Acta Sci. Math. (Szeged)
60 (1995), 619–636

On the structure of inner derivations in partial Jordan-triple algebras

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To the honour of Prof. Tandori's 70-th and Prof. Leindler's 60-th birthdays

Communicated by P. T. Nagy

Absract. Partial Jordan-triples are algebras with three variables occuring naturally in the description of the holomorphic automorphism groups of bounded circular domains. The following question is studied: under which conditions can all inner derivations of a partial Jordan-triple be recovered from their restrictions to the complete algebraic base of the triple.

1. Introduction

Since the works [6,8,2] it is well-understood that the key to any kind of holomorphic classification of bounded circular domains in Banach spaces is the study of the so-called partial Jordan-triples. By a *partial Jordan-triple* we mean an algebraic structure $\mathbf{E} = (E, E_0, \{\ *\ \})$ with a product of 3-variables where E is a complex Banach space, E_0 is a complex (closed) subspace of E and $\{\ *\ \}$ is a continuous real-trilinear operation

$$E \times E_0 \times E \to E \qquad (x, a, y) \mapsto \{x, a^*, y\}$$

Received October 19, 1994.
AMS Subject Classification (1991): 17C65, 17C30, 32M05, 32M15, 32H20.
* Supported by the Alexander von Humboldt Foundation and by the Hungarian NFS, Grant No. 7292.

symmetric complex bilinear in the outer variables x, y and conjugate linear in a such that the linear operators

$$(a\Box b^*)x := \{a, b^*, x\} \qquad (a, b \in E_0\,,\ x \in E)\ ,$$

satisfy the axioms

$$(J1) \qquad (a\Box a^*)E_0 \subset E_0\ ,\quad (a\Box a^*|E_0) \in \mathrm{Her}_+(E_0)\ ,\quad \|a\Box a^*|E_0\| = \|a\|^2$$

$$(J2) \qquad a\Box a^* \in \mathrm{Der}(\mathbf{E})$$

for all $a \in E_0$. Here $\mathrm{Her}_+(E_0)$ is the family of all E_0-*Hermitian* operators*) with non-negative spectrum and $\mathrm{Der}(E)$ denotes the set of all *derivations* of $\mathbf{E}$. By a derivation of $\mathbf{E}$ we mean a linear mapping $D: E \to E$ such that $D(E_0) \subset E_0$ and

$$D\{x, a^*, y\} = \{(Dx), a^*, y\} + \{x, (Da)^*, y\} + \{x, a^*, (Dy)\} \qquad (x, y \in E\,,\ a \in E_0)\ .$$

The *inner derivations* of $\mathbf{E}$ are the finite *real*-linear combinations of the derivations of the form $ia\Box a^*$ ($a \in E_0$). In the sequel we shall call the space E_0 the *base* of the partial Jordan-triple $\mathbf{E}$ and we set $\mathbf{E}_0 := (E_0, E_0, \{\ *\ \})$. We shall use the notations $\mathrm{Der}_0(\mathbf{E}) := \{$inner derivations of $\mathbf{E}\}$. Remark that the partial Jordan-triples coinciding with their base spaces are the widely studied JB*-triples.

In 1990, Panou [11] obtained the complete partial Jordan-triple-axiomatics along with a structure description of all the possible partial Jordan-triples of the form $E = E_0 \oplus E_1$, $\{E_1, E_0^*, E_1\} = 0$ which can be associated with some finite dimensional bounded bicircular domain. His main tool to the structure description was the following fact [11, Prop. 1.6]: every inner derivation of the base of a partial Jordan-triple associated a finite dimensional bounded circular (not only bicircular) domain admits a unique inner derivative extension to the whole space.

In 1991, [13] gave the complete partial Jordan-triple-axiomatics of the so-called *geometric partial Jordan-triples* which can be associated with some bounded circular domain in a Banach space. To obtain finer results concerning the structure of such partial Jordan-triples, it seems to be useful to look for infinite dimensional generalizations of [11, Prop. 1.6]. Notice that Panou's proof relies heavily upon the fact that, in a finite dimensional geometric partial Jordan-triple $\mathbf{E}$, the norm closure $\mathcal{K}$ of the group generated by $\exp(\mathrm{Der}_0(\mathbf{E}))$ is a compact subgroup of the unitary group with Lie-algebra $\mathrm{Der}_0(\mathbf{E})$. In infinite dimensions, even if the base

*) i.e. bounded linear operators $A: E_0 \to E_0$ with $\|\exp(i\tau A)\| = 1$ ($\tau \in \mathbb{R}$). We shall always write $\|\cdot\|$ for the norm in any Banach space in consideration. For linear operators we take the usual operator-norm automatically.

of the geometric partial Jordan-triple is finite dimensional, the group $\mathcal{K}$ may be non-compact. Before the appearence of [11], there was already a positive infinite dimensional result [12, Lemma 2.1] concerning the extendibility of inner derivations in partial Jordan-triples with commutative base.

In this short note first we prove the analog of [11, Prop. 1.6] to infinite dimensional partial Jordan-triples where the base is a finite dimensional Cartan factor. Our arguments are completely Jordan-theoretical and require no further assumptions beyond the axioms (J1),(J2). Moreover, the version of axiom (J1) used here requiring only $a\Box a^*|E_0 \in \mathrm{Her}_+(E_0)$ is somewhat weaker than its usual form in the literature postulating $a\Box a^* \in \mathrm{Her}_+(E)$. Then, by introducing the concept of *quasigrids* and using norm-density considerations, we generalize the result to partial Jordan-triples whose base spaces are c_0-direct sums of elementary JB*-triples. In particular we get that the restriction mapping $D \mapsto D|E_0$ is a Lie-algebra isomorphism between $\mathrm{Der}_0(\mathbf{E})$ and $\mathrm{Der}_0(\mathbf{E}_0)$ whenever the base space E_0 is a so-called compact JB*-triple (for def. see [3]).

2. The case of finite dimensional factor base

Lemma 2.1. *Given a derivation D of a partial Jordan-triple $\mathbf{E}$ and an element $a \in E_0$ in the base E_0 of $\mathbf{E}$, we have*

$$[D, a\Box a^*] = (Da)\Box a^* + a\Box(Da)^* \ .$$

Proof. For any $x \in E$, by axiom (J2) we have

$$\begin{aligned}[D, a\Box a^*]x &= D\{a, a^*, x\} - \{a, a^*, (Dx)\} = \{(Da), a^*, x\} + \{a, (Da)^*, x\} \\ &= ((Da)\Box a^* + a\Box(Da)^*)x.\end{aligned}$$

■

Corollary 2.2. *A derivation $D \in \mathrm{Der}(E)$ vanishing on the base E_0 commutes with every inner derivation of $\mathbf{E}$.*

Proposition 2.3. *Suppose the base E_0 of the partial Jordan-triple $\mathbf{E}$ is a finite dimensional Cartan factor. Then for any inner derivation D_0 of $\mathbf{E}_0$ there is a unique inner derivation D of $\mathbf{E}$ with $D_0 = D|E_0$.*

Proof. Let $D_0 \in \mathrm{Der}(\mathbf{E}_0)$. Using spectral decomposition [9], we can write $D_0 = i\sum_{j=1}^n \alpha_j a_j \Box a_j^* | E_0$ for some minimal tripotents $a_1, \dots, a_n \in E_0$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Now the operator $i\sum_{j=1}^n \alpha_j a_j \Box a_j^*$ is an inner derivation extending D_0 to the whole E. Since real linear combinations of inner derivations are inner derivations, it suffices to show that

$$\sum_{j=1}^n \alpha_j a_j \Box a_j^* = 0 \quad \text{whenever} \quad \sum_{j=1}^n \alpha_j \{a_j, a_j^*, c\} = 0 \qquad (c \in E_0)$$

and $a_0, \dots, a_n$ are minimal tripotents in E_0.

Since $\dim(E_0) < \infty$, the group of all linear isometries of E_0 is a compact finite dimensional Lie group whose Lie algebra is $\mathrm{Der}(E_0)$. Consider its identity component $\mathcal{U}$. That is, $\mathcal{U}$ is the closed subgroup of $\mathcal{L}(E_0)$ generated by the operators $\exp(ia\Box a^*)$ $(a \in E_0)$. Then

$$\mathrm{Span}(\mathcal{U}a) = E_0 \qquad (0 \neq a \in E_0) .$$

Indeed, the subspace $\mathrm{Span}(\mathcal{U}a)$ is $\mathrm{Der}(E_0)$-invariant. It is well-known [5] that Der-invariant subspaces are ideals in JB*-triples, and the only ideals in the factor E_0 are $\{0\}$ and E_0. By writing simply $\int dU$ for the integration with respect to the normalized Haar measure*) on $\mathcal{U}$,

$$\int (Ue)\Box(Ue)^* dU = \int (Uf)\Box(Uf)^* dU \qquad (e, f \text{ minimal tripotents in } E_0)$$

because the group $\mathcal{U}$ is transitive (i.e. $f = Ue$ for some $U \in \mathcal{U}$ above) on the manifold of minimal tripotents in factors [9].

Let us fix $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and minimal tripotents $a_1, \dots, a_n \in E_0$ such that

$$Dc = 0, \qquad (c \in E_0) \quad \text{where} \quad D := \sum_{j=1}^n \alpha_j a_j \Box a_j^* .$$

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*) For Borel-measurable functions $\phi: \mathcal{U} \to \mathcal{L}(E)$ with finite range, $\int \psi dU := \sum_{L \in \mathrm{ran}\psi} dU(\phi^{-1}(L)) \cdot L$. For continuous $\Phi: \mathcal{U} \to \mathcal{L}(E)$, the Cauchy-type definition $\int \Phi dU := \lim_n \int \phi_n dU$ whenever $\left(\phi_n\right)_{n=1}^\infty$ is a sequence of Borel functions of finite range tending uniformly to Φ makes sense.

Since $D|E_0 = 0$, by 2.2, D commutes with any inner derivation of E. Hence

$$D = \exp\,\mathrm{ad}(ia\Box a^*)D = \exp(ia\Box a^*)D\exp(ia\Box a^*)^{-1} \qquad (a \in E_0),$$
$$D = UDU^{-1} \qquad (U \in \mathcal{U}),$$
$$D = \int UDU^{-1}dU = \sum_{j=1}^{n} \alpha_j U(a_j\Box a_j^*)U^{-1}dU = \sum_{j=1}^{n} \alpha_j (Ua_j)\Box(Ua_j)^* dU$$
$$= \sum_{j=1}^{n} \alpha_j Z$$

where
$$Z := \left[\int (Ue)\Box(Ue)^* dU \,:\, e \text{ min.trip.} \in E_0\right] .$$

By Schur's lemma, the operator Z is a multiple of the identity when restricted to E_0 (since $Z|E_0$ commutes with $\mathcal{U}$). Morever

$$Z|E_0 = \gamma\ \mathrm{id}_{E_0} \qquad \text{with some } \gamma > 0,$$

because $e\Box e^* \neq 0$ is a positive E-Hermitian operator whenever $0 \neq e \in E_0$. Consequently, from the assumption $D|E_0$ it follows $\sum_j \alpha_j = 0$ whence also $D = \sum_j \alpha_j Z = 0$. ∎

Corollary 2.4. *If the base E_0 of $\mathbf{E}$ is a finite dimensional Cartan factor, then the restriction map $D \mapsto D|E_0$ of the complex operator Lie algebra $\mathcal{D} := \mathrm{Span}_{\mathbb{C}}\, E_0\Box E_0^*$ is injective. The center $\mathcal{Z}$ of $\mathcal{D}$ is 1-dimensional. The mapping*

$$P\colon \sum_j \lambda_j e_j\Box e_j^* \mapsto \sum_j \lambda_j Z \qquad (\lambda_j \in \mathbb{C},\ e_j \text{ min.trip.} \in E_0)$$

is a well-defined contractive projection of $\mathcal{D}$ onto $\mathcal{Z}$.

Proof. To prove the first statement, we only need to notice that any operator $D \in \mathcal{D}$ has the form

$$D = A + iB\ , \quad A = \sum_{j=1}^{n} \alpha_j a_j\Box a_j^*\ , \quad B = \sum_{j=1}^{n} \beta_j b_j\Box b_j^*$$

with $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \mathbb{R}$. Since the operators $A|E_0, B|E_0$ are E_0-Hermitian, $D|E_0$ vanishes if and only if $A|E_0 = B|E_0 = 0$. By Proposition 2.3, the latter is equivalent to $A = B = 0$.

To see that the mapping P is a projection, observe that if $a \in E_0$ and $U := \widehat{U}|E_0$ where $\widehat{U} := \exp(i\tau a\Box a^*)$ and with $\tau \in \mathbb{R}$ and $a \in E_0$ then $(Ue)\Box(Ue)^* = \widehat{U}(e\Box e^*)\widehat{U}^{-1}$ $(e \in E_0)$. Therefore every operator $U \in \mathcal{U}$ admits a (not necessarily unique) surjective isomorphic extension $\widehat{U}$ from E_0 to E such that $(Ue)\Box(Ue)^* = \widehat{U}(e\Box e^*)\widehat{U}^{-1}$ $(e \in E_0)$. With such an extension operation

$$PD = \int \widehat{U} D \widehat{U}^{-1} dU \qquad (D \in \mathcal{D}) .$$

Since $\|\widehat{U}D\widehat{U}^{-1}\| = \|D\|$ for all $U \in \mathcal{U}$, the mapping P is contractive. ∎

Remark 2.5. Although the closed subgroup $\mathcal{U}$ of $\mathcal{L}(E_0)$ generated by the family $\{\exp(ia\Box a^*)|E_0 : a \in E_0\}$ of operators is a finite dimensional compact Lie group if $\dim(E_0) < \infty$, the closed subgroup $\widetilde{U}$ of $\mathcal{L}(E)$ generated by $\{\exp(ia\Box a^*) : a \in E_0\}$ need not be a finite dimensional compact Lie group if the partial Jordan-triple E is infinite dimensional. The proof by Panou [11] using Levi–Malcev decomposition for the special case $\dim(E) < \infty$ of the Proposition relies heavily upon the compactness of the group of surjective linear isomerties of E.

3. Extendibility of derivations of c_0-direct sums

Definition 3.1. A partial Jordan-triple $\mathbf{E} := (E, E_0, \{ * \})$ has the *inner derivation extension property* (IDEP for short) if for every inner derivation $D_0 \in \mathrm{Der}_0(\mathbf{E}_0)$ of the base E_0 there is only a unique inner derivation $D \in \mathrm{Der}_0(\mathbf{E})$ with $D_0 = D|E_0$.

Proposition 3.2. *Suppose the base E_0 of the partial Jordan-triple $\mathbf{E} = (E, E_0, \{ * \})$ is the c_0-direct sum of a family $\mathcal{F}$ of closed ideals such that the subtriples $(E, F, \{ * \})$ have* IDEP *for all $F \in \mathcal{F}$. Then $\mathbf{E}$ has* IDEP .

Proof. It suffices to see that

$$D := \sum_{k=1}^{n} \alpha_k a_k \Box a_k^* = 0 \quad \text{whenever} \quad D_0 := D|E_0 = 0,$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $a_1, \ldots, a_n \in E_0$. For any subfamily $\mathcal{Z} \subset F$, let $P_{\mathcal{Z}}$ denote the projection of E_0 onto the closed ideal $\otimes^{c_0}\mathcal{Z}$ along the complementer ideal $\otimes^{c_0}\left(\mathcal{F} \setminus \mathcal{Z}\right)$. By the definition of c_0-direct sums, $\lim_{\mathcal{Z} \text{ finite} \subset \mathcal{F}} \|a - P_{\mathcal{Z}}a\| = 0$ for any $a \in E_0$). Therefore

$$\lim_{\mathcal{Z} \text{ finite} \subset \mathcal{F}} \|P_{\mathcal{Z}}a\Box P_{\mathcal{Z}}a^* - a\Box a^*\| = 0 \qquad (a \in E_0) .$$

On the other hand, the operators $a\Box a^*$ act componentwise on E_0, i.e.

$$P_{\mathcal{Z}}a\Box P_{\mathcal{Z}}a^*|E_0 = P_{\mathcal{Z}}\Big(a\Box a^*\Big)|E_0 \qquad (a \in E_0\ ,\ \mathcal{Z} \subset \mathcal{F})\ .$$

By passing to finite linear combinations, it follows

$$D = \lim_{\mathcal{Z}\ \text{finite}\subset\mathcal{F}} \sum_{k=1}^{n} \alpha_k P_{\mathcal{Z}} a_k \Box P_{\mathcal{Z}} a_k^* = \lim_{\mathcal{Z}\ \text{finite}\subset\mathcal{F}} \sum_{k=1}^{n} \alpha_k \sum_{F\in\mathcal{Z}} P_{\{F\}} a_k \Box P_{\{F\}} a_k^*.$$

Here we have

$$0 = P_{\{F\}} D_0 | E_0 = \sum_{k=1}^{n} \alpha_k P_{\{F\}} a_k \Box P_{\{F\}} a_k^* | E_0 \qquad (F \in \mathcal{F})\ .$$

Hence, by assumption, for the inner derivative extension of $D_0|F$ to $(E, F, \{\ *\ \})$ we get

$$0 = \sum_k \alpha_k P_{\{F\}} a_k \Box P_{\{F\}} a_k^* \qquad (F \in \mathcal{F})\ .$$

Thus

$$D = \lim_{\mathcal{Z}\ \text{finite}\subset\mathcal{F}} \sum_{k=1}^{n} \alpha_k \sum_{F\in\mathcal{Z}} P_{\{F\}} a_k \Box P_{\{F\}} a_k^*.$$

■

Remark 3.3. By a result of Bunce-Chu [3], the c_0-direct sums of finite dimensional Cartan factors are the so-called *compact* JB*-*triples* i.e. JB*-triples whose inner derivations are compact operators. Thus in view of Proposition 2.3 we have the following.

Corollary 3.4. *A partial Jordan-triple whose base is a compact* JB*-*triple, has* IDEP . *In particular, partial Jordan-triples with finite dimensional base have* IDEP .

4. Quasigrids

Definition 4.1. A subset Q of the base E_0 of a partial Jordan-triple $(E, E_0, \{\ *\ \})$ is a *quasigrid* if every finite subset of Q generates a finite dimensional subtriple of E_0.

Remark 4.2. A family G consisting of tripotents in E_0 is called [10] a *grid* if $(e\Box e^*)f = \lambda f$ for some $\lambda \in \{0, \frac{1}{2}, 1\}$ whenever $e, f \in G$. Clearly, grids consisting of minimal tripotents are quasigrids.

A further important example of quasigrids is given by the following two lemmas.

Lemma 4.3. *Let H, K be Hilbert spaces. Then the family $r(H,K)$ of all operators with finite rank (finite dimensional range) is a quasigrid in $\mathcal{L}(H,K)$ (equipped with the canonical triple product $\{a,b^*,c\} := \frac{1}{2}ab * c + \frac{1}{2}cb^*a$ where b^* means the adjoint of the operator b).*

Proof. Any finite rank operator is a finite linear combination of 1-rank operators of the form $e \otimes f^* : h \mapsto \langle h, f\rangle e$ (where $e \in K$, $f \in H$). If $e_1, \ldots, e_N \in K$ and $f_1, \ldots, f_N \in H$ then $\{e_i \otimes f_j^*, (e_k \otimes f_\ell^*)^*, e_m \otimes f_n^*\} \in \operatorname{Span}\{e_p \otimes f_q^* : p, q = 1, \ldots, N\} =: S$. Thus if the finite family $F \in r(H,K)$ consists of finite linear combinations of the operators $e_1 \otimes f_1^*, \ldots, e_N \otimes f_N^*$ then the subtriple generated by F is contained in the N^2-dimensional subtriple S of $\mathcal{L}(H,K)$. ■

Corollary 4.4. *If $\dim(H) < \infty$ or $\dim(K) < \infty$ then $\mathcal{L}(H,K)$ is a quasigrid itself. In particular H ($\simeq \mathcal{L}(H,\mathbb{C})$) is a quasigrid itself.*

Proof. In this case we have $\mathcal{L}(H,K) = r(H,K)$. ■

Lemma 4.5. *A spin factor is a quasigrid itself.*

Proof. According to [7,(4.11)], in general, *Jordan-triples of finite rank are quasigrids* in our terminology. A direct proof giving more insight is also very simple:

A spin factor can be viewed as a Hilbert space H endowed with the triple product

$$\{a,b^*,c\} := \frac{1}{2}\langle a,b\rangle c + \frac{1}{2}\langle c,b\rangle a - \frac{1}{2}\langle \overline{a},\overline{c}\rangle \overline{b}$$

where the operation $^{-}: \sum_i \xi_i e_i \mapsto \sum_i \overline{\xi_i} e_i$ is a conjugation with respect to a given orthonormed bases $\left\{e_i\right\}_{i\in I}$. It is immediate that the subtriple generated by the family $\{a_1, \ldots, a_n\}$ is contained in $\operatorname{Span}\{a_1, \overline{a_1}, \ldots, a_n, \overline{a_n}\}$. ■

Definition 4.6. Given a partial Jordan-triple $\mathbf{E} := (E, E_0, \{\ *\ \})$ and a quasigrid $Q \subset E_0$, the finite real linear combinations of derivations from $\{ia\Box a^*|E_0 :\ a \in Q\}$ (respectively $\{ia\Box a^*|E_0 :\ a \in Q\}$) will be called *Q-derivations of* $\mathbf{E}_0$ (respectively *Q-derivations of* $\mathbf{E}$).

Theorem 4.7. *If Q is a quasigrid in the base E_0 of the partial Jordan-triple $\mathbf{E} = (E, E_0, \{\ *\ \})$ then every inner Q-derivation A of $\mathbf{E}_0$ admits a unique extension $\widetilde{A}$ to E wich is an inner derivation of $\mathbf{E}$.*

Proof. Suppose

$$iA = \sum_{j=1}^{n} \alpha_j a_j \Box a_j^* |E_0 = \sum_{k=1}^{m} \beta_k b_k \Box b_k^* |E_0$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m \in Q$ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathbb{R}$. Let F_0 denote the subtriple of E_0 generated by the family $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$. Since Q is a quasigrid, $\dim(F_0) < \infty$. On the other hand the mapping $A_0 := A|F_0$ is an inner derivation of F_0. Thus we may apply Proposition 2.3 to the partial Jordan-triple $(E, F_0, \{\ *\ \})$ and the derivation A_0 of F_0 to conclude that $\sum_{j=1}^{n} \alpha_j a_j \Box a_j^* = \sum_{k=1}^{m} \beta_k b_k \Box b_k^*$ on the whole E. ■

Corollary 4.8. *If the base E_0 is isomorphic to $\mathcal{L}(H, K)$ and one of the Hilbert spaces H, K is finite dimensional then the partial Jordan-triple $(E, E_0, \{\ *\ \})$ has* IDEP .

Definition 4.9. Given a Q-derivation D of the base E_0, we call the unique Q-derivation $\widetilde{D}$ of E with $D = \widetilde{D}|E_0$ the *Q-extension* of D.

5. Elementary triples in the base

In [4] one has introduced the concept of *elementary* JB*-*triples* as the smallest infinite dimensional analogues of the classical finite dimensional Cartan factors. Namely, an elementary JB*-triple of Type I is an isometric copy of a space $c_0(H, K)$ of all compact operators acting between two Hilbert spaces H, K (regarded as a subtriple of $\mathcal{L}(H, K)$ with the mentioned operator triple product). An elementary JB*-triple of Type II is an isometric copy of the subtriple $c_0^{\mathcal{B}+}(H)$ of some $c_0(H, H)$ consisting of the operators with *symmetric matrix* with respect to some given orthonormed bases $\mathcal{B}$ in H. Similarly, an elementary JB*-triple of Type III is a copy of some space $c_0^{\mathcal{B}-}(H)$ (the subtriple of $c_0(H, H)$ consisting of the operators with *antisymmetric matrix* with respect to some orthonormed basis $\mathcal{B}$). Elementary JB*-triples of Type IV are spin factors (possibly infinite dimensional) and those of Type V and VI are the 18- and 27-dimensional expectional Cartan factors.

Thus, in terms of quasigrids, one can characterise elementary JB*-triples as irreducible JB*-triples admitting a norm-dense quasigrid. Indeed, if we consider an elementary JB*-triple F of Type I, II or III as a subtriple of a $c_0(H,K)$-space as described above then $F \cap r(H,K)$ is a norm-dense quasigrid in F. (Triples of Types IV,V,VI are quasigrids).

Our next aim will be to extend the result of the Proposition to elementary JB*-triples of Type I, II, III.

Remark 5.1. Recall that if H_0, K_0 are finite dimensional Hilbert spaces then the inner derivations of the typical factor $F_1 := c_0(H_0, K_0)$ of Type I are the mappings of the form $x \mapsto iBx + ixA$ whith any couple of self-adjoint operators $A \in \mathcal{L}(K_0, K_0)$, $B \in \mathcal{L}(H_0, H_0)$ such that trace(A) = trace(B). The inner derivations of the triples of Type II, resp. III $F_2 := c_0^{\mathcal{B}+}(H_0)$ and $F_3 := c_0^{\mathcal{B}-}(H_0)]$ are the mappings of the form $x \mapsto iAx + ixA^T$ (defined on F_2 and F_3, resp.) with any self-adjoint $A \in \mathcal{L}(H_0)$ where A^T is the operator whose matrix is the *transpose* of that of A with respect to the basis $\mathcal{B}$.

It follows immediately that for the norm closure of the inner derivations of the infinite dimensional analogues of the factors F_1, F_2, F_3 (i.e. the spaces $c_0(H,K), c_0^{\mathcal{B}\pm}$ with infinite dimensional Hilbert spaces H, K) are the mappings of the form $x \mapsto iBx + ixA$ resp. $x \mapsto iAx + ixA^T$ (with arbitrary self-adjoint compact operators $A: H \to H$ and $B: K \to K$).

Lemma 5.2. *Given a self-adjoint operator A with finite rank on a Hilbert space H, there exists a finite sequence $\delta_1, \dots, \delta_m \in \mathbb{R}$ and there are orthogonal projections $P_1, \dots, P_m$ with finite rank such that*

$$A = \sum_{k=1}^{m} \delta_k P_k , \qquad \sum_{k=1}^{m} |\delta_k| \le 2\|A\| .$$

Proof. We can write

$$A = \sum_{j=1}^{n} \Big(\lambda_j R_j + \mu_j S_j \Big) \qquad \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n = 0 = \mu_n \ge \mu_{n-1} \ge \dots \ge \mu_1$$

with projections $R_1, S_1, \dots, R_n, S_n$ to pairwise orthogonal finite dimensional subspaces. Here $||A|| = \max \lambda_1, -\mu_1$. Thus the choice $m := 2n-2$,

$$\delta_k := \lambda_k - \lambda_{k+1}\ , \qquad P_k := \sum_{j\leq k} Q_j\ ,$$
$$\delta_{k+n-1} := \mu_k - \mu_{k+1}\ , \qquad P_{k+n-1} := \sum_{j\leq k} S_j \quad (k = 1, \dots, n-1)$$

suits our requirements. ∎

Lemma 5.3. *Let H, K be infinite dimensional Hilbert spaces with an orthonormed basis $\mathcal{B}$ in H. For $k = 1, 2, 3$ let $\mathbf{E}^k$ be a partial Jordan-triple whose base E_0^k is the elementary* JB*-*triple*

$$E_0^1 := c_0(H, K)\ , \qquad E_0^2 := c_0^{\mathcal{B}+}(H)\ , \qquad E_0^3 := c_0^{\mathcal{B}-}(H)\ ,$$

and in each case let Q^k be the quasigrid of all finite rank operators in E_0^k. Suppose $A \in r(H, H)$ is a self-adjoint operator with trace$(A) = 0$. *Then the Q^k-extensions $\widetilde{D}_k$ of the derivations*

$$D_1 \colon E_0^1 \ni a \mapsto iaA\ , \qquad D_2 \colon E_0^2 \ni a \mapsto iAa + iaA^T, \qquad D_2 \colon E_0^3 \ni a \mapsto iAa + iaA^T,$$

where ^{T} denotes operator transposition with respect to $\mathcal{B}$, satisfy the norm-estimate

$$||\widetilde{D}_k|| \leq 4M||A||$$

whenever $||\{x, a^*, x\}|| \leq M||x||^2||a|| \quad (a \in E_0^k, x \in E^k, \quad k = 1, 2, 3)$.

Proof. Let us write the operator A in the form

$$A = \sum_{j=1}^{m} \delta_j P_j\ , \qquad \sum_{j=1}^{m} |\delta_j| \leq 2||A||$$

with orthogonal projections $P_j \in r(H, H)$ and coefficients $\delta_j \in \mathbb{R}$. Choose a a finite dimensional subspace H_0 in H such that $N := \dim(H_0) \geq \operatorname{rank}(P_j)$ $(j = 1, \dots, n)$ and let P_0 denote the orthogonal projection of H onto H_0. Consider the derivations

$$D_{1,j} \colon E_0^1 \ni a \mapsto ia\Big(P_j - \frac{\operatorname{rank}(P_j)}{N} P_0\Big),$$
$$D_{2,j} \colon E_0^2 \ni a \mapsto iP_j a + iaP_j^T,$$
$$D_{3,j} \colon E_0^3 \ni a \mapsto i\Big(P_j + \frac{\operatorname{rank}(P_j)}{N} P_0\Big)a + ia\Big(P_j + \frac{\operatorname{rank}(P_j)}{N} P_0\Big)^T$$

for any index j. Observe that

$$D_k = \sum_{j=1}^{n} i\delta_j D_{k,j} \qquad (k = 1, 2, 3)$$

due to the assumption $\mathrm{trace}(A) = \sum_{j=1}^{n} \delta_j \mathrm{rank P_j} = 0$. Thus, by the previous lemma, it suffices to check that each $D_{k,j}$ is a Q^k-derivation of $\mathbf{E}_0^k$ whose Q^k-extension $\widetilde{D}_{k,j}$ to E^k satisfies the norm estimate

$$\|\widetilde{D}_{k,j}\| \leq 4M \; .$$

Let us fix an index j arbitrarily and write

$$P := P_j, \quad r := \mathrm{rank}(P).$$

Remark that $r \leq N$. Let us fix orthonormed bases $\{e_1, \ldots, e_r\}$ and $\{f_1, \ldots, f_N\}$ in PH and P_0H, respectively.

Case of $\mathbf{E}^1$. Choose also an orthonormed set $\{g_1, \ldots, g_N\}$ in the space K and write R_0 for the orthogonal projection of K onto the span of $\{g_1, \ldots, g_N\}$. For any system $1 \leq i_1 < i_2 < \cdots < i_r \leq N$, the operators

$$I_{i_1,\ldots,i_r} := \sum_{s=1}^{r} g_{i_s} \otimes e_s^*, \quad I := \sum_{t=1}^{N} g_t \otimes f_t^*$$

are partial isometries belonging to the quasigrid Q^1. Indeed, an immediate calculation shows that $I_{i_1,\ldots,i_r} I^*_{i_1,\ldots,i_r} I_{i_1,\ldots,i_r} = I_{i_1,\ldots,i_r}$. For any $a \in E_0^1$ we have

$$\begin{aligned}
2 \sum_{1\leq i_1<\cdots<i_r\leq N} & \left(I_{i_1,\ldots,i_r} \Box I^*_{i_1,\ldots,i_r}\right) a \\
&= \sum_{1\leq i_1<\cdots<i_r\leq N} \left(I_{i_1,\ldots,i_r} I^*_{i_1,\ldots,i_r} a + a I^*_{i_1,\ldots,i_r} I_{i_1,\ldots,i_r}\right) \\
&= \Big(\sum_{1\leq i_1<\cdots<i_r\leq N} \sum_{s=1}^{r} g_{i_s} \otimes g_{i_s}^*\Big) a + a \Big(\sum_{1\leq i_1<\cdots<i_r\leq N} \sum_{s=1}^{r} e_s \otimes e_s^*\Big) \\
&= \binom{N-1}{r-1} \Big(\sum_{t=1}^{N} g_t \otimes g_t^*\Big) a + a \binom{N}{r} \sum_{s=1}^{r} e_s \otimes e_s^* \\
&= \binom{N-1}{r-1} R_0 a + a \binom{N}{r} P, \\
2\left(I \Box I^*\right) a &= II^* a + a I^* I = R_0 a + a P_0.
\end{aligned}$$

Hence

$$D_{1,j}a = 2i\binom{N}{r}^{-1} \sum_{1\le i_1<i_2<\cdots<i_r\le N} (I_{i_1,\dots,i_r}\Box I^*_{i_1,\dots,i_r})a - \frac{2ir}{N}(I\Box I^*)a \quad (a \in E^1_0).$$

Therefore for the Q^1-extension of D_1 to E^1 we get

$$\widetilde{D}_{1,j} = 2i\binom{N}{r}^{-1} \sum_{1\le i_1<i_2<\cdots<i_r\le N} I_{i_1,\dots,i_r}\Box I^*_{i_1,\dots,i_r} - \frac{2ir}{N}I\Box I^*.$$

Since $||b\Box b^*|| \le M$ for any $b \in E^1_0$ with $||b|| = 1$, it follows

$$||\widetilde{D}_{1,j}|| \le 2\binom{N}{r}^{-1} \sum_{1\le i_1<i_2<\cdots<i_r\le N} M + \frac{2r}{N}M \le 4M.$$

Case of $\mathbf{E}^2$. Now we have simply

$$D_{2,j}a = 2i\Big(J\Box J^*\Big)a \qquad (a \in E^2_0)$$

with the partial isometry

$$J := \sum_{s=1}^{n} e_s \otimes \overline{e_s}^* .$$

Hence the Q^2-extension of D_2, j to E^2 is $\widetilde{D}_{2,j} = 2iJ\Box J^*$ with $||\widetilde{D}_2|| \le 2M$.

Case of $\mathbf{E}^3$. For any system of indices $1 \le i_1 < i_2 < \cdots < i_r \le N$ set

$$J_{i_1,\dots,i_r} := \sum_{s=1}^{r}\Big(e_s \otimes \overline{f_{i_s}}^* - f_{i_s} \otimes \overline{e_s}^*\Big).$$

Since the set $\{e_1, \dots, e_r, f_1, \dots, f_N\}$ is orthonormed in H, we have $J_{i_1,\dots,i_r} = J_{i_1,\dots,i_r}J^*_{i_1,\dots,i_r}J_{i_1,\dots,i_r}$, that is the operators $J_{i_1,\dots,i_r} \in Q^3$ are partial isometries.

Observe that, for any $a \in E_0^3$,

$$
\begin{aligned}
2 \sum_{1 \le i_1 < \cdots < i_r \le N} & \left(J_{i_1,\dots,i_r} \Box J^*_{i_1,\dots,i_r} \right) a \\
&= \sum_{1 \le i_1 < \cdots < i_r \le N} \left(J_{i_1,\dots,i_r} J^*_{i_1,\dots,i_r} a + a J^*_{i_1,\dots,i_r} J_{i_1,\dots,i_r} \right) \\
&= \sum_{1 \le i_1 < \cdots < i_r \le N} \sum_{s=1}^{r} \left(e_s \otimes e_s^* + f_{i_s} \otimes f_{i_s}^* \right) a + \\
&\qquad + a \sum_{1 \le i_1 < \cdots < i_r \le N} \sum_{s=1}^{r} \left(\overline{e_s} \otimes \overline{e_s}^* + \overline{f_{i_s}} \otimes \overline{f_{i_s}}^* \right) \\
&= \binom{N}{r} \left(\sum_{s=1}^{r} (e_s \otimes e_s^*) a + a \sum_{s=1}^{r} (e_s \otimes e_s^*)^T \right) + \\
&\qquad + \binom{N-1}{r-1} \left(\sum_{t=1}^{N} (f_t \otimes f_t^*) a + a \sum_{t=1}^{N} (f_t \otimes f_t^*)^T \right) \\
&= \binom{N}{r} (Pa + aP^T) + \binom{N-1}{r-1} (P_0 a + a P_0^T).
\end{aligned}
$$

Thus

$$
D_{3,j} a = 2i \binom{N}{r}^{-1} \sum_{1 \le i_1 < i_2 < \cdots < i_r \le N} \left(J_{i_1,\dots,i_r} \Box J^*_{i_1,\dots,i_r} \right) a \qquad (a \in E_0^3)
$$

whence for the Q^3-extension to E^3 we obtain

$$
\widetilde{D}_{3,j} = 2i \binom{N}{r}^{-1} \sum_{1 \le i_1 < \cdots < i_r \le N} J_{i_1,\dots,i_r} \Box J^*_{i_1,\dots,i_r},
$$

$$
\|\widetilde{D}_{3,j}\| \le 2 \binom{N}{r}^{-1} \sum_{1 \le i_1 < \cdots < i_r \le N} M = 2M.
$$

■

Remark 5.4. By changing the roles of the spaces K and H in the treatment of the case of $\mathbf{E}^1$ above, we can see that also that the Q^1-extension $\widetilde{D_1'}$ of a Q^1-derivation

$$
D_1' : E_0^1 \ni a \mapsto iBa \qquad (B \text{ self-adjoint} \in r(K,K)\,,\ \operatorname{trace}(B) = 0)
$$

satisfies the norm estimate

$$
\|\widetilde{D_1'}\| \le 4M\|B\|.
$$

Lemma 5.5. *With the notations of Lemma* 5.3 *and Remark* 5.4, *we have the norm estimates*

$$4M(\|A\|+\|B\|) \geq \|D_1 + D_1'\| \geq \max\{\|A\|, \|B\|\},$$
$$4M\|A\| \geq \|D_2\| \geq \|A\|,$$
$$4M\|A\| \geq \|D_3\| \geq \|A\|$$

even if $\mathrm{trace}(A), \mathrm{trace}(B) \neq 0$*).

Proof. For the cases $\mathrm{trace}(A), \mathrm{trace}(B) = 0$, the upper estimates are already established. Each of the quasigrids $c_0^{\mathcal{B}\pm}(H) \cap r(H,H), r(H,K)$ contains partial isometries of arbitrarily large rank whenever the underlying Hilbert spaces H, K are infinite dimensional. Hence, for $k = 1,2,3$, any Q^k-derivation of $\mathbf{E}^k$ can be perturbed by a Q^k-derivation of the form $\varepsilon I \square I^*$ of arbitrarily small norm in a manner such that the perturbed derivation have the form $a \mapsto iB'a + iaA'$ with $\mathrm{trace}(A') = \mathrm{trace}(B') = 0$. Therefore the upper norm estimates extend also to the case. Next we proceed to the lower estimates.

Case of $\mathbf{E}^1$. Given a couple of unit vectors $u \in H$ and $v \in K$, we have $-i(D_1 + D_1')(v \otimes u^*) = v \otimes (Au)^* + (Bv) \otimes u^*$, $\|v \otimes u^*\| = 1$. Choosing first u to be an eigenvector of A with eigenvalue of highest modul and v from the kernel of B (by assumption A, B have finite rank), we have $\|(D_1 + D_1')(v \otimes u^*)\| = \|(Au) \otimes v^*\| = \|Au\| \cdot \|v\| = \|A\|$. Therefore $\|D_1 + D_1'\| \geq \|A\|$. Choosing v to be an eigenvector of B with eigenvalue of highest module and u from the kernel of A, similarly we get $\|D_1 + D_1'\| \geq \|B\|$.

Case of $\mathbf{E}^2$. If $u, v \in H$ are unit vectors and u is an eigenvector of A with eigenvalue $\lambda \in \{\pm\|A\|\}$ and v belongs to the kernel of A then

$$D_2(u \otimes \overline{v}^* + v \otimes \overline{u}^*) = i\lambda(u \otimes \overline{v}^* + v \otimes \overline{u}^*).$$

Hence immediately $\|D_2\| \geq \|A\|$.

Case of $\mathbf{E}^3$. A similar argument to the one used for $\mathbf{E}^2$ with $u \otimes \overline{v}^* - v \otimes \overline{u}^*$ instead of $u \otimes \overline{v}^* + v \otimes \overline{u}^*$.

■

Proposition 5.6. *A partial Jordan-triple* $\mathbf{E} := (E, E_0, \{\ *\ \})$ *whose base* E_0 *is an elementary* JB*-*triple has* IDEP .

---

*) For any operator $c \in r(H,K)$ we have $\mathrm{trace}(c^*c) = \mathrm{trace}(cc^*)$. Since $(c \square c^*)a = (cc^*a + ac^*c)/2$ $(a, c \in c_0(H,K))$, it follows that the mapping $[c_0(H,K) \ni a \mapsto iBa + iaA]$ with $A \in r(H,H)$, $B \in r(K,K)$ is an $r(H,K)$-derivation if and only if $\mathrm{trace}(A) = \mathrm{trace}(B)$.

Proof. If E_0 itself is a quasigrid then the statement is already contained in Proposition 2.3.

If E_0 is not a quasigrid then E_0 is isometrically isomorphic to some space of the form $c_0(H,K)$ or $c_0^{\mathcal{B}\pm}(H)$ with infinite dimensional underlying Hilbert spaces H, K. Thus we may assume that E_0 coincides with one of the mentioned operator spaces. These cases are considered in Lemmas 5.3, 5.5. In any case, let Q denote the quasigrid of all finite rank elements of E_0. Furthermore, let $\mathcal{D}$ resp. $\mathcal{D}_0$ be the families of all Q-derivations of E resp. the base space E_0. According to the norm estimates of the previous lemma, the Q-extension

$$\begin{aligned} T:&\mathcal{D}_0 \to \mathcal{D} \\ &D \mapsto [\widetilde{D} \in \mathcal{D} : \ \widetilde{D}|E_0 = D] \end{aligned}$$

is a bijective linear operation $\mathcal{D}_0 \leftrightarrow \mathcal{D}$ such that

$$\|T\| \leq 4M, \qquad \|T^{-1}\| \leq 1.$$

(The inverse $T^{-1} : \widetilde{D} \mapsto \widetilde{D}|E_0$ is obviously contractive). Therefore the operation T admits a unique linear bicontinuous extension

$$\overline{T}: \overline{\mathcal{D}_0} \leftrightarrow \overline{\mathcal{D}}$$

where $\overline{\mathcal{D}_0}$ resp. $\overline{\mathcal{D}}$ denote the closures of the linear submanifolds $\mathcal{D}_0$ resp. $\mathcal{D}$ in the spaces $\mathrm{Der}(E_0)$ resp. $\mathrm{Der}(E)$ with respect to operator norm. Since the quasigrid Q is norm dense in E_0, and since $\{ia\Box a^* : \ a \in Q\} \subset \mathcal{D}_0$, by the continuity of the mapping $a \mapsto a\Box a^*$ we have

$$\begin{aligned} a\Box a^* &= \lim_{Q\ni a_i \to a} a_i\Box a_i^* = \lim_{Q\ni a_i \to a} T(a_i\Box a^*|E_0) = \overline{T}(\lim_{Q\ni a_i \to a} a_i a|E_0) \\ &= \overline{T}(a\Box a^*|E_0) \qquad (a \in E_0). \end{aligned}$$

Hence, by the linearity of the mapping $\overline{T}$,

$$\sum_{k=1}^{n} \alpha_k a_k \Box a_k^* = \overline{T}\Big(\sum_{k=1}^{n} \alpha_k a_k \Box a_k^* |E_0\Big) = 0 \quad \text{whenever} \quad \sum_{k=1}^{n} \alpha_k a_k \Box a_k^* |E_0 = 0.$$

Thus inner derivations of $\mathbf{E}_0$ admit unique inner derivative extensions to $\mathbf{E}$. ■

In view of Proposition 3.2 our main result is immediate.

Theorem 5.7. *A partial Jordan-triple whose base is a c_0-direct sum of a family of elementary* JB*-*triples has* IDEP .

Definition 5.8. Let us call a derivation of a partial Jordan-triple a *σ-inner derivation* if it is the norm limit of some sequence of inner derivations.

As a by-product of the proof of Theorem 5.6, we have also obtained the following result.

Corollary 5.9. *Let $(E, E_0, \{\ *\ \})$ be a partial Jordan-triple whose base E_0 is an elementary* JB*-*triple with infinite rank or finite dimensions**). *Then any σ-inner derivation of E_0 admits a unique σ-inner derivative extension to E*

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) In other words, the elementary JB-triple E_0 is not isomorphic to some infinite dimensional spin factor or to some $c_0(H, K)$ with finite dimensional H and infinite dimensional K.

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