it follows that if $S \subset \partial\Delta$ is such that $T(S)$ differs just in a 0-set with respect to λ from S $\big($i.e. $\lambda([S \cup T(S)] \setminus [S \cap T(S)]) = 0\big)$ then either $\lambda(S) = 0$ or $\lambda(S) = 1$.

Thus if for a Borel set $\Gamma \subset C$ we have $N(\Gamma) = \Gamma\ \ \forall N \in G$, then $\phi_0^{-1}(\Gamma)$ is either a 0-set or the complementer in $\partial\Delta$ of some 0-set (wrt λ).

4) If $\zeta, \eta \in \bar{\Delta} \setminus \{-1, 1\}$ and $\eta \notin G(\zeta)$ then there exist G-invariant neighbourhoods U, V of ζ and η, respectively, that are disjoint.

Proof. – Observe that for any $t \in \boldsymbol{Z}$, $M_2^t\colon 1 \mapsto 1, (-1) \mapsto (-1), [-1, 1] \mapsto [-1, 1]$ circle $\mapsto$ circle. So from the conformity of Aut $\boldsymbol{C}$ it easily follows that, for every $t \in \bar{L}$, M_2^t maps the bounded domain $D \equiv \{\zeta \in \boldsymbol{C}\colon |\zeta - i| < \sqrt{2}|\} \cup \{|\zeta \in \boldsymbol{C}\colon |\zeta + i| < < \sqrt{2}\}$ onto itself. Thus $ND = D\ \ \forall N \in G$ (cf. 1)$\big)$. Let d_D denote the Kobayashi distance on D $\big($for its definition see [11]$\big)$ and consider the orbit $G(\zeta)$. From (8) we deduce that $G(\zeta) = \{\pm M_2^t\zeta\colon t \in \boldsymbol{Z}\} \subset \Delta \setminus \{-1, 1\} \subset D$. Since $M_2^t\zeta = \dfrac{\zeta + \text{th}\,(t)}{1 + \zeta\,\text{th}\,(t)} \to \pm 1$ according to $t \to \pm\infty$, the set $G(\zeta)$ has no cluster point in D. Hence $d_D(\eta, G(\zeta)) > 0$. Thus the choices $U \equiv \{\zeta' \in D\colon d_D(\zeta', G(\zeta)) < \frac{1}{2} d_D(\eta, G(\zeta))\}$ and $V \equiv \{\eta' \in D\colon d_D(\eta', G(\zeta) > \frac{1}{2} d_D(\eta, G(\zeta))\}$ fulfill our requirements.

We show now that $\lambda\big(\varphi_0^{-1}(G(\zeta_0))\big) = 1$ for some $\zeta_0 \in \bar{\Delta}$.

Proof. – The last remark and 3) exclude that for every pair $\zeta, \eta \in \bar{\Delta} \setminus \{-1, 1\}$ and for all neighbourhoods U, V of $G(\zeta)$ and $G(\eta)$, respectively, we have $\lambda(\varphi_0^{-1}(U)) > 0$ and $\lambda(\varphi_0^{-1}(V)) > 0$ in the same time. If for any $\zeta \in \bar{\Delta} \setminus \{-1, 1\}$, one can find a neighbourhood U of $G(\zeta)$ such that $\lambda(\varphi_0^{-1}(U)) = 0$ then the separability of $\bar{\Delta}$ implies that $\lambda\big(\varphi_0^{-1}(\bar{\Delta} \setminus \{-1, 1\})\big) = 0$, whence $\lambda\big(\varphi_0^{-1}(\{-1, 1\})\big) = \lambda(\varphi_0^{-1}(\bar{\Delta})) - \lambda\big(\varphi_0^{-1}\big(\bar{\Delta} \setminus \{-1, 1\}\big)\big) = 1$. Now we can choose e.g. $\zeta_0 = 1$. If for some $\zeta_1 \in \bar{\Delta} \setminus \{-1, 1\}$, any neighbourhood U of $G(\zeta_1)$ satisfies $\lambda(\varphi_0^{-1}(U)) > 0$ then for any G-invariant neighbourhood of this ζ_1 we necessarily have by 3) that $\lambda(\varphi^{-1}(U)) = 1$. Therefore $1 = \lambda\big(\varphi_0^{-1}(\{\zeta \in \in D\colon d_D(\zeta, G(\zeta_1)) < 1/n\})\big) \to \lambda\big(\varphi_0^{-1}(G(\zeta_1))\big)$ $(n \to \infty)$. Thus, in this case, taking $\zeta_0 = \zeta_1$, the requirements are satisfied.

Henceforth we assume that

$$\text{range}\ \varphi_0 = \{c_1, c_2, \dots\} \subset G(c) \subset \bar{\Delta}$$

(where $c, c_1, c_2, \dots$ are given constants). Our previous observation ensures that this can be done without loss of generality.

5) Step 1) directly implies the existence of a unique pair of Borel functions $s_n\colon \partial\Delta \to \{0,1\}$ and $t_n\colon \partial\Delta \to \boldsymbol{Z}$ for each $n \in \boldsymbol{N}$, such that

$$M_1^{s_n(x)} M_2^{t_n(x)} = M(T^{n-1}x)^{-1} \dots M(x)^{-1} \qquad \forall x \in \partial\Delta .$$

Thus by (9) we have

$$(9') \qquad \varphi_0(T^n x) = M_1^{s_n(x)} M_2^{t_n(x)} \varphi_0(x) \qquad \forall_\lambda x \in \partial\Delta \ \forall n \in \boldsymbol{N} .$$

Introducing the functions $s \equiv 1_{\{\exp(2\pi i\tau):\, 0 \leqslant \tau < \delta\}}$ and $t \equiv 1_{\partial\Delta} - s$, we also have $M(x) = M_1^{s(x)} M_2^{t(x)}|_{\bar\Delta}$, $\forall x \in \partial\Delta$. Now (8) enables us to express s_n and t_n in terms of s and t. In particular, one sees by induction on n that $s_n(x) = \mathrm{mod}_2\,[s(x) + \dots + s(T^{n-1}x)]$. Thus

$$M_1^{s_n(x)} = [\zeta \mapsto (-1)^{s(x)+\dots+s(T^{n-1}x)}\zeta] \qquad \forall x \in \partial\Delta \ \forall n \in \boldsymbol{N} .$$

6) We achieve a stronger control over the functions $(-1)^{s_n(\cdot)}$. Consider the function $\tilde{s}\colon \boldsymbol{R} \to \{0,1\}$ defined by $\tilde{s}(\tau) \equiv s(\exp(2\pi i\tau))$. Thus $\tilde{s}(\tau) = \sum_{m=-\infty}^{\infty} 1_{[0,\delta)}(\tau + m)$ $\forall \tau \in \boldsymbol{R}$. Introducing the functions $\tilde{s}_n(\tau) \equiv s(\exp(2\pi i\tau)) + s(T\exp(2\pi i\tau)) + \dots + s(T^{n-1}\exp(2\pi i\tau))$, we have

$$\tilde{s}_n(\tau) = \tilde{s}(\tau) + \tilde{s}(\tau - \delta) + \dots + \tilde{s}(\tau - (n-1)\delta) = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} 1_{[0,\delta)}(\tau + m - k\delta) =$$
$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} 1_{[k\delta,(k+1)\delta)}(\tau + m) = \sum_{m=-\infty}^{\infty} 1_{[0,n\delta)}(\tau + m) .$$

Therefore, $\tilde{s}_n$ is a periodic continuation (with period-length 1) of the function

$$\tau \mapsto \begin{cases} \text{Integral part of } (n\delta + 1) \text{ if } 0 \leqslant \tau < n\delta\text{-entier}\,(n\delta). \\ \text{Integral part of } (n\delta) \quad \text{ if } n\delta \text{ entier}\,(n\delta) \leq \tau < 1 \end{cases}$$

Since $\int_{\partial\Delta} (-1)^{s_n}\, d\lambda = \int_0^1 (-1)^{s_n(\tau)}\, d\tau$, this means that if $n_m \to \infty$ is a sequence in $\boldsymbol{N}$ such that $\mathrm{dist}(n_m\delta, \{2k-1\colon k \in \boldsymbol{N}\}) \to 0$ $(m \to \infty)$ then $\int_{\partial\Delta} (-1)^{s_{n_m}}\, d\lambda \to -1$, i.e. the sequence of the functions $(-1)^{s_{n_m}}$ converges in measure to the identically -1 function on $\partial\Delta$. So, by the classical Riesz-Weyl Lemma, there is a subsequence $(n_{m_j}\colon j \in \boldsymbol{N})$ with $(-1^{s_{n_{m_j}}(\tau)} \to -1$ $(j \to \infty)$ $\forall_\lambda x \in \partial\Delta$, or which is the same, $s_{n_{m_j}}(x) \to 1$ $(j \to \infty)$ $\forall_\lambda x \in \partial\Delta$.

Similarly, $\mathrm{dist}(n'_m\delta, \{2k\colon k \in \boldsymbol{N}\}) \to 0$ $(m \to \infty)$ implies the existence of a subsequence $(n'_{m_j}\, j \in \boldsymbol{N})$ with $s_{n'_{m_j}}(x) \to 0$ $(j \to \infty)$ $\forall_\lambda x \in \partial\Delta$.

7) A sequence $n_m \to \infty$ for which $\mathrm{dist}(n_m \cdot \delta, \{2k-1\colon k \in \boldsymbol{N}\}) \to 0$ $(m \to \infty)$ certainly exists. (Proof: The set $\{\exp(\pi i n\delta)\colon n \in \boldsymbol{N}\}$ is dense in $\partial\Delta$ and the relation $\mathrm{dist}(n_m\delta, \{2k-1\colon k \in \boldsymbol{N}\}) \to 0$ is equivalent to $\exp[2\pi i(\delta/2)n_m] \to -1$.) Clearly, for any such a sequence $(n_m\colon m \in \boldsymbol{N})$ we have $\exp(2\pi i\delta_{n_m}) \to 1$ i.e. $T^{n_m} \to \mathrm{id}_{\partial\Delta}$ (if $m \to \infty$).

From now on, let $(n'_m\colon m \in \boldsymbol{N})$ denote a fixed sequence in $\boldsymbol{N}$ such that $n'_m \to \infty$, $T^{n'_m} \to \mathrm{id}_{\partial\Delta}$ and $\forall_\lambda x \in \partial\Delta$ $s_{e'_m}(x) \to 1$ $(m \to \infty)$.

Suppose then that $(n_m\colon m \in \boldsymbol{N})$ is a sequence with $n_m \to \infty$, $T^{n_n} \to \mathrm{id}_{\partial\Delta}$ and $\forall_\lambda x \in \partial\Delta$, $s_{n_m}(x) \to 0$ $(m \to \infty)$. Since $\mathrm{range}(\varphi_0) \subset G(c) = \{\pm M_2^t\colon t \in \boldsymbol{Z}\}$ (cf. conclusion of 4)) and since $G(c)$ has two cluster points outside of itself whenever $c \neq \pm 1$ (namely the points -1 and 1), $\mathrm{range}(\varphi_0)$ is a discrete subset of $\boldsymbol{C}$. By the Lebesgue Shift Theorem, the fact $T^{n'_m} \to \mathrm{id}_{\partial\Delta}$ implies $\varphi_0(T^{n'_m}x) \to \varphi_0(x)$ $\forall_\lambda x \in \partial\Delta$. Similarly, $\varphi_0(T^{n_m}x) \to \varphi_0(x)$, $\forall_\lambda x \in \partial\Delta$. By the discreteness of $\mathrm{range}(\varphi_0)$, we have then

$$(10) \qquad \forall_\lambda x \in \partial\Delta \quad \exists m_0(x) \quad \forall m > m_0(x)$$

$$\varphi_0(x) = \varphi_0(T^{n'_m}x) = M_1^{s_{n'_m}(\tau)} M_2^{t_{n'_m}(x)} \varphi_0(x) = M_1 M_2^{t_{n'_m}(x)} \varphi_0(x)$$

and

$$\varphi_0(x) = \varphi_0(T^{n_m}x) = M_1^{r_{T_m}(x)} M_2^{t_{T_m}(x)} \varphi_0(x) = M_2^{t_{n_m}(x)} \varphi_0(x) .$$

Thus $\forall_\lambda x \in \partial\Delta$ $\exists t', t'' \in \boldsymbol{Z}$ $M_1 M_2^{t'} \varphi_0(x) = M_2^{t''} \varphi_0(x) = \varphi_0(x)$. For each $t'' \neq 0$ and $\zeta \in \boldsymbol{C}$, it follows from the relation $M_2^{t''}\zeta = \zeta$ that $\zeta = -1$ or $\zeta = 1$. Therefore (10) can be valid only if $\forall_\lambda x \in \partial\Delta$ $\exists m_0(x)$ $\forall m > m_0(x)$ $t_{n_m}(x) = 0$. Thus

(10′) If $n_m \to \infty$ is a sequence with $T^{n_m} \to \mathrm{id}_{\partial\Delta}$ and

$\forall_\lambda x \in \partial\Delta$ $s_{n_m}(x) \to 0$, then $t_{n_m}(x) \to 0$ $\forall_\lambda x \in \partial\Delta$.

8) We shall arrive at a contradiction, by showing that (10′) is impossible. In fact, we shall prove that

a) There exists a sequence $n_m \to \infty$ consisting of *odd* numbers such that $T^{n_m} \to \mathrm{id}_{\partial\Delta}$ and $s_{n_m}(x) \to 0$ $\forall_\lambda x \in \partial\Delta$.

b) $\mathrm{mod}_2 [s_n(x) + t_n(x)] = \mathrm{mod}_2\, n$ $\forall x \in \partial\Delta$ $\forall n \in \boldsymbol{N}$.

By *b*), for any sequence $(n_m\colon m \in \boldsymbol{N})$ as in *a*), we have that $t_{n_m}(x)$ is odd for all $m \in \boldsymbol{N}$ and $x \in \partial\Delta$. But hence $t_{n_m}(x) \to 0$ $\forall x \in \partial\Delta$. This contradiction proves the theorem.

Proof of *a*): The conclusion of 6) tells us that *a*) is equivalent to the existence of a sequence $n_m^* \to \infty$ of odd numbers such that $\mathrm{dist}\,(n_m^* \cdot \delta, \{2k\colon k \in \boldsymbol{N}\}) \to 0$ $(m \to \infty)$. But this latter property is equivalent to $\exp\,[2\pi i n_m^*(\delta/2)] \to 1$ which can be easily satisfied by some odd sequence $(n_m^*\colon m \in \boldsymbol{N})$, since the set $\{\exp\,[2\pi i(2l+1)(\delta/2)]\colon l \in \boldsymbol{N}\}$ is dense in $\partial\Delta$ (for δ is irrational).

Proof of *b*): Proceed by induction on n. For $n = 1$, $M_1^{s_1(x)} M_2^{t_1(x)} = M(x)^{-1}$ $(= M_1^{-1}$ or $M_2^{-1})$. Thus either $s_1(x) = 1$ and $t_1(x) = 0$ or $s_1(x) = 0$ and $t_1(x) = 1$. Anyway, $s_1(x) + t_1(x)$ is odd, similarly to 1 $(= n)$ for all $x \in \partial\Delta$.

To perform the inductive step, observe that

$$M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} = M(T^n x)^{-1} M(T^{n-1}x)^{-1} \dots M(x)^{-1} = M(T^n x)^{-1} M_1^{s_n(x)} M_2^{t_n(x)} .$$

Now there are three cases:

i) If $M(T^n x) = M_1$ then $M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} M_1^{s_n(x)-1} M_2^{t_n(x)} = M_1^{\mathrm{mod}_2[s_n(x)-1]} M_2^{t_n(x)}$, i.e. $\mathrm{mod}_2[s_{n+1}(x) + t_{n+1}(x)] = \mathrm{mod}_2[s_n(x) - 1 + t_n(x)] =$ by the induction hypotheses $= \mathrm{mod}_2(n-1) = \mathrm{mod}_2(n+1)$.

ii) If $M(T^n x) = M$ and $s_n(x) = 0$ then $M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} = M_2^{-1} M_2^{t_n(x)}$, i.e. $0 = s_{n+1}(x)$ and $t_{n+1}(x) = t_n(x) - 1$. Thus $\mathrm{mod}_2[s_{n+1}(x) + t_{n+1}(x)] = \mathrm{mod}_2[s_n(x) + t_n(x) - 1] = \mathrm{mod}_2(n-1) = \mathrm{mod}_2(n+1)$.

iii) If $M(T^n x) = M_2$ and $s_n(x) = 1$ then $M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} = M_2^{-1} M_1 M_2^{t_n(x)} =$ by (8) $= M_1 M_2^{t_n(x)+1}$, i.e.

$$\mathrm{mod}_2[s_{n+1}(x) + t_{n+1}(x)] = \mathrm{mod}_2[s_n(x) + t_n(x) + 1] = \mathrm{mod}_2(n+1) .$$

The proof of Theorem 4 is complete. □

The seemingly too particular statement of Theorem 4 enables us to reach a general conclusion:

A theorem of D. MAHARAM (cf. [7], [10]) asserts that for any σ-finite measure μ, there exists a sequence $\varrho_1, \varrho_2, \dots > 0$ and a sequence of cardinalities $\mathfrak{a}_1, \mathfrak{a}_2, \dots$ such that $L^1(\mu) \simeq L^1\left(\bigoplus_{n=1}^{\infty} \varrho_n \lambda^{\mathfrak{a}_n}\right)$ (for $\mathfrak{a} > 0$, $\lambda^{\mathfrak{a}}$ denotes the $\mathfrak{a}$-th power of the measure λ; $\lambda^0 \equiv$ [atom with weight 1]). This fact yields an application of Theorem 4 to decide the fixed point problem of Aut $\bar{B}(E)$ even for the most general L^∞-spaces E (and hence, by a theorem of M. RIEFFEL [8], for all M-lattices admitting a predual).

LEMMA 11. – Let X be a discrete topological space. Then for all $F \in \mathrm{Aut}\,\bar{B}(C_b(X))$ there exists a (unique) permutation T of X and a function $M\colon X \to \mathrm{Aut}\,\bar{\Delta}$ such that $F = [f \mapsto [x \mapsto M(x) f(Tx)]]$.

PROOF. – Let ϕf denote the (unique) continuous extension to βX (the Stone-Čech compactification of X) of any $f \in C_b(X)$. Now the map $\hat{F} \equiv \phi F \phi^{-1}$ is a biholomorphic automorphism of $\bar{B}(C(\beta X))$. Since the isolated points of βX are exactly the points of X and since any automorphism of a topological space sends the set of its isolated points onto itself, we have $T_{\hat{F}}(X) = X$. Hence $(Ff)(x) = (\phi^{-1} \hat{F} \phi f)(x) = (\hat{F}\phi f)|_X(x) = (\hat{F}\phi f)(x) = [\hat{F}(\phi f)](x) = M_{\hat{F}}(x)[(\phi f)(T_{\hat{F}} x)] =$ since $T_{\hat{F}} x \in X = M_{\hat{F}}(x) \cdot (T_{\hat{F}} x)$ $\forall x \in X$. □

COROLLARY 2. – For a discrete space X, all the members of Aut $\bar{B}(C_b(X))$ have fixed point.

PROOF. – Let τ denote the topology of pointwise convergence on $C_b(X)$ (i.e. by definition, $f_j \xrightarrow{\tau} f$ iff $\forall x \in X\ f_j(x) \to f(x)$, for every net $(f_j: j \in J)$ and function f in $C_b(X)$). Observe that $\bar{B}(C_b(X))$ endowed with the topology τ coincides (set theoretically) with the topological product space $\bar{\Delta}$: which is compact by Tychonoff's Product Space Theorem. On the other hand, it readily follows from Lemma 11 that any $F \in \operatorname{Aut} \bar{B}(C_b(X))$ is also $\tau \to \tau$ continuous (the definition of F requires only its continuity for the norm topology). Hence the Schauder-Tychonoff Fixed Point Theorem establishes (cf. [1]) that each $F \in \operatorname{Aut} \bar{B}(C_b(X))$ has fixed point. $\square$

THEOREM 5. – Let E be an M-lattice (for definition see [8], [9]) having a predual $^*E$. Then the following properties are equivalent:

a) Any $F \in \operatorname{Aut} \bar{B}(E)$ has a fixed point.

b) $E \simeq C_b(X)$ for some discrete topological space X.

PROOF. – By a theorem of M. RIEFFEL [8], the M-lattices with predual are exactly the L^∞-spaces. Thus we may assume without loss of generality that $^*E = L^1(X, \mu)$ and $E = L^\infty(X, \mu)$ for some fixed measure space (X, μ). If the measure μ is atomic then obviously *b*) holds and hence Corollary 2 implies *a*). Suppose μ is non-atomic. Then *b*) is false, thus it suffices to find an $F \in \operatorname{Aut} \bar{B}(L^\infty(X, \mu))$ free of fixed points. Fix a μ-measurable subset $X' \subset X$ such that the measure $\mu|_{X'}$ be non-atomic and we have $0 < \mu(X') < \infty$. By Maharam's Isomorphism Theorem (cf. [7], [10]), there exists a μ-measurable subset $Y \subset X'$ and a cardinality $\mathfrak{a} > 0$ such that $\mu(Y) > 0$ and $L^1(Y, \mu|_Y) \simeq L^1(\mu(Y) \cdot \lambda)$. Therefore $L^\infty(X, \mu)$ is isometrically isomorphic with the direct sum of $L^\infty(\lambda^{\mathfrak{a}})$ and some other L^∞ space $\hat{E}$ where the norm of a generic element (f, g) (f in $L^\infty(\lambda^{\mathfrak{a}})$, g in $\hat{E}$) is defined by $\|(f, g)\| \equiv \max\{\|f\|, \|g\|\}$. Hence, to prove Theorem 5, it suffices to show that some $F \in \operatorname{Aut} \bar{B}(L^\infty(\lambda^{\mathfrak{a}}))$ has no fixed point. Let

$$F_0: \bar{B}(L^\infty(\lambda^{\mathfrak{a}})) \to L^\infty(\lambda^{\mathfrak{a}}) \quad \text{be the mapping defined by}$$

$$F_0: f \mapsto \left[(\partial\Delta)^{\mathfrak{a}} \in (\xi_\alpha: \alpha < \mathfrak{a}) \mapsto M(\xi_0)\varphi_f((T\xi_0), \xi_\alpha: 0 < \alpha < \mathfrak{a})\right]^\sim$$

where $M: \partial\Delta \to \operatorname{Aut} \mathbf{C}$ and $T: \partial\Delta \to \partial\Delta$ are the same as in Theorem 4 and φ_f denotes a (fixed) Borel measurable representant with range in $\bar{\Delta}$ of f, for any $f \in \bar{B}(L^\infty(\lambda^{\mathfrak{a}}))$. Then it follows from Theorem 4 that F_0 has no fixed point and belongs to $\operatorname{Aut} \bar{B}(L^\infty(\lambda^{\mathfrak{a}}))$. $\square$

REFERENCES

[1] N. DUNFORD - J. T. SCHWARTZ, *Linear operators*, Interscience, New York, 1958.
[2] C. J. EARLE - R. S. HAMILTON, *A fixed point theorem for holomorphic mappings*, Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, R.I. (1970), pp. 61-65.

[3] L. Gillman - M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Prin
1960.

[4] T. L. Hayden - T. J. Suffridge, *Fixed points of holomorphic maps in Banach s*
Proc. Amer. Math. Soc., **60** (1976), pp. 95-105.

[5] K. Jacobs, *Neuere Methoden und Ergebnisse der Ergodentheorie*, Ergebnisse d.
Wiss., **29**, Springer, Berlin, 1960.

[6] W. Kaup - H. Upmeier, *Banach spaces with biholomorphically equivalent unit ba*
isomorphic, Proc. Amer. Math. Soc., **58** (1976), pp. 129-133.

[7] D. Maharam, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. USA, **28** (
pp. 108-111.

[8] M. A. Rieffel, *A characterization of commutative group algebras and measure al*
Trans. Amer. Math. Soc., **116** (1965), pp. 32-65.

[9] H. Schaeffer, *Banach lattices and positive operators*, Grundl. d. Math. Wiss.,
Springer, Berlin, 1974.

[10] Z. Semadeni, *Banach Spaces of Continuous Functions*, Monografie Matemat
Tom. **55**, PWN-Polisch Scientific Publishers, Warszava, 1971.

[11] E. Vesentini - T. Franzoni, *Holomorphic maps and invariant distances*, North-H
Publishing Company, Amsterdam, 1980.