Acta Sci. Math. (*Szeged*)
57 (1993), 555–567

On weakly and weakly* continuous elements in Jordan triples

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 80th birthday

Introduction

In a previous paper [8] we have studied the structure of elements in dual Jordan triple algebras whose squaring operation was assumed to be continuous from various topologies into the weak* topology. Such elements admit interesting compactness properties with a direct physical interpretation in case of (bounded) weak*-weak* continuous squaring. The obvious disadvantage of such a concept is that it works only in the presence of a predual. In [6] the notion of weakly continuous elements (i.e. elements with weak-weak continuous squaring) in JB*s was introduced to characterize the one-parameter groups of weakly continuous holomorphic automorphisms of the unit ball. This concept applies to every JB* but we have no nice spectral representations in general. Recently, a connection was established in [3] between the weak and weak* continuity of elements in terms of bidual embedding. However, it may be a quite hard problem to interpret the results with seemingly simple structure gained by bidual embedding in the original setting. This paper is devoted to the study of such a situation. We shall describe the weakly continuous elements of spaces of continuous JB*-valued functions and we shall completely characterize the ideals $\mathrm{Cont}_w(E)$ and $\mathrm{Cont}_{w^*}(E)$ of all weak- resp. weak*-continuous elements in a dual JB*-triple E.

We recall that a JB* is a complex Banach space E equipped with a ternary operation

$$E \times E \times E \longrightarrow E \quad (x, y, z) \longmapsto \{xy^*z\}$$

Received February 19, 1992.

* Supported by the Xunta de Galicia, project XUGA 20702 B 90.
⋆ Supported by the Hungarian National Science Foundation, grant No. 91/3.

called the *triple product* satisfying the following conditions

(J1) $\{xy^*z\}$ is symmetric and bilinear in the outer variables x, z and conjugate linear in the inner variable y,

(J2) $\|\{xx^*x\}\| = \|x\|^3$,

by writing $a \diamond b^*$ for the operator $x \mapsto \{ab^*x\}$ on E,

(J3) $a \diamond b^*\{xy^*z\} = \{(a \diamond b^*x)y^*z\} - \{x(a \diamond b^*y)^*z\} + \{xy^*(a \diamond b^*z)\}$ (Jordan identity),

(J4) $a \diamond a^*$ is an E-hermitian operator with non-negative spectrum

for all $a, b, x, y, z \in E$. It is well-known that (J2) is equivalent to the relation $\|a \diamond b^*\| = \|a\|\|b\|$ $(a, b \in E)$.

A closed linear subspace $I \subset E$ is called a subtriple if $\{II^*I\} \subset E$ and an ideal if $\{EE^*I\}, \{EI^*E\} \subset I$. Two subsets $A, B \subset E$ are said to be orthogonal if $A \diamond B^* = 0$. In that case also $B \diamond A^* = 0$ and we write $A \perp B$. By $A^\perp$ we denote the subspace $\{x \in E : x \perp A\}$. A vector $e \in E$ is a tripotent if $\{ee^*e\} = e$, and the tripotent e is called an atom in E if $\{eE^*e\} = \mathbb{C}E$.

The simplest example of a non-trivial JB* is the complex line $\mathbb{C}$ with the triple product $\{\zeta\xi^*\eta\} := \zeta\overline{\xi}\eta$. Every C*-algebra is a JB* in the triple product $2\{xy^*z\} := xy^*z + zy^*x$. Given a locally compact topological space Ω and a topological vector space V, we write $\mathcal{C}_0(\Omega, V)$ for the space of all continuous functions $f: \Omega \to V$ such that for each neighborhood U of 0 in V the inverse image $f^{-1}(V \setminus U)$ is precompact in Ω. If E is a JB* then $\mathcal{C}_0(\Omega, E)$ becomes a JB* under the pointwise triple product. In particular the space $\mathcal{C}_0(\Omega) := \mathcal{C}_0(\Omega, \mathbb{C})$ is a JB*.

A JB*-triple E is called a JBW*-triple if E is a dual Banach space. In that case it has a unique predual E_*. and we refer to $w^* := \sigma(E, E_*)$ as the weak* topology on E. The bidual E^{**} of a JB* is a JBW* whose triple product extends that in E. Any JBW* E admits a unique decomposition into the orthogonal ℓ^∞-direct sum of its atomic and purely non-atomic ideals

$$E = E_{\mathrm{a}} \oplus E_{\mathrm{a}}^\perp .$$

Here E_{a} is the weak* closed linear hull of the set of the atoms of E, and $E_{\mathrm{a}}^\perp$ has no atoms at all. Besides, E_{a} has the orthogonal ℓ^∞-direct sum representation

$$E_{\mathrm{a}} = \oplus_{m \in \mathcal{M}} E_m$$

where $\{E_m : m \in \mathcal{M}\}$ is the set of all weak* closed minimal ideals in E. Each E_m is isomorphic to some finite or infinite dimensional Cartan factor and $E_{\mathrm{a}}^\perp$ is isomorphic to a weak-operator closed subtriple of $\mathcal{L}(K)$ for some Hilbert space K.

We refer to [7] for a systematic study of JB*s, and to [5] for a survey and more complete references of the theory.

If E is a JB* $a \in E$ and τ is a linear topology on E then we say that the element a is τ-continuous in E if the squaring operation $x \mapsto \{xa^*x\}$ associated with a is τ-τ continuous at 0 when restricted to bounded subsets of E. We write

$$\mathrm{Cont}_\tau(E) := \{\,\tau - \text{continuous elements of } E\,\}\ .$$

Without danger of confusion we shall denote by w the weak topology $\sigma(E, E^*)$ on a JB* E. We say that E is a weakly continuous JB* if $E = \mathrm{Cont}_w(E)$.

Finally we recall that a topological space T without isolated points is said to be perfect. A topological space S is called scattered if S has no non-empty perfect subspaces. Scattered spaces were thoroughly studied under the name dispersed spaces in [1].

1. The dual of $\mathcal{C}_0(S, F)$ for a scattered space S

Lemma 1.1. *Let Ω and V be a locally compact totally disconnected topological space and a topological vector space, respectively. Then the linear submanifold*

$$F := \{\varphi \in \mathcal{C}_0(\Omega, V) : \varphi(\Omega) \qquad \textit{is finite}\}$$

is dense in $\mathcal{C}_0(\Omega, V)$ with respect to the topology of uniform convergence.

Proof. Let $f \in \mathcal{C}_0(\Omega, V)$ and a balanced neighborhood W of 0 in V be arbitrarily given. Since the space Ω is totally disconnected, for every $x \in \Omega$ there exists an open-closed neighborhood U_x of x such that $f(U_x) \subset f(x) + W$. Since $f \in \mathcal{C}_0(\Omega, V)$, we can choose a finite family $\{x_1, \ldots, x_n\}$ of points such that $\bigcup_{k=1}^n U_{x_k} \supset \{x \in \Omega : f(x) \notin W\}$. Thus, by setting $G_0 := \Omega \setminus \bigcup_{k=1}^n U_{x_k}$ and defining recursively $G_k := U_{x_k} \setminus \bigcup_{j=0}^{k-1} G_j \quad (k = 1, \ldots, n)$, we obtain a disjoint open-closed covering $\{G_0, G_1, \ldots, G_n\}$ of Ω with the property

$$f(x) - \sum_{k=1}^{n} f(x_k) 1_{G_k}(x) \in W \qquad\qquad (x \in \Omega)$$

where the symbol 1_G stands for the indicator function of the set G (i.e. $1_G(x) :=$ [1 for $x \in G$ and 0 else]). Since the sets G_k are open-closed, we have $\sum_{k=1}^n f(x_k) 1_{G_k} \in \mathcal{C}_0(\Omega, V)$ which completes the proof.

■

Henceforth let S and E denote a locally compact scattered topological space and a Banach space, respectively. For any function $\varphi \in \mathcal{C}_0(S)$ and vector $v \in E$ we shall write φv for the vector valued function $x \mapsto \varphi(x)v$ on S.

Since scattered spaces are totally disconnected, we have in particular the following.

Corollary 1.2. *The linear submanifold $\mathcal{C}_0(S) \otimes E$ spanned by the set $\{\varphi v : \varphi \in \mathcal{C}_0(S), v \in E\}$ is dense in $\mathcal{C}_0(S, E)$.*

Proposition 1.3. *For the dual of $\mathcal{C}_0(S, E)$ we have*

$$\mathcal{C}_0(S,E)^* = \Big\{[f \mapsto \sum_{x\in S}\langle \phi_x, f(x)\rangle] : \phi_x \in E^*(x \in S), \sum_{x\in S} \|\phi_x\| < \infty\Big\}.$$

Proof. Let $\Phi \in \mathcal{C}_0(S, E)^*$ be arbitrarily fixed and let us write $\mathcal{G} :=$ { open-closed subsets of S}. Define

$$\mu(G) := \|\Phi|\{f : f(S \setminus G) = 0\}\| \qquad (G \in \mathcal{G}) .$$

Observe that μ is a finitely additive bounded non-negative measure on $\mathcal{G}$. Therefore there exists a bounded positive linear functional Λ_0 on the linear submanifold $L := \{\varphi : \varphi(S) \text{ is finite}\}$ of $\mathcal{C}_0(S)$ such that

$$\Lambda_0(1_G) = \mu(G) \qquad\qquad (G \in \mathcal{G}).$$

By Lemma 1.1 the functional Λ_0 admits a unique continuous extension $\Lambda \in \mathcal{C}_0(S)^*_+$. According to a well-known result [1] on scattered spaces,

$$\Lambda(\varphi) = \sum_{k=1}^{\infty} \alpha_k \varphi(x_k) \qquad\qquad (\varphi \in \mathcal{C}_0(S))$$

for some sequences $x_1, x_2, \ldots \in S$ and $\alpha_1, \alpha_2, \ldots \geq 0$ with $\sum_k \alpha_k < \infty$.

Consider any function $f \in \mathcal{C}_0(S, E)$ with finite range. We can write $f = \sum_{k=1}^n 1_{G_k} v_k$ with a suitable disjoint family $G_1, \ldots, G_n \in \mathcal{G}$ and vectors $v_1, \ldots, v_n \in E$. Hence

$$\begin{aligned}|\langle \Phi, f\rangle| &\leq \sum_{k=1}^{n} |\langle \Phi, 1_{G_k} v_k\rangle| \leq \sum_{k=1}^{n} \mu(G_k)\|v_k\| \\ &= \int \|f(x)\| d\mu(x) = \langle \Lambda_0, [x \mapsto \|f(x)\|]\rangle .\end{aligned}$$

By the density of $\{g : \mathrm{ran}(g) \text{ is finite}\}$ in $\mathcal{C}_0(S, E)$ it follows

$$|\langle \Phi, g \rangle| \leq \langle \Lambda, [x \mapsto \|g(x)\|] \rangle = \sum_{k=1}^{\infty} \alpha_k \|g(x_k)\| \qquad (g \in \mathcal{C}_0(S, E)) .$$

In particular

$$(1) \qquad |\langle \Phi, \varphi v \rangle| \leq \sum_{k=1}^{\infty} \alpha_k \varphi(x_k) \|v\| \qquad (\varphi \in \mathcal{C}_0(S, E),\ v \in E) .$$

For fixed $v \in E$ we have $[\varphi \mapsto \langle \Phi, \varphi v \rangle] \in \mathcal{C}_0(S)^*$. Thus, by the Riesz-Kakutani representation theorem,

$$\langle \Phi, \varphi v \rangle = \int \varphi \, d\mu_v \qquad (\varphi \in \mathcal{C}_0(S, E),\ v \in E)$$

for suitable (uniquely defined) Radon measures μ_v of bounded variation on S. Taking into account (1), the Radon-Nikodým theorem implies the existence of bounded sequences $\beta_1^v, \beta_2^v, \ldots$ $(v \in E)$ such that

$$\mu_v = \sum_{k=1}^{\infty} \beta_k^v \delta_{x_k}, \quad |\beta_1^v| \leq \alpha_1 \|v\|,\ |\beta_2^v| \leq \alpha_2 \|v\|, \ldots \qquad (v \in E),$$

where δ_x denotes the usual Dirac measure of weight 1 at the point $x \in S$.

For fixed $\varphi \in \mathcal{C}_0(S)$, the functional $v \mapsto \langle \Phi, \varphi v \rangle$ is linear on E. Hence the mapping $v \mapsto \mu_v$ is also linear. It follows

$$\phi_k := [v \mapsto \beta_k^v] \in E^* , \quad \|\phi_k\| \leq \alpha_k \qquad (k = 1, 2, \ldots) .$$

That is

$$(2) \qquad \langle \Phi, g \rangle = \sum_{k=1}^{\infty} \langle \phi_k, f(x_k) \rangle \qquad (g \in \mathcal{C}_0(S, E))$$

which shows that any bounded linear functional on $\mathcal{C}_0(S, E)$ is of the form $f \mapsto \sum_{x \in S} \langle \phi_x, f(x) \rangle$ with $\sum_{x \in S} \|\phi_x\| < \infty$. The converse of this statement is trivial. ∎

Corollary 1.4. *If S is a locally compact scattered topological space and E is a Banach space, a bounded net $(f_i : i \in I)$ in $\mathcal{C}_0(S,E)$ converges weakly to 0 if and only if for every fixed $x \in S$ the net $(f_i(x) : i \in I)$ converges weakly to 0 in E.*

Proof. If $(f_i : i \in I)$ converges weakly to 0 in $\mathcal{C}_0(S,E)$ then for the functionals $\Phi_{x,\phi} : f \mapsto \langle \phi, f(x)\rangle$ we have $0 = \lim_i \Phi_{x,\phi} = \lim_i \langle \phi, f(x)\rangle$ whenever $x \in S$ and $\phi \in E^*$. Thus $(f_i(x) : i \in I)$ converges weakly to 0 for each $x \in S$ in this case.

Suppose $\lim_i \langle \phi, f(x)\rangle = 0 \quad (x \in S, \phi \in E^*)$. Let Φ be any bounded linear functional on $\mathcal{C}_0(S,E)$. By Proposition 1.3, we have a representation of the form (2) with suitable $x_1, x_2, \ldots \in S$ and $\phi_1, \phi_2, \ldots \in E^*$ such that $\sum_k \|\phi_k\| < \infty$. Given any $\varepsilon > 0$, we can choose n such that $\sum_{k>n} \|\phi_k\| \sup_i \|f_i\| < \varepsilon/2$. Then we can choose $i_0 \in I$ such that $|\sum_{k=1}^n \langle \phi_k, f_i(x_k)\rangle| < \varepsilon/2$ whenever $i \geq i_0$ in I. It follows

$$|\langle \Phi, f\rangle| \leq |\sum_{k=1}^{n} \langle \phi_k, f_i(x_k)\rangle| + \sum_{k>n} \|\phi_k\| \|f_i(x_k)\| < \varepsilon \qquad (i > i_0)$$

which completes the proof. ∎

2. Weakly continuous elements in $\mathcal{C}_0(\Omega, E)$

Lemma 2.1. *Let E, F be* JB**s, $a \in \mathrm{Cont}_w(E)$ and let T be a J*-homomorphism of E onto F. Then $T(a) \in \mathrm{Cont}_w(F)$.*

Proof. Consider any weak neighborhood W of 0 in F. Since $T^{-1}(W)$ is a weak neighborhood and since $a \in \mathrm{Cont}_w(E)$, there exists a weak neighborhood U of 0 in E such that

$$\{[U \cap \mathrm{B}(E)]a^*[U \cap \mathrm{B}(E)]\} \subset T^{-1}(W)\,.$$

By a well-known theorem of Dang–Horn [2], by writing H for the kernel $H := T^{-1}\{0\}$, the JB* F is isometrically isomorphic to the factor space E/H by the factor mapping T/H. Therefore we have $T(\mathrm{B}(E)) = \mathrm{B}(F)$ and the image $V := T(U)$ is a weak neighborhood of 0 in F. Hence

$$T\{[U \cap \mathrm{B}(E)]a^*[U \cap \mathrm{B}(E)]\} = \{[V \cap \mathrm{B}(F)]T(a)^*[V \cap \mathrm{B}(F)]\} \subset W$$

proving $T(a) \in \mathrm{Cont}_w(F)$. ∎

Since J*-homomorphic images of JB*s are JB*s, the above Lemma can be interpreted as follows.

Corollary 2.2. *The J*-homomorphic image of a weakly continuous* JB* *is a weakly continuous* JB*.

Lemma 2.3. *Assume* $a \in \mathrm{Cont}_w \mathcal{C}_0(\Omega, E)$ *where* Ω *is a locally compact topological space and* E *is some* JB*. *Then the function* $\varphi : \omega \mapsto \|a(\omega)\|$ *belongs to* $\mathrm{Cont}_w \mathcal{C}_0(\Omega)$.

Proof. Given any bounded linear functional Φ on $\mathcal{C}_0(\Omega)$, by the Riesz-Kakutani representation theorem there is a unique Radon measure ν_Φ of bounded variation on Ω such that

$$\langle \Phi, \psi a \rangle = \int \psi d\nu_\Phi \qquad (\psi \in \mathcal{C}_0(\Omega))$$

where ψa denotes the function $\omega \mapsto \psi(\omega)a(\omega)$.

Suppose $\varphi \notin \mathrm{Cont}_w \mathcal{C}_0(\Omega)$. Then there exists a bounded net $(\psi_i : i \in I)$ converging weakly to 0 in $\mathcal{C}_0(\Omega)$ and a Radon measure μ of bounded variation on Ω such that

$$\lim_{i \in I} \int \psi_i^2 \varphi d\mu \neq 0 \,.$$

Define $\Omega_0 := \{\omega \in \Omega : \varphi(\omega) \neq 0\}$ and let $T : \mathcal{C}_0(\Omega_0) \to \mathcal{C}_0(\Omega, E)$ be the isometric embedding

$$T(\psi) := \Big[\psi(\omega)\varphi(\omega)^{-1}a(\omega) \text{ if } \omega \in \Omega_0, \quad 0 \text{ if } \omega \in \Omega \setminus \Omega_0\Big] \,.$$

The functional

$$\Psi_0(T(\psi)) := \int \psi d\mu \qquad (\psi \in \mathcal{C}_0(\Omega_0)$$

is a well-defined bounded linear functional on the subspace $T(\mathcal{C}_0(\Omega_0))$ of $\mathcal{C}_0(\Omega, E)$. Let Ψ denote any Hahn-Banach extension of Ψ_0 to $\mathcal{C}_0(\Omega, E)$.

Consider the net $a_i := \psi_i a \;\; (i \in I)$ in $\mathcal{C}_0(\Omega, E)$. It is obviously bounded and for every $\Phi \in \mathcal{C}_0(\Omega, E)^*$

$$\langle \Phi, a_i \rangle = \int \psi_i d\nu_\Phi \longrightarrow 0$$

since $\psi_i \to 0$ weakly in $\mathcal{C}_0(\Omega)$. Thus $a_i \to 0$ weakly in $\mathcal{C}_0(\Omega, E)$. However, this contradicts the weak continuity of the element a in $\mathcal{C}_0(\Omega, E)$ because

$$\langle \Psi, \{a_i a^* a_i\} \rangle = \int \psi_i \varphi \psi_i d\mu \not\to 0 \,.$$

■

Theorem 2.4. *Let Ω be a locally compact topological space and let P be the maximal perfect subset of Ω. Then we have*

$$\mathrm{Cont}_w \mathcal{C}_0(\Omega, E) = \{f \in \mathcal{C}_0(\Omega, E) : f|P = 0 \quad and \ = f(\Omega) \subset \mathrm{Cont}_w(E)\}.$$

Proof. Let us write $U := \{f \in \mathcal{C}_0(\Omega, E) : f|P = 0,\ f(\Omega) \subset \mathrm{Cont}_w(E)\}$. With the aid of the homomorphisms $T_\omega(f) := f(\omega)$ $(\omega \in \Omega, f \in \mathcal{C}_0(\Omega, E))$ it is immediate from Lemma 2.1 that $f(\Omega) \subset \mathrm{Cont}_w(E) = T_\omega(\mathcal{C}_0(\Omega, E))$ for every $\omega \in \Omega$. In [3] it is proved that $\mathrm{Cont}_w \mathcal{C}_0(\Omega) = \{f \in \mathcal{C}_0(\Omega, E) : f|P = 0\}$. Hence by Lemma 2.3 it follows $\mathrm{Cont}_w \mathcal{C}_0(\Omega, E) \subset U$

To prove the converse inclusion, notice first that U is a closed ideal in $\mathcal{C}_0(\Omega, E)$. Therefore, by [3], it suffices to see that U is a weakly continuous JB*.

Define $S := \Omega \setminus P$. It is well-known that S is an open scattered subset of the space Ω. It is also immediate that the ideal U is isometrically isomorphic to $\mathcal{C}_0(S, \mathrm{Cont}_w(E))$ by the restriction $f \mapsto f|S$.

In general, let F be a weakly continuous JB*. We prove that $\mathcal{C}_0(S, F)$ is a weakly continuous JB* as follows. Let $(f_i : i \in I)$ be a bounded net in $\mathcal{C}_0(S)$ converging weakly to 0. By Corollary 1.4 this means that

$$\langle \phi, f_i(x) \rangle \longrightarrow 0 \qquad (x \in S,\ \phi \in F^*)\ .$$

Since F is a weakly continuous JB*, for every fixed $f \in \mathcal{C}_0(S, F)$ we have hence

$$\langle \phi, \{f_i(x) f(x)^* f_i(x)\} \rangle \longrightarrow 0 \qquad (x \in S,\ \phi \in F^*)\ .$$

Again by Corollary 1.4, this latter relation means that the net $(\{f_i f^* f_i\} : i \in I)$ converges weakly to 0 in $\mathcal{C}_0(S, F)$. That is, any element of $\mathcal{C}_0(S, F)$ is weakly continuous.

■

3. Weak*-continuous elements of JBW*s

Throughout this section let E be a JBW* with the decomposition

$$E = E_{\mathrm{a}} \oplus E_{\mathrm{a}}^{\perp}$$

into atomic and completely non-atomic parts (see [4],[8]) and with the Cartan factor decomposition

$$E_{\mathrm{a}} = \oplus_{m \in \mathcal{M}} F_m\ .$$

According to this decomposition we write every $x \in E$ in the form

$$x = x_a^{\perp} \oplus x_a = x_a^{\perp} \oplus_{m \in \mathcal{M}} x_m\ .$$

Remark 3.1. For the Cartan factors F_m the ideals $\mathrm{Cont}_{w^*}F_m$ are completely described in [8]. Namely, infinite dimensional spin factors admit no non-trivial weak*-continuous elements and the weak*-continuous elements of factors of type I,II,III correspond to compact operators in the canonical operator representation. The exceptional factors of type V,VI are finite dimensional and hence obviously weak*-continuous.

Concerning the weakly continuous part of Cartan factors, it is shown in [6] that $\mathrm{Cont}_w F_m = \mathrm{Cont}_{w^*}F_m$ for every $m \in \mathcal{M}$.

Lemma 3.2. *A bounded net $(x_i : i \in I)$ in E converges weakly* to 0 if and only if*

$$\text{w}*-\lim_i x_{i\mathrm{a}}^{\perp} = 0, \qquad \text{w}*-\lim_i x_{im} = 0 \quad (m \in \mathcal{M}). \tag{3}$$

Proof. The necessity of these relations is clear.

Suppose (3) holds and let $\Phi \in E^*$ be a linear functional which is continuous also with respect to the weak* topology in E. Remark (cf. [4]) that the family of all w*-continuous linear functionals of E can be identified with the predual of E, and by setting

$$\phi_{\mathrm{a}}^{\perp} := \Phi \big| E_{\mathrm{a}}^{\perp} \qquad \phi_m := \Phi \big| F_m \quad (m \in \mathcal{M})$$

we have

$$\|\Phi\| = \|\phi_{\mathrm{a}}^{\perp}\| + \sum_{m\in\mathcal{M}} \|\phi_m\| \,.$$

We may assume without loss of generality that $\|x_{i\mathrm{a}}\|, \|x_{im}\| \leq 1 \quad (i \in I)$. Given $\varepsilon > 0$ arbitrarily, we can find a finite subset $\mathcal{N}$ of $\mathcal{M}$ such that

$$\sum_{m\in\mathcal{M}\setminus\mathcal{N}} \|\phi_m\| < \varepsilon/2$$

and then we can find $i_0 \in I$ with

$$|\langle \phi_{i\mathrm{a}}^{\perp} \rangle| + \sum_{m\in\mathcal{N}} |\langle \phi_m, x_{im} \rangle| < \varepsilon/2 \,.$$

Hence

$$|\langle \Phi, x_i \rangle| < \varepsilon \qquad\qquad (i \geq i_0)$$

which proves $\text{w}*-\lim_i x_i = 0$ by the arbitrariness of $\Phi \in E_*$ and $\varepsilon > 0$.

■

Proposition 3.3. *We have*

$$\mathrm{Cont}_{w^*}(E) = \oplus_{m\in\mathcal{M}} \mathrm{Cont}_{w^*}(F_m).$$

Proof. According to the main theorem of [8], by writing $\{F_m : \mathcal{M}_0\}$ for the family of all infinite dimensional spin factors of E, we have

$$\mathrm{Cont}_{w^*}(E) = \mathrm{comp}_{w*}(E) \subset \oplus_{m\in\mathcal{M}\setminus\mathcal{M}_0} \mathrm{comp}_{w*}(F_m) = \oplus_{m\in\mathcal{M}} \mathrm{Cont}_{w^*}(F_m)\ .$$

To prove the converse inclusion, fix $a = \oplus_{m\in\mathcal{M}} a_m \in \oplus_{m\in\mathcal{M}} \mathrm{Cont}_{w^*}(F_m)$ arbitrarily. Furthermore let

$$x_i = x_{\mathrm{a}i}^{\perp} \oplus x_{\mathrm{a}i} = x_{\mathrm{a}i}^{\perp} \oplus_{m\in\mathcal{M}} x_{im} \qquad (i \in I)$$

be a bounded net in E converging to 0 with respect to the weak* topology. Since the net $(x_i : i \in I)$ is bounded, the net $(\{x_i a^* x_i\} : i \in I)$ is also bounded in E. By assumption $a_m = 0$ for $m \in \mathcal{M}_0$ and $a_{\mathrm{a}}^{\perp} = 0$. Hence, for every index $i \in I$, $\{x_{im} a_m^* x_{im}\} = 0 \ \ (m \in \mathcal{M}_0)$ and $\{x_{i\mathrm{a}}^{\perp} a_{\mathrm{a}}^{\perp *} x_{i\mathrm{a}}^{\perp}\} = 0$. On the other hand, we know [8] that, in the usual Hilbert space operator representation, $\mathrm{Cont}_{w^*}(F_m)$ consists of compact operators for all infinite dimensional factors F_m with $m \in \mathcal{M} \setminus \mathcal{M}_0$. Therefore

$$\mathrm{w*-}\lim_i \{x_{i\mathrm{a}}^{\perp} (a_{\mathrm{a}}^{\perp})^* x_{i\mathrm{a}}^{\perp}\} = 0, \qquad \mathrm{w*-}\lim_i \{x_{im} a_m^* x_{im}\} = 0 \quad (m \in \mathcal{M}).$$

Hence Lemma 3.2 establishes $\mathrm{w*-}\lim_i \{x_i a^* x_i\} = 0$ which completes the proof. ■

4. Weakly continuous elements of JBW*s

Lemma 4.1. *Let J be an arbitrary non-empty set. Then*

$$\mathrm{Cont}_w(\ell^\infty(J)) = c_0(J).$$

Proof. Regarding J as a discrete topological space, we have $\mathcal{C}_0(J) = c_0(J)$. Since J is obviously scattered, hence $c_0(J)$ is a weakly continuous JB*. Since $c_0(J)$ is an ideal in $\ell^\infty(J)$, by [3] it follows $c_0(J) \subset \mathrm{Cont}_w(\ell^\infty(J))$.

If $a \in \ell^\infty(J) \setminus c_0(J)$ then for some infinite sequence $j_1, j_2, , \ldots$ we have $\inf_{n=1}^\infty |a_{j_n}| > 0$. Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers lying in the interval $[0,1]$. Given any continuous function $f \in \mathcal{C}([0,1])$, define

$x^f_{j_n} := f(q_n)/\overline{a_{j_n}}$ and $z^f_{j_n} := f(q_n)$ for $(n = 1, 2, \ldots)$ and let $z^f_{j_n} := x^f_{j_n} := 0$ for $j \in J \setminus \{j_1, j_2, \ldots\}$. Then for the sequences $x^f := (x^f_{j_n} : j \in J)$ and $z^f := (z^f_{j_n} : j \in J)$ we have $x^f, z^f \in \ell^\infty(J)$ and $z^f = \{x^f a^* 1_J\}$. Thus for any $f \in \mathcal{C}([0,1])$ the sequence z^f belongs to the ideal U_a of $\ell^\infty(J)$ generated by the singleton a. It is proved in [3] that $a \in \mathrm{Cont}_w(\ell^\infty(J))$ if and only if U_a is a weakly continuous JB*. However, U_a cannot be weakly continuous because $\{z^f : f \in \mathcal{C}([0,1])\}$ is a subtriple of U_a which is isometrically isomorphic to $\mathcal{C}([0,1])$ by the mapping $f \mapsto z^f$ and (also by [3]) the JB* $\mathcal{C}([0,1])$ is not weakly continuous.

■

Proposition 4.2. *Suppose $E = \oplus_{i\in I} E_i$ where $\oplus$ means ℓ^∞-direct sum of pairwise orthogonal ideals. Then* $\mathrm{Cont}_w(E)$ *coincides with the c_0-direct sum* $\oplus^{c_0}_{i\in I}\mathrm{Cont}_w(E_i)$.

Proof. By [3], $\mathrm{Cont}_w(E)$ is the largest weakly continuous closed ideal of E. Since each $\mathrm{Cont}_w(E_i)$ is a weakly continuous ideal of E, hence $\oplus^{c_0}_{i\in I}\mathrm{Cont}_w(E_i) \subset \mathrm{Cont}_w(E)$.

To see the converse inclusion, assume $a = \oplus_{i\in I} a_i$ (with $a_i \in E_i \quad (i \in I)$) belongs to $\mathrm{Cont}_w(E)$. One verifies directly that $a_i \in \mathrm{Cont}_w(E_i)$ for every $i \in I$. Suppose additionally that $a \notin \oplus^{c_0}_{i\in I}\mathrm{Cont}_w(E_i)$. Then we can find $\varepsilon > 0$ and a sequence $i_1, i_2, \ldots$ in I such that $\|a_{i_n}\| \geq \varepsilon$ for $n = 1, 2, \ldots$. It is established in [3] that finite linear orthogonal combinations of weakly continuous tripotents are dense in $\mathrm{Cont}_w(E)$. Therefore there exists a finite orthogonal family $u^1, u^2, \ldots, u^N$ of tripotents in $\mathrm{Cont}_w(E)$ and constants $\alpha_1, \ldots, \alpha_N$ with $\|a - \sum_{k=1}^N \alpha_k u^k\| < \varepsilon/2$. Using the canonical decompositions $u^k = \oplus_{i\in I} u^k_i \ (k = 1, \ldots, N)$, it follows

$$\max\{|\alpha_k| : u^k_{i_n} \neq 0\} = \|\sum_{k=1}^N \alpha_k u^k_{i_n}\| \geq \varepsilon/2 \qquad (n = 1, 2, \ldots)\,.$$

In particular, for some index k_0, $I_0 := \{i \in I : u^{k_0}_i \neq 0\}$ is infinite. Since $u^{k_0} \in \mathrm{Cont}_w(E)$ and since $\mathrm{Cont}_w(E)$ is an ideal in E, the subtriple

$$U := \oplus_{i\in I_0}\mathbb{C}u^{k_0}_i = \{\oplus_{i\in I_0}\lambda_i u^{k_0}_i : \sup_{i\in I_0}|\lambda_i| < \infty\}$$

is contained in $\mathrm{Cont}_w(E)$. Consequently U is a weakly continuous JB*. However, U is isometrically isomorphic to $\ell^\infty(I_0)$ whence, by 4.1 and the infiniteness os I_0, we cannot have $U = \mathrm{Cont}_w(U)$, a contradiction.

■

Theorem 4.3. *Let E be a* JBW* *with the factor decomposition*

$$E = E_{\rm a} \oplus E_{\rm a}^{\perp} = \oplus_{m \in \mathcal{M}} F_m \oplus E_{\rm a}^{\perp}$$

described at the beginning of Section 3. *Then*

$$\mathrm{Cont}_w(E) = \oplus_{m \in \mathcal{M}}^{co} \mathrm{Cont}_w(F_m) = \oplus_{m \in \mathcal{M}}^{co} \mathrm{Cont}_{w^*}(F_m)$$

Proof. In view of Remark 3.1 and Proposition 4.2 it suffices to see that

$$\mathrm{Cont}_w(E_{\rm a}^{\perp}) = \{0\} .$$

Since the weakly continuous tripotents in $\mathrm{Cont}_w(E_{\rm a}^{\perp})$ span a dense linear submanifold (see [3]), it suffices to prove only that non-trivial tripotents in $\mathrm{Cont}_w(E_{\rm a}^{\perp})$ are not weakly continuous.

Let $u \neq 0$ be any tripotent in $E_{\rm a}^{\perp}$. Since $E_{\rm a}^{\perp}$ is an atom free JBW*, by [4] there exist an orthogonal couple u_1, u_1' of non-zero tripotents in $E_{\rm a}^{\perp}$ such that

$$u = u_1 + u_1' \qquad \{uu^*u_1\} = u_1,\ \{uu^*u_1'\} = u_1' .$$

Similarly we can split u_1' into a non-trivial orthogonal tripotent sum $u_2 + u_2'$ with the property $\{u_1'u_1'^*u_2\} = u_2$, $\{u_1'u_1'^*u_2'\} = u_2'$. Thus, continuing in this manner, we can construct an orthogonal sequence $u_1, u_2, u_3, \ldots$ of non-zero tripotents such that

$$u \diamond u^*(u_k) = u_k \qquad (k = 1, 2, \ldots) .$$

Since $E_{\rm a}^{\perp}$ has a predual, for any bounded sequence of constants $\lambda = (\lambda_1, \lambda_2, \ldots)$ the element

$$x^{\lambda} := \mathrm{w}*\text{–} \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k u_k$$

is well defined (see [4]). The mapping $\lambda \mapsto x^{\lambda}$ is an isometric isomorphism between the space ℓ^{∞} and the subtriple $X := \{x^{\lambda} : \lambda \in \ell^{\infty}\}$. Observe that X is a closed subtriple of the ideal U_u of $E_{\rm a}^{\perp}$ generated by the singleton u. By Lemma 4.1 the JB* X is not weakly continuous. Therefore U_u is no weakly continuous JB* and hence the tripotent u cannot belong to $\mathrm{Cont}_w(E_{\rm a}^{\perp})$. ∎

Corollary 4.4. *If E is a* JBW* *then any weakly continuous element of E is the c_0-direct sum of an orthogonal family of scalar multiples of weakly continuous atoms.*

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