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A Banach–Stone type theorem for lattice norms in C_0-spaces

L.L. Stachó

Dedicated to Prof. S. Csörgő on the occasion of his 60-th birthday

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Abstract. We consider the space $E = E(\Omega, \|.\|)$ as the commutative C*-algebra $\mathcal{C}_0(\Omega)$ equipped with a norm $\|.\|$ having the monotonicity property $\|f\| \geq \|g\|$ if $|f| \geq |g|$. We show there exists a finest partition Π of the underlying space Ω along with a function $m\colon \Omega \to \mathbb{R}_+$ with the following properties: $\sup_{S\in\Pi} \#S < \infty$, $0 < \inf m \leq \sup m < \infty$ and each E-Hermitian operator A can be written in the matrix form $Af(\omega) = \sum_{\eta\in S} a^{(S)}_{\omega\eta} f(\eta)$, $\omega \in S \in \Pi_E$ with some system $[a^{(S)} : S \in \Pi]$ of matrices $a^{(S)} = [a^{(S)}_{\omega\eta}]_{\omega,\eta\in S}$ indexed with the elements of Ω and we have $\{f|_S : \ \|f\| \leq 1\} = \{\varphi \in \mathcal{C}(S) : \sum_{\omega\in S} |\varphi(\omega)|^2 \leq 1\}$ for any partition member $S \in \Pi$. Hence, generalizing the Banach–Stone theorem, we obtain matrix descriptions for surjective isometries $E(\Omega, \|.\|) \to E(\widetilde{\Omega}, \|.\|^\sim)$. We apply this result to show that unlike in the classical case of spectral norms, the linear isometric equivalence of the spaces $E(\Omega, \|.\|)$ and $E(\widetilde{\Omega}, \|.\|^\sim)$ does not imply the existence of a positive surjective linear isometry in general, disproving a conjecture on Sunada type theorems for generalized Reinhardt domains.

1. Introduction

Given two locally compact topological Hausdorff spaces Ω and $\widetilde{\Omega}$, the classical Banach–Stone theorem asserts that any surjective isometry $U\colon \mathcal{C}_0(\widetilde{\Omega}) \to \mathcal{C}_0(\Omega)$ with

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respect to the spectral norms $\|f\|_\infty = \max|f|$ has the form $Uf(\omega) = u(\omega)f(T\omega)$, $f \in \mathcal{C}_0(\Omega)$, $\omega \in \Omega$ with some bijection $T{:}\,\Omega \leftrightarrow \widetilde{\Omega}$ and a function $u{:}\,\Omega \to \mathbb{C}$ with $|u| = 1$. Our aim in this paper is to achieve an analogous description if we replace spectral norms with arbitrary Banach lattice norms with respect to the natural pointwise ordering of the functions. We conclude the following main results.

Theorem 1.1. *Given a complex lattice norm $\|.\|$ on $\mathcal{C}_0(\Omega)$, there is a (unique) finest partition Π of the space Ω into pairwise disjoint finite subsets such that the restrictions of any $\|.\|$-Hermitian operator $A{:}\,\mathcal{C}_0(\Omega) \to \mathcal{C}_0(\Omega)$ have the form*

$$Af|_S = \mathbf{a}^A(S)f|_S, \quad f \in \mathcal{C}_0(\Omega), \quad S \in \Pi \tag{1.2}$$

with a (unique) family of linear maps $\mathbf{a}^A(S){:}\,\mathcal{C}(S) \to \mathcal{C}(S)$. Given any partition member $S \in \Pi$, there exists a (unique) inner product $\langle .|.\rangle_S$ on the finite-dimensional function space $\mathcal{C}(S)$ such that

$$\{f|_S:\ \|f\| \le 1\} = \{\varphi \in \mathcal{C}(S):\ \langle \varphi|\varphi\rangle_S \le 1\}, \quad S \in \Pi. \tag{1.3}$$

Theorem 1.4. *Let $U{:}\,\mathcal{C}_0(\widetilde{\Omega}) \to \mathcal{C}_0(\Omega)$ be a surjective linear isometry with respect to two complex Banach lattice norms $\|.\|^\sim$ and $\|.\|$. Write $\Pi, [\langle .|.\rangle_S : S \in \Pi]$ and $\widetilde{\Pi}, [\langle .|.\rangle^\sim_Z :\ Z \in \widetilde{\Pi}]$ for the respective partitions and families of inner products associated with these norms by Theorem 1.1. Then there exists a bijection $T{:}\,\Pi \leftrightarrow \widetilde{\Pi}$ along with a family $[\mathbf{u}(S) :\ S \in \Pi]$ of surjective linear $\langle .|.\rangle^\sim_{T(S)} \to \langle .|.\rangle_S$ unitary operators $\mathbf{u}(S){:}\,\mathcal{C}(T(S)) \to \mathcal{C}(S)$ such that the sets S and $T(S)$ have always the same cardinalities and*

$$U\widetilde{f}|_S = \mathbf{u}(S)\widetilde{f}|_{T(S)}, \quad \widetilde{f} \in \mathcal{C}_0(\widetilde{\Omega}), \quad S \in \Pi. \tag{1.5}$$

In the course of the proofs we achieve a more detailed description of the partition Π and the inner products $\langle .|.\rangle_S$ in terms of some geometrical data of the unit ball of the norm $\|.\|$. In several steps we follow a remarkably similar pattern to some arguments appearing also in [1], [2], [5], [6], [8] and [11]. A heuristical reason for this fact is that Theorems 1.1 and 1.4 can be formulated in terms of the atomic part of the dual lattice. However, we have to establish that dual Hermitian operators preserve both atomic and continuous parts which requires new arguments in Section 3. It seems also (see Remark 4.5) that even earlier results on generalized orthogonal systems [5], [8], [11] in atomic lattices cannot reduce our treatment essentially.

Though the issue may have interest for all researchers in Banach space geometry, this paper was originally motivated by problems in infinite-dimensional complex analysis concerning generalized Reinhardt domains. A classical Reinhardt domain is an open connected subset D in the space $\mathbb{C}^n$ of all complex n-tuples, being invariant under all coordinate multiplications $M_{\lambda_1,\dots,\lambda_n}\colon(z_1,\dots,z_n) \mapsto (\lambda_1 z_1,\dots,\lambda_n z_n)$ with $|\lambda_1|,\dots,|\lambda_n| = 1$. The Reinhardt domain $D \subset \mathbb{C}^n$ is said to be complete if $M_{\lambda_1,\dots,\lambda_n} D \subset D$ whenever $|\lambda_1|,\dots,|\lambda_n| \le 1$. Regarding $\mathbb{C}^n$ as the complex ordered space of the functions $z\colon\{1,\dots,n\} \to \mathbb{C}$, these properties can be stated as

(1.6) $f \in D$ and $|g| = |f| \;\Rightarrow\; g \in D$; (1.7) $f \in D$ and $|g| \le |f| \;\Rightarrow\; g \in D$.

Observe that we can define Reinhardt domains (resp. complete Reinhardt domains) in any complex topological vector lattice by requiring D to be an open connected set satisfying (1.6) (resp. (1.7)) in terms of the order absolute value. In particular a bounded convex complete Reinhardt domain in a normed complex vector lattice is the unit ball of some equivalent lattice norm (a norm with $|f| \le |g| \Rightarrow \|f\| \le \|g\|$).

In 1974 Sunada [13], [14] investigated the structure of bounded classical Reinhardt domains from the viewpoint of holomorphic equivalence. He established that holomorphically equivalent bounded Reinhardt domains containing the origin in $\mathbb{C}^n$ admit linear equivalences which preserve the positive cone $\mathbb{R}^n_+$. In the light of later developments, holomorphic equivalence is nothing more than linear equivalence in the category of bounded convex Reinhardt domains in Banach lattices. Indeed, since 1976 we know [9], [3], [15] that holomorphically equivalent bounded circular domains in Banach spaces are linearly isomorphic. Sunada's Lie algebraic methods were peculiar to finite dimensions. Motivated by this fact, several concepts of infinite-dimensional Reinhardt domains appeared soon. The results ranged in contexts of various sequence spaces, separable Banach spaces with unconditional basis and atomic Banach lattices [5], [8], [11], [16], [2], [1] with the common features that they entailed positive linear equivalence from holomorphic equivalence as a consequence of a direct decomposability of the underlying space to so-called Hilbert components. In 2003, inspired by an interesting work of Vigué [17] on the possible lack of symmetry of continuous products of discs with different radius, in [12] we introduced the concept of continuous Reinhardt domains (CRD for short). By definition, CRDs are Reinhardt domains in the sense (1.6) with the natural ordering in a space of the type $\mathcal{C}_0(\Omega)$ or which is the same, in a commutative C*-algebra. It seems, so far only symmetric complete CRDs were intensively investigated. In [12] we achieved a rather precise description for them by showing they are some topological mixture of finite-dimensional Euclidean balls, essentially more involved than direct sums of topological products of balls. Namely a symmetric complete CRD in $\mathcal{C}_0(\Omega)$ with locally compact Ω is the unit ball of a norm $\|f\| = \sup_{S\in\Pi} \sum_{\omega\in S} m(\omega)|f(\omega)|^2$ with some partition Π of Ω and a weight function $m:\Omega \to \mathbb{R}_+$ such that $\sup_{S\in\Pi} \#S < \infty$ and $0 < \inf m, \sup m < \infty$. Later on [7] matrix representations were found for the linear isomorphisms between two symmetric CRDs. To prove these results, we intensively used the Jordan theory of the bidual embedding of symmetric domains. However, the main points of both Sunada's and Vigué's papers concern the non-symmetric case which we settle in Theorems 1.1 and 1.4 with completely different tools. We finish the paper by showing that the matrix form for linear isomorphisms of CRDs given by Theorem 1.4 leads to a disproof of the seemingly plausible continuous analog of Sunada's theorem.

Theorem 1.8. *There are linearly isomorphic bounded CRDs without admitting a linear isomorphism which maps real valued functions to real valued functions.*

The construction with Möbius twist, a continuation of [7, Example 2.10], sheds light to possible connections between combinatorial topology and CRDs.

2. Notations, preliminaries

Throughout this work, let Ω be an arbitrarily fixed locally compact topological Hausdorff space. As usually, $\mathbb{R}$ and $\mathbb{C}$ are the fields of all real resp. complex numbers and $\mathcal{C}_0(\Omega), \mathcal{C}_b(\Omega)$ and $\mathcal{B}(\Omega)$ will denote the complex Banach spaces of all continuous functions vanishing at infinity resp. all bounded continuous resp. all bounded Borel-measurable functions $\Omega \to \mathbb{C}$ equipped with the spectral norm $\|f\|_\infty := \sup |f|$. We keep fixed the notations $\|.\|$ and D for another complex Banach lattice norm on $\mathcal{C}_0(\Omega)$ and its open unit ball

$$D := \{f \in \mathcal{C}_0(\Omega) : \ \|f\| < 1\},$$

respectively. According to [10, Cor. 4 of Thm. 5.3] the norm $\|.\|$ is necessarily equivalent to $\|.\|_\infty$. Therefore D is a bounded open convex set in $\mathcal{C}_0(\Omega)$ with the CRD-property (1.7). Conversely, given any convex complete CRD in $\mathcal{C}_0(\Omega)$, its gauge function is a Banach lattice norm on $\mathcal{C}_0(\Omega)$. Moreover, it is worth noticing that the convex hull of an open set in $\mathcal{C}_0(\Omega)$ satisfying only (1.6), is necessarily a complete CRD satisfying (1.7) as well.* Since the norms $\|.\|$ and $\|.\|_\infty$ are equ-

* Proof. Assume $C \subset \mathcal{C}_0(\Omega)$ is an open convex set satisfying (1.6) and let $f, g \in \mathcal{C}_0(\Omega)$ with $f \in C$ and $|g| \le |f|$. For $k = 1, 2$ define $f_k := T_k(f, g)$ where $T_1, T_2 : \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : \ |\zeta_1| \ge |\zeta_2|\} \to \mathbb{C}$ are the continuous transformations $T_k(\zeta_1, \zeta_2) := \zeta_2 + (-1)^k i[\zeta_2/|\zeta_1|]\sqrt{|\zeta_1|^2 - |\zeta_2|^2}$ for $(\zeta_1, \zeta_2) \ne (0, 0)$ and $T_k(0, 0) := 0$. Then we have $f_1, f_2 \in \mathcal{C}_0(\Omega)$ and $|f_1| = |f_2| = |f|$. Property (1.6) of C implies $f_1, f_2 \in C$. On the other hand $g = \frac{1}{2} f_1 + \frac{1}{2} f_2 \in \frac{1}{2} C + \frac{1}{2} C = C$.

ivalent, their continuous linear functionals coincide and the common dual space admits the Riesz-Kakutani representation

$$\mathcal{C}_0(\Omega)' = \mathbf{d}\mathcal{M}(\Omega) = \{\mathbf{d}\mu : \ \mu \in \mathcal{M}(\Omega)\}, \quad \mathbf{d}\mu : \mathcal{C}_0(\Omega) \ni f \mapsto \textstyle\int f \ d\mu$$

where $\mathcal{M}(\Omega)$ denotes the family of all complex Radon measures with bounded total variation on Ω. For any Borel set $S \subset \Omega$ we let $\mathcal{M}(S) := \{\mu \in \mathcal{M}(\Omega) : \mu(U) = 0 \text{ for } U \text{ Borel} \subset \Omega \setminus S\}$. In the sequel we shall use the notations

$$\|\mathbf{d}\mu\|_* := \sup_{f\in D} |\textstyle\int f \ d\mu|, \quad \mu \in \mathcal{M}(\Omega) \quad \text{and} \quad D_* := \{\mathbf{d}\mu : \ \mu \in \mathcal{M}(\Omega), \ \|\mathbf{d}\mu\|_* < 1\}$$

for the dual norm of $\|.\|$ and its open unit ball, respectively. To simplify formulas, in later calculations we shall write $\int fLd\mu$ instead of the operator form $[L\mathbf{d}\mu]f$ whenever L is any self-mapping of $\mathbf{d}\mathcal{M}(\Omega)$. Furthermore we write ϕ instead of M_ϕ. With these conventions we have the handsome formal identity $[\phi \ \mathbf{d}\mu]f = \int f\phi \ \mathbf{d}\mu$ for any $f \in \mathcal{C}_0(\Omega)$. Given a bounded linear functional Φ on $\mathcal{C}_0(\Omega)$, we introduce the formal notations $\int_S \Phi$ and $\mathrm{supp}(\Phi)$ for the value $\mu(S)(= \int_S d\mu)$ respectively the support of the unique measure $\mu \in \mathcal{M}(\Omega)$ satisfying $\Phi = \mathbf{d}\mu$. Observe that $\omega \in \mathrm{supp}(\Phi)$ if and only if for every neighborhood U of the point ω there is a function $f \in \mathcal{C}_0(\Omega)$ vanishing outside U and such that $\Phi(f) \neq 0$. In particular $\mathrm{supp}(g\Phi) = \mathrm{supp}(g) \cap \mathrm{supp}(\Phi)$ if $g \in \mathcal{C}_b(\Omega)$ where $\mathrm{supp}(g) := \mathrm{closure}\{\omega : \ g(\omega) \neq 0\}$. We shall denote with δ_ω the measure with unit mass supported on $\{\omega\}$ and 1_S stands for the indicator function of a Borel subset S in Ω (that is $\int f \ d\delta_\omega = f(\omega)$ for $f \in \mathcal{B}(\Omega)$ and $1_S(\eta) = [1 \text{ if } \eta \in S, \ 0 \text{ else}]$). Notice that $1_{\{\omega\}}\Phi = M_{1_{\{\omega\}}}\Phi = (\int_{\{\omega\}} \Phi) \ \mathbf{d}\delta_\omega$ for any $\Phi \in \mathcal{C}_0(\Omega)'$. For later use we remark the following.

Lemma 2.1. *Assume F is a subspace in $\mathcal{C}_0(\Omega)'$ such that $fF \subset F$, $f \in \mathcal{C}_0(\Omega)$. We have $\dim(F) \geq n$ if and only if there are functionals $0 \neq \Phi_1, \ldots, \Phi_n \in F$ with pairwise disjoint support. If $\dim(F) < \infty$ then $F = \mathbf{d}\mathcal{M}(S)$ for some set $S \subset \Omega$ with $\#S = \dim(F)$.*

Proof. Define $S := \bigcup_{\Phi\in F} \mathrm{supp}(\Phi)$ and consider a sequence $\omega_1, \ldots, \omega_m \in S$. There are open sets $U_1, \ldots, U_m \subset \Omega$ with pairwise disjoint closures such that $\omega_k \in U_k$. We can choose $\Psi_1, \ldots, \Psi_m \in F$ and $g_1, \ldots, g_m \in \mathcal{C}_0(\Omega)$ with $\omega_k \in \mathrm{supp}(\Psi_k)$, $g_k(\Omega \setminus U_k) = 0$ and $\int g_k\Psi_k = \Psi_k(g_k) \neq 0$. Then the functionals $\Phi_k := g_k\Psi_k$ have pairwise disjoint supports and hence they are are linearly independent. Thus $\dim(F) \geq \#S \geq \sup\{m : \exists \Phi_1, \ldots, \Phi_m \in F\setminus\{0\} \ \mathrm{supp}(\Phi_k)\cap\mathrm{supp}(\Phi_\ell) = \emptyset \ (k \neq \ell)\}$. Suppose $\#S < \infty$. By choosing m to be maximal with $S = \{\omega_1, \ldots, \omega_m\}$, we see that $F \subset \{\Phi : \ \mathrm{supp}(\Phi) \subset S\} = \mathbf{d}\mathcal{M}(S) = \sum_{k=1}^m \mathbb{C}\mathbf{d}\delta_k$.

■

Finally we recall the concept of Hermitian operators in Banach spaces. Following the terminology established in [1], given a bounded domain G in a Banach space E, we say that a continuous linear map $A{:}\,E \to E$ is G-Hermitian if $\exp(itA)G = G$ for all $t \in \mathbb{R}$. We shall write $\mathrm{Her}(G)$ for the family of all G-Hermitian operators. Though this notation does not contain any explicit hint to the space E and the underlying norm, the domain G itself determines E up to norm equivalence unambiguously.

3. Hermitian operators in the dual space

Lemma 3.1. *Let $L \in \mathrm{Her}(D_*)$, $\omega \in \Omega$ and $f{:}\,\Omega \to \mathbb{R}$ be a bounded Borel function such that $f(\omega) = 0$. Then the operator $\widetilde{L} := fL1_{\{\omega\}} + 1_{\{\omega\}}Lf$ is D_*-Hermitian. We have $\widetilde{L} = 0$ if $fL\mathbf{d}\delta_\omega = 0$. Otherwise the constant $\lambda := \int_{\{\omega\}} Lf^2 L\mathbf{d}\delta_\omega$ is strictly positive and*

$$\exp(it\widetilde{L})\ \mathbf{d}\delta_\omega = \cos(\lambda^{1/2}t)\mathbf{d}\delta_\omega + i\lambda^{-1/2}\sin(\lambda^{1/2}t)fL\mathbf{d}\delta_\omega, \quad t \in \mathbb{R}.$$

Proof. The multiplication operators with the bounded real valued Borel functions f and $1_{\{\omega\}}$ are D_*-Hermitian. The commutator of two D_*-Hermitian operators is i-times a D_*-Hermitian operator and hence $\widetilde{L} = -[f,[1_{\{\omega\}},L]] \in \mathrm{Her}(D_*)$. By writing $\widetilde{\delta}$ for the unique measure with $\mathbf{d}\widetilde{\delta} = fL\mathbf{d}\delta_\omega$, direct calculation yields that $\widetilde{L}\mathbf{d}\delta_\omega = \mathbf{d}\widetilde{\delta}$ and $\widetilde{L}\mathbf{d}\widetilde{\delta} = \lambda\mathbf{d}\delta_\omega$. In particular the two-dimensional subspace spanned by $\{\mathbf{d}\delta_\omega, \mathbf{d}\widetilde{\delta}\}$ is $\widetilde{L}$-invariant and for all $t \in \mathbb{R}$ we have

$$\exp(it\widetilde{L})\mathbf{d}\delta_\omega = \begin{cases} \mathbf{d}\delta_\omega + it\mathbf{d}\widetilde{\delta} & \text{if } \lambda = 0, \\ \cos(\lambda^{1/2}t)\mathbf{d}\delta_\omega + i\lambda^{-1/2}\sin(\lambda^{1/2}t)\mathbf{d}\widetilde{\delta} & \text{if } \lambda \neq 0. \end{cases}$$

Since $\widetilde{L} \in \mathrm{Her}(D_*)$, the orbit $\{\exp(it\widetilde{L})\mathbf{d}\delta :\ t \in \mathbb{R}\}$ must be bounded. This is possible only if $\lambda = 0$ and $fL\mathbf{d}\delta_\omega = \mathbf{d}\widetilde{\delta} = 0$ or if $\lambda > 0$.

■

Corollary 3.2. *Given a D_*-Hermitian operator L and a point $\omega \in \Omega$, we have*

$$\|gL\mathbf{d}\delta_\omega\|_*^2 = \langle g|g\rangle^{(L,\omega)}, \qquad g \in \mathcal{B}(\Omega)$$

in terms of the sesquilinear form $\langle g|h\rangle^{(L,\omega)} := \int_{\{\omega\}} Lg\overline{h}L\mathbf{d}\delta_\omega\ \|\mathbf{d}\delta_\omega\|_*^2$ on $\mathcal{B}(\Omega)$.

Proof. For short, write $\langle .|. \rangle$ instead of $\langle .|. \rangle^{(L,\omega)}$. Consider any function $g \in \mathcal{B}(\Omega)$ and define $f := 1_{\Omega\setminus\{\omega\}} g$. Thus $g = g(\omega)1_{\{\omega\}} + f$ and

$$gL\mathbf{d}\delta_\omega = \alpha\mathbf{d}\delta_\omega + fL\mathbf{d}\delta_\omega \quad \text{where } \alpha := g(\omega) \textstyle\int_{\{\omega\}} L\mathbf{d}\delta_\omega. \tag{3.3}$$

According to [12], multiplication operators with Borel functions of module 1 are $\|.\|_*$-isometries. In particular $\|gL\mathbf{d}\delta_\omega\|_* = \||g|L\mathbf{d}\delta_\omega\|_*$. Thus we may assume without loss of generality $g = |g| \geq 0$ and $f \geq 0$. Furthermore $\alpha \in \mathbb{R}$ in (3.3) for the following reason. The functional $0 \neq \Delta_\omega \colon \mathbf{d}\mu \mapsto \int_{\{\omega\}} \mathbf{d}\mu = \mu\{\omega\}$ supports the unit ball D_* of the norm $\|.\|_*$ at the point $\mathbf{d}\delta_\omega^0 := \|\mathbf{d}\delta_\omega\|_*^{-1}\mathbf{d}\delta_\omega$ (that is $\|\mathbf{d}\delta_\omega^0\|_* = 1$ and $\Delta_\omega(\mathbf{d}\delta_\omega^0) = \sup_{\|\mathbf{d}\mu\|_*\leq 1} |\Delta_\omega(\mathbf{d}\mu)| = \|\mathbf{d}\delta_\omega\|_*^{-1}$). Since $L \in \mathrm{Her}(D_*)$, the numerical range characterization [4] of Hermitian operators establishes $\int_{\{\omega\}} L\mathbf{d}\delta_\omega = \Delta_\omega(L\mathbf{d}\delta_\omega) \in \mathbb{R}$. Then we can apply Lemma 3.1 with $\lambda := \langle f|f\rangle \|\mathbf{d}_\omega\|_*^{-2} = \int_{\{\omega\}} Lf^2 L\mathbf{d}\delta_\omega$. We have the only alternatives $\lambda > 0$ or $\lambda = 0$.

If $\lambda > 0$ then, with the choice $t := \frac{1}{\sqrt{\lambda}} \arccos \frac{\alpha}{\sqrt{\lambda+\alpha^2}}$, Lemma 3.1 implies $\exp(it\widetilde{L})\mathbf{d}\delta_\omega = \alpha[\lambda+\alpha^2]^{-1/2}\mathbf{d}\delta_\omega + i\lambda^{-1/2}\lambda^{1/2}[\lambda+\alpha^2]^{-1/2} fL\mathbf{d}\delta_\omega$. It follows

$$\|gL\mathbf{d}\delta_\omega\|_*^2 = \|\alpha\mathbf{d}\delta_\omega + fL\mathbf{d}\delta_\omega\|_*^2 = \|\alpha\mathbf{d}\delta_\omega + ifL\mathbf{d}\delta_\omega\|_*^2 = [\lambda+\alpha^2]\|\mathbf{d}\delta_\omega\|_*^2.$$

If $\lambda = 0$ then $fL\mathbf{d}\delta_\omega = 0$ and $gL\mathbf{d}\delta_\omega = \alpha\mathbf{d}\delta_\omega$. Thus in both cases we have

$$\|gL\mathbf{d}\delta_\omega\|_*^2 = [\lambda+\alpha^2]\|\mathbf{d}\delta_\omega\|_*^2 = \langle f|f\rangle + g(\omega)^2 [\textstyle\int_{\{\omega\}} fL\mathbf{d}\delta_\omega]^2 \|\mathbf{d}\delta_\omega\|_*^2.$$

Since $g = g(\omega)1_{\{\omega\}} + f$ and $1_{\{\omega\}}f = 0$, $\langle g|g\rangle = g(\omega)^2\langle 1_{\{\omega\}}|1_{\{\omega\}}\rangle + \langle f|f\rangle$. Hence we complete the proof with the observation

$$\begin{aligned}\langle 1_{\{\omega\}}|1_{\{\omega\}}\rangle &= \|\mathbf{d}\delta_\omega\|_*^2 \textstyle\int_{\{\omega\}} L1_{\{\omega\}}L\mathbf{d}\delta_\omega = \|\mathbf{d}\delta_\omega\|_*^2 \int_{\{\omega\}} L[\int_{\{\omega\}} L\mathbf{d}\delta_\omega]\mathbf{d}\delta_\omega = \\ &= \|\mathbf{d}\delta_\omega\|_*^2 [\textstyle\int_{\{\omega\}} L\mathbf{d}\delta_\omega]^2.\end{aligned}$$

■

Recall that the unit ball D of the norm $\|.\|$ is a bounded open neighborhood of the origin with respect to the natural maximum-norm $\|.\|_\infty$ in $\mathcal{C}_0(\Omega)$. Therefore we can fix a natural number N_D such that

$$N_D^{-1}\|.\|_\infty \leq \|.\| \leq N_D\|.\|_\infty. \tag{3.4}$$

Lemma 3.5. *Given a D_*-Hermitian operator L and a point $\omega \in \Omega$, the support of the measure $\mu \in \mathcal{M}(\Omega)$ with $\mathbf{d}\mu = L\mathbf{d}\delta_\omega$ consists of at most N_D^4 points.*

Proof. Define $F := \{fL\mathbf{d}\delta_\omega : \ f \in \mathcal{C}_0(\Omega)\}$ and consider a sequence $\Phi_k := f_k L\mathbf{d}\delta_\omega$, $k = 1, \dots, n$ with functionals of pairwise disjoint non-empty support and such that $\|\Phi_k\|_* = 1$. In view of Lemma 2.1, we have to see only that $n \le N_D^4$. Observe that the family $\{f_1, \dots, f_n\}$ is orthonormed with respect to the sesquilinear form $\langle .|. \rangle := \langle .|. \rangle^{(L,\omega)}$. Indeed, if $k \ne \ell$ we have $\mathrm{supp}(f_k \overline{f_\ell} L\mathbf{d}\delta_\omega) = \mathrm{supp}(f_k \overline{f_\ell}) \cap \mathrm{supp}(L\mathbf{d}\delta_\omega) \subset \bigcap_{j=k,\ell} \mathrm{supp}(f_j) \cap \mathrm{supp}(L\mathbf{d}\delta_\omega) = \bigcap_{j=k,\ell} \mathrm{supp}(\Phi_j) = \emptyset$ entailing $f_k \overline{f_\ell} L\mathbf{d}\delta_\omega = 0$ and $\langle f_k | f_\ell \rangle = \int_{\{\omega\}} L f_k \overline{f_\ell} L\mathbf{d}\delta_\omega = 0$. Also, for any index, $\langle f_k | f_k \rangle = \|\Phi_k\|_*^2 = 1$. Therefore, for the functional $\Phi := \sum_{k=1}^n \Phi_k$ we have $\|\Phi\|_* = \langle \Phi | \Phi \rangle^{1/2} = n^{1/2}$. On the other hand, the disjointness of the sets $\mathrm{supp}(\Phi_k)$ implies that the total variation norms $\|\Phi_k\|_1$ sum up in the sense that $\|\Phi\|_1 = \sum_{k=1}^n \|\Phi_k\|_1$. Furthermore from (3.4) it follows $N_D^{-1} \|.\|_1 \le \|.\|_* \le N_D \|.\|_1$. Hence we get the conclusion $n \le N_D^4$ from the estimates

$$n^{1/2} = \|\Phi\|_* \ge N_D^{-1} \|\Phi\|_1 = N_D^{-1} \textstyle\sum_{k=1}^n \|\Phi_k\|_1 \ge N_D^{-2} \sum_{k=1}^n \|\phi_k\|_* = n N_D^{-2}.$$

■

Recall that Radon measures with finite total variation admit a unique decomposition into *atomic* and *continuous* part. That is $\mathcal{M}(\Omega) = \mathcal{M}_{\mathrm{at}}(\Omega) \oplus \mathcal{M}_{\mathrm{c}}(\Omega)$ where $\mathcal{M}_{\mathrm{at}}(\Omega) := \{\sum_{n=1}^\infty \alpha_n \delta_{\omega_n} : \ \sum_n |\alpha_n| < \infty, \ \omega_1, \omega_2, \ldots \in \Omega\}$ and $\mathcal{M}_{\mathrm{c}}(\Omega) := \{\mu \in \mathcal{M}(\Omega) : \ \ \mu\{\omega\} = 0 \ \ \forall \ \omega \in \Omega\}$. For any $\mu \in \mathcal{M}(\Omega)$, the set $\mathrm{At}(\mu) := \{\omega \in \Omega : \ \ \mu\{\omega\} \ne 0\}$ is countable with $\sum_{\omega \in \mathrm{At}(\mu)} |\mu\{\omega\}| < \infty$ and the measure $\mu_{\mathrm{at}} := \sum_{\omega \in \mathrm{At}(\mu)} \mu\{\omega\} \delta_\omega$ is the unique element $\nu \in \mathcal{M}_{\mathrm{at}}(\Omega)$ with $\mu - \nu \in \mathcal{M}_{\mathrm{c}}(\Omega)$.

Theorem 3.6. *Any D_*-Hermitian operator L preserves the subspaces $\mathbf{d}\mathcal{M}_{\mathrm{at}}(\Omega)$ and $\mathbf{d}\mathcal{M}_{\mathrm{c}}(\Omega)$ of $\mathcal{C}_0(\Omega)'$.*

Proof. Assume $L \in \mathrm{Her}(D_*)$. The relation $L\mathbf{d}\mathcal{M}_{\mathrm{at}}(\Omega) \subset \mathbf{d}\mathcal{M}_{\mathrm{at}}(\Omega)$ is established by Lemma 3.5. Let $\mu \in \mathcal{M}_{\mathrm{c}}(\Omega)$ and suppose indirectly that $L\mathbf{d}\mu \notin \mathbf{d}\mathcal{M}_{\mathrm{c}}(\Omega)$. Then $\int_{\{\omega\}} L\mathbf{d}\mu \ne 0$ for some point $\omega \in \Omega$. By Lemma 3.5 we can write $L\mathbf{d}\delta_\omega = \sum_{k=1}^n \alpha_k \mathbf{d}\delta_{\omega_k}$ with suitable finite systems $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{C}$ and $\{\omega_1, \dots, \omega_n\} \subset \Omega$ such that $\omega_1 = \omega$ and $\alpha_k \ne 0$, $1 \le k \le n$. Define

$$S := \Omega \setminus \{\omega_1, \dots, \omega_n\}, \quad \widetilde{L} := 1_S L 1_{\{\omega\}} + 1_{\{\omega\}} L 1_S L.$$

Then $\widetilde{L} \in \mathrm{Her}(D_*)$. Furthermore $1_{\{\omega\}}\mathbf{d}\mu = 0$, $1_S\mathbf{d}\mu = \mathbf{d}\mu$, $1_S\mathbf{d}\delta_{\omega_k} = 0$ and

$$\widetilde{L}\mathbf{d}\mu = 1_S L(1_{\{\omega\}}\mathbf{d}\mu) + 1_{\{\omega\}} L(1_S\mathbf{d}\mu) = \textstyle\sum_{k=1}^n \alpha_k 1_{\{\omega\}}\mathbf{d}\delta_{\omega_k} = \alpha_1 \mathbf{d}\delta_\omega \ne 0 \ ,$$

$$\widetilde{L}^2\mathbf{d}\mu = \alpha_1 \widetilde{L}\mathbf{d}\delta_\omega = \alpha_1 1_S L\mathbf{d}\delta_\omega = \alpha_1 \textstyle\sum_{k=1}^n \alpha_k 1_S \mathbf{d}\delta_{\omega_k} = 0 \ .$$

Thus $\exp(it\widetilde{L})\mathbf{d}\mu = \mathbf{d}\mu + it\alpha_1\mathbf{d}\delta_\omega$, $t \in \mathbb{R}$ which contradicts the $\|.\|_*$-isometry of the operators $\exp(it\widetilde{L})$, $t \in \mathbb{R}$.

■

Lemma 3.7. *Given any operator $L \in \mathrm{Her}(D_*)$, the matrix*

$$a^{(L)} = [a^{(L)}_{\eta\omega}]_{\eta,\omega\in\Omega}\ ,\quad a^{(L)}_{\eta\omega} := \int_{\{\eta\}} L\mathbf{d}\delta_\omega \tag{3.8}$$

indexed with the points of the space Ω has at most N_D^4 non-zero entries in every column $[a^{(L)}_{\eta\omega}]_{\eta\in\Omega}$ respectively every row $[a^{(L)}_{\eta\omega}]_{\omega\in\Omega}$. Furthermore $a^{(L)}$ is self-adjoint with respect to the inner product

$$\langle\varphi|\psi\rangle := \textstyle\sum_{\omega\in\Omega} m(\omega)\varphi(\omega)\overline{\psi(\omega)}\ ,\quad m(\omega) = \|\mathbf{d}\delta_\omega\|_*^2 \tag{3.9}$$

defined on $\mathcal{F}(\Omega) := \{\text{functions } \Omega\to\mathbb{C} \text{ with finite support}\}$ in the sense that $\langle a^{(L)}\varphi|\psi\rangle = \langle\varphi|a^{(L)}\psi\rangle$ with the identification $a^{(L)}\varphi \equiv \Big[\eta\mapsto \sum_\omega a^{(L)}_{\eta\omega}\varphi(\omega)\Big]$.

Proof. We know already that, given any point $\omega\in\Omega$, $\#\mathrm{supp}(L\mathbf{d}\delta_\omega)\le N_D^4$ and

$$a^{(L)}_{\eta\omega} = 0 \quad \text{for } \eta\notin\mathrm{supp}(L\mathbf{d}\delta_\omega) \quad\text{and}\quad L\mathbf{d}\delta_\omega = \textstyle\sum_{\eta\in\Omega} a^{(L)}_{\eta\omega}\mathbf{d}\delta_\eta.$$

In particular $\#\{\eta:\ a^{(L)}_{\eta\omega}\neq 0\}\le N_D^4$. It follows also $L\sum_{\omega\in\Omega}\varphi(\omega)\mathbf{d}\delta_\omega = \sum_{\eta\in\Omega}[\sum_{\omega\in\Omega}\varphi(\omega)]\mathbf{d}\delta_\eta = \sum_{\eta\in\Omega}[a^{(L)}\varphi](\eta)\mathbf{d}\delta_\eta$, $\varphi\in\mathcal{F}(\Omega)$. Thus an application of Corollary 3.2 with the indicator function $f := 1_{\{\eta\}}$ yields $\|1_{\{\eta\}}L\mathbf{d}\delta_\omega\|_*^2 = \int_{\{\omega\}} L1_{\{\eta\}}L\mathbf{d}\delta_\omega\|\mathbf{d}\delta_\omega\|_*^2$. That is $|a^{(L)}_{\eta\omega}|^2\|\mathbf{d}\delta_\eta\|_*^2 = \int_{\{\omega\}} La^{(L)}_{\eta\omega}\mathbf{d}\delta_\eta\|\mathbf{d}\delta_\omega\|_*^2$ or equivalently $|a^{(L)}_{\eta\omega}|^2 m(\eta) = a^{(L)}_{\eta\omega}a^{(L)}_{\omega\eta}m(\omega)$. With the change $\omega\leftrightarrow\eta$ we get also $|a^{(L)}_{\omega\eta}|^2 m(\omega) = a^{(L)}_{\omega\eta}a^{(L)}_{\eta\omega}m(\eta)$. Therefore

$$\overline{a^{(L)}_{\eta\omega}}m(\eta) = a^{(L)}_{\omega\eta}m(\omega),\qquad \eta,\omega\in\Omega. \tag{3.10}$$

From (3.10) it immediately follows that $\#\{\omega:\ a^{(L)}_{\eta\omega}\neq 0\}\le N_D^4$ and the matrix $a^{(L)}$ is $\langle .|.\rangle$-selfadjoint.

■

Remark 3.11. By the Alaoglu-Bourbaki theorem, the unit ball D_* of the dual norm $\|.\|_*$ is weak*-compact in $\mathcal{C}_0(\Omega)' = \mathbf{d}\mathcal{M}(\Omega)$. We have not applied this property during the considerations in Section 3. Therefore the statements in 3.1–3.10 hold when $\|.\|_*$ denotes any lattice norm on $\mathbf{d}\mathcal{M}(\Omega)$ with respect to the natural ordering and D_* is the unit ball of $\|.\|_*$. By [10, Cor. 4 of Thm. 5.3], the norm $\|.\|_*$ is necessarily equivalent to $\|.\|_1$ even in this more general setting and the constant N_D in (3.4) can be replaced with some N without reference to a ball in the predual.

4. Proof of Theorems 1.1 and 1.4

Definition 4.1. We are now in a position of being able to specify the main objects Π and $[\langle.|.\rangle_S:\ S\in\Pi]$ in Theorems 1.1 and 1.4. Henceforth let

$$\begin{aligned}&\Pi:=\{\Omega_\omega:\ \omega\in\Omega\}\ \text{ where }\ \Omega_\omega:=\{\eta\in\Omega:\ \exists A\in\mathrm{Her}(D)\ \ 1_{\{\eta\}}A^*\mathbf{d}\delta_\omega\neq 0\},\\ &m(\omega):=\|\mathbf{d}\delta_\omega\|_*^2\quad\text{for}\quad \omega\in\Omega,\\ &\langle\varphi|\psi\rangle_S:=\sum_{\omega\in S}m(\omega)\varphi(\omega)\overline{\psi(\omega)}\quad\text{for}\quad S\text{ finite}\subset\Omega\ \text{ and }\ \varphi,\psi\in\mathcal{C}(S).\end{aligned}$$

Lemma 4.2. *Let $Z\subset\Omega$ be a finite set. Then given any complex matrix $[a_{\eta\omega}]_{\eta,\omega\in Z}$ with the symmetry property* (3.10) *and such that $a_{\eta\omega}=0$ whenever $\eta\neq\omega\notin\Omega_\omega$, there exists an operator $A\in\mathrm{Her}(D)$ such that $A^*\mathbf{d}\delta_\omega=\sum_{\eta\in Z}a_{\eta\omega}\mathbf{d}\delta_\eta$, $\omega\in Z$.*

Proof. Let $Z=\{\omega_1,\dots,\omega_N\}$ and $W:=\{(\omega_k,\omega_\ell):\ k<\ell,\ \omega_k\in\Omega_{\omega_\ell}\}$. For any couple $(\eta,\omega)\in W$ choose an operator $B_{\eta\omega}\in\mathrm{Her}(D)$ such that $\beta_{\eta\omega}:=\int_{\{\eta\}}B^*_{\eta\omega}\mathbf{d}\delta_\omega\neq 0$. Define $\widehat{Z}:=Z\cup\bigcup_{(\eta,\omega)\in W}\bigcup_{\zeta\in Z}\mathrm{supp}B^*_{\eta\omega}\mathbf{d}\delta_\zeta$. The set $\widehat{Z}$ is finite by Lemma 3.5. Therefore, for any point $\zeta\in Z$ we can fix a bounded continuous function $f_\zeta:\Omega\to\mathbb{R}$ such that $f_\zeta(\zeta)=1$ and $f(\widehat{Z}\setminus\{\zeta\})=0$. Observe that the operator C_ζ of the multiplication with f_ζ on $\mathcal{C}_0(\Omega)$ belongs to $\mathrm{Her}(D)$ its adjoint C^*_ζ is the multiplication with f_ζ on $\mathcal{C}_0(\Omega)'$ belonging to $\mathrm{Her}(D_*)$. It is also well-known that $i\mathrm{Her}(D)$ with the usual commutator product is a Lie subalgebra in $\mathcal{L}(\mathcal{C}_0(\Omega))$. Consider the operators

$$A_{\eta\omega}:=-[C_\eta,[C_\omega,B_{\eta\omega}]],\quad \widetilde{A}_{\eta\omega}:=i[C_\eta,A_{\eta\omega}]\quad\text{for}\quad(\eta,\omega)\in W.$$

Furthermore write $A_{\omega\omega}:=C_\omega$, $\omega\in Z$. Then all of them are D-Hermitian and direct calculation shows the following relations: $A^*_{\omega\omega}\mathbf{d}\delta_\omega=\mathbf{d}\delta_\omega$, $A^*_{\omega\omega}\mathbf{d}\delta_\zeta=0$ if $\zeta\in Z\setminus\{\omega\}$ and, for all $(\eta,\omega)\in W$, $A^*_{\eta\omega}\mathbf{d}\delta_\zeta=0$ if $\zeta\in Z\setminus\{\eta,\omega\}$ and $A^*_{\eta\omega}\mathbf{d}\delta_\omega=\beta_{\eta\omega}\mathbf{d}\delta_\eta$. By Lemma 3.7, the matrix of the operator $A^*_{\eta\omega}$ satisfies (3.10). Therefore $A^*_{\eta\omega}\mathbf{d}\delta_\eta=m(\eta)m(\omega)^{-1}\overline{\beta}_{\eta\omega}\mathbf{d}\delta_\omega\neq 0$. Hence we get $\widetilde{A}^*_{\eta\omega}\mathbf{d}\delta_\zeta=0$ for $\zeta\in Z\setminus\{\eta,\omega\}$, $\widetilde{A}^*_{\eta\omega}\mathbf{d}\delta_\omega=i\beta_{\eta\omega}\mathbf{d}\delta_\eta$ and $A^*_{\eta\omega}\mathbf{d}\delta_\eta=-im(\eta)m(\omega)^{-1}\overline{\beta}_{\eta\omega}\mathbf{d}\delta_\eta$. We complete the proof with the observation that the real linear combination $A:=\sum_{\omega\in Z}\alpha_\omega A_{\omega\omega}+\sum_{(\eta,\omega)\in W}(\alpha_{\eta\omega}A_{\eta\omega}+\widetilde{\alpha}_{\eta\omega}\widetilde{A}_{\eta\omega})$ satisfying the relations $\alpha_\omega=a_{\omega\omega}$ and $(\alpha_{\eta\omega}+i\widetilde{\alpha}_{\eta\omega})\beta_{\eta\omega}=a_{\eta\omega}$, suits the requirements of the lemma. ∎

Proposition 4.3. *The family* Π *is a partition of* Ω *into sets of* $\leq N_D^4$ *elements with a constant* N_D *satisfying* (3.4). *The subspaces* $\mathbf{d}\mathcal{M}(S) := \sum_{\eta\in S}\mathbb{C}\mathbf{d}\delta_\eta$, $S\in\Pi$ *of* $\mathcal{C}_0(\Omega)'$ *are the minimal finite-dimensional subspaces of* $\mathcal{C}_0(\Omega)'$ *being invariant under the operators in* $\mathrm{Her}^*(D) := \{A^* :\ A\in\mathrm{Her}(D)\}$ *and we have*

$$\left\|\sum_{\eta\in S}\varphi(\eta)\mathbf{d}\delta_\eta\right\|_*^2 = \langle\varphi|\varphi\rangle_S\ ,\quad S\in\Pi,\ \varphi\in\mathcal{C}(S). \tag{4.4}$$

Proof. The space $\mathbf{d}\mathcal{M}_{\mathrm{at}}(\Omega)$ is an atomic Banach lattice when equipped with the norm $\|.\|_*$. In [11] we have shown an analogous result on general atomic Banach lattices which can be applied if we replace the sets Ω_ω with the adjacency classes $\widehat{\Omega}_\omega := \{\eta\in\Omega :\ \omega\sim\eta\}$ of the relation $\omega\sim\eta :\Leftrightarrow \exists\ L\in\mathrm{Her}(D_*^{\mathrm{at}})\ \ 1_{\{\eta\}}L\mathbf{d}\delta_\omega\neq 0$ where $D_*^{\mathrm{at}} := D_*\cap\mathbf{d}\mathcal{M}_{\mathrm{at}}(\Omega)$. Hence the family $\widehat{\Pi} := \{\widehat{\Omega}_\omega :\ \omega\in\Omega\}$ is a partition of Ω and each subspace $\mathbf{d}\mathcal{M}_{\mathrm{at}}(\widehat{\Omega}_\omega)$ is a Hilbert space with orthogonal basis $\{\mathbf{d}\delta_\eta :\ \eta\in\widehat{\Omega}_\omega\}$. According to Theorem 3.6, the restriction of each D_*-Hermitian operator to $\mathbf{d}\mathcal{M}_{\mathrm{at}}(\Omega)$ is D_*^{at}-Hermitian. Therefore we have $\Omega_\omega\subset\widehat{\Omega}_\omega$, $\omega\in\Omega$ and (4.4) holds. Thus, taking into account Lemma 2.1, it remains to prove only that each set Ω_ω consists of at most N_D^4 points and the relation $\omega\approx\eta\ \ :\Leftrightarrow \eta\in\Omega_\omega$ ($\Leftrightarrow \exists\ A\in\mathrm{Her}(D)\ \ 1_{\{\eta\}}A^*1_{\{\eta\}}\mathbf{d}\delta_\omega\neq 0$) is an equivalence.

In terms of supports of measures, we have $\Omega_\omega = \bigcup_{A\in\mathrm{Her}(D)}\mathrm{supp}(A^*\mathbf{d}\delta_\omega)$. According to Lemma 3.5, the sets $\mathrm{supp}(A^*\mathbf{d}\delta_\omega)$ consist of at most N_D^4 elements for any operator $A\in\mathrm{Her}(D)$ and each point $\omega\in\Omega$. Since real linear combinations of D-Hermitian operators are D-Hermitian, it is just elementary linear algebra to conclude hence that also $\#\Omega_\omega\leq N_D^4$ and there exists $A_\omega\in\mathrm{Her}(D)$ such that $\Omega_\omega = \mathrm{supp}(A_\omega^*\mathbf{d}\delta_\omega)$.

The symmetry and reflexivity of $\approx$ is immediate from Lemma 4.2. To establish its transitivity, assume $\omega\approx\eta\approx\xi$ for three distinct points in Ω. Applying Lemma 4.2 with the set $Z := \{\omega,\eta,\xi\}$, we can find a couple of operators $A, B\in\mathrm{Her}(D)$ such that $A^*\colon\mathbf{d}\delta_\omega\mapsto\mathbf{d}\delta_\eta\mapsto m(\eta)m(\omega)^{-1}\mathbf{d}\delta_\omega,\ \mathbf{d}\delta_\xi\mapsto 0$ and $B^*\colon\mathbf{d}\delta_\eta\mapsto\mathbf{d}\delta_\xi\mapsto m(\xi)m(\eta)^{-1}\mathbf{d}\delta_\eta,\ \mathbf{d}\delta_\omega\mapsto 0$. By setting $C := -i[A,B]$, it follows $C\in\mathrm{Her}(D)$ and $C^*\mathbf{d}\delta_\omega = i\mathbf{d}\delta_\xi\neq 0$ entailing $\omega\approx\xi$.

■

Remark 4.5. 1) As soon as we know that the set Ω_ω is finite, by the aid of Corollary 3.2 we can establish (4.3) in a self-contained manner as follows. Let $\omega\in\Omega$ and a function $\varphi\in\mathcal{C}(\Omega_\omega)$ be arbitrarily given. An application of Lemma 4.2 with $Z := \Omega_\omega$ yields the existence of an operator $A\in\mathrm{Her}(D)$ such that $A^*\mathbf{d}\delta_\omega = \sum_{\eta\in\Omega_\omega}\mathbf{d}\delta_\eta$ and $A\mathbf{d}\delta_\eta = m(\eta)m(\omega)^{-1}\mathbf{d}\delta_\omega$ for $\omega\neq\eta\in\Omega_\omega$. Thus, with the function $f := \sum_{\eta\in\Omega_\omega}\varphi(\eta)1_{\{\eta\}}\in\mathcal{B}(\Omega)$ we have $\sum_{\eta\in\Omega_\omega}\varphi(\eta)\mathbf{d}\delta_\eta = fA^*\mathbf{d}\delta_\omega$. By Corollary 3.2, we get

$$\|\sum_{\eta\in\Omega_\omega}\varphi(\eta)\mathbf{d}\delta_\eta\|_*^2 = \langle f|f\rangle^{(A^*,\omega)} = \sum_{\eta\in\Omega_\omega}|\varphi(\eta)|^2\langle 1_{\{\eta\}}|1_{\{\eta\}}\rangle^{(A^*,\omega)}.$$

By definition, $\langle 1_{\{\eta\}}|1_{\{\eta\}}\rangle^{(A^*,\omega)} = \int_{\{\omega\}} A^* 1_{\{\eta\}} A^* \mathbf{d}\delta\omega \|\mathbf{d}\delta_\omega\|_*^2$. Since $m(\omega) = \|\mathbf{d}\delta_\omega\|_*^2$ and $1_{\{\eta\}} A^* \mathbf{d}\delta_\omega = \mathbf{d}\delta_\eta$, on the right-hand side above we can write

$$\langle 1_{\{\eta\}}|1_{\{\eta\}}\rangle^{(A^*,\omega)} = \int_{\{\omega\}} A^* \mathbf{d}\delta_\eta m(\omega) = a_{\omega\eta}^{(A^*)} m(\omega) = m(\eta) m(\omega)^{-1} m(\omega) = m(\eta).$$

2) The partition Π may be strictly finer than the partition $\widehat{\Pi} = \{\widehat{\Omega}_\omega : \ \omega \in \Omega\}$ borrowed from [11] in the proof of Proposition 4.2. We mention the following example without the straightforward but tedious proof. Let D be the unit ball of the norm $\|f\| := \sup\{|f(\omega)|, (|f(0)|^2 + |f(1)|^2)^{1/2} : \ 0 < \omega < 1\}$ on $\mathcal{C}[0,1]$. Then $\|\mathbf{d}\mu\|_* = (|\mu\{0\}|^2 + |\mu\{1\}|^2)^{1/2} + \|1_{(0,1)} \mathbf{d}\mu\|_1$, for any measure $\mu \in \mathcal{M}[0,1]$. Thus $D_*^{\mathrm{at}} = \left\{ \sum_{n=0}^\infty \alpha_n \mathbf{d}\delta_{\omega_n} : (|\alpha_0|^2 + |\alpha_1|^2)^{1/2} + \sum_{n=2}^\infty |\alpha_n| < 1, \ 0 < \omega_2, \omega_3, \ldots < 1 \right\}$ where $\omega_0 := 0$ and $\omega_1 := 1$. For the corresponding partitions of $\Omega := [0,1]$ we have $\Omega_\omega = \{\omega\}$, $\omega \in [0,1]$ while $\widehat{\Omega}_0 = \widehat{\Omega}_1 = \{0,1\}$ and $\widehat{\Omega}_\omega = \{\omega\}$ only if $0 < \omega < 1$.

4.6. End of the proof of Theorem 1.1. Observe that, given a finite subset S in Ω and an operator $A \in \mathrm{Her}(D)$, we have $Af|_S = \mathbf{a}f|_S$, $f \in \mathcal{C}_0(\Omega)$ with a suitable linear mapping $\mathbf{a}: \mathcal{C}(S) \to \mathcal{C}(S)$ if and only if $S = \bigcup_{\omega\in S} \Omega_\omega$ and $\mathbf{a}1_{\{\omega\}} = \sum_{\eta} a_{\eta\omega}^{(A^*)} 1_{\{\eta\}}$, $\omega \in S$. This fact is an immediate consequence of the relations $Af(\omega) = [A^* \mathbf{d}\delta_\omega] f = \left[\sum_{\eta\in\Omega_\omega} a_{\eta\omega}^{(A^*)} \mathbf{d}\delta_\eta \right] f$ and $[\mathbf{a}f|_S](\omega) = \sum_{\eta\in S} f(\eta) \alpha_{\eta\omega} = \left[\sum_{\eta\in S} \alpha_{\eta\omega} \mathbf{d}\delta_\eta \right] f$ where $\alpha_{\eta\omega} := [\mathbf{a}1_{\{\eta\}}](\omega)$. Therefore, taking into account Proposition 4.3, the partition Π has property (1.2), moreover Π is the only finest partition satisfying (1.2). The fact that (1.3) holds as well, follows directly from (4.4). ■

4.7 End of the proof of Theorem 1.4. Let us write $\widetilde{D}$ and D for the unit balls of the norms $\|.\|^\sim$ and $\|.\|$, respectively. Observe that, in terms of the Lie adjoint $U^\# X := U^{-1} X U$, $X \in \mathcal{L}(\mathcal{C}_0(\Omega))$ of the surjective $\|.\|^\sim \to \|.\|$ isometry U, we have $A \in \mathrm{Her}(D)$ if and only if $U^\# A \in \mathrm{Her}(\widetilde{D})$. Therefore the operation $[U^*]^\#: Y \mapsto [U^*]^{-1} Y U^*$ establishes a one-to-one correspondence $\mathrm{Her}^*(\widetilde{D}) \leftrightarrow \mathrm{Her}^*(D)$ and the $\mathrm{Her}^*(\widetilde{D})$-invariant subspaces of $\mathbf{d}\mathcal{M}(\widetilde{\Omega}) = \mathcal{C}_0(\widetilde{\Omega})'$ are exactly the U^*-images of the $\mathrm{Her}^*(D)$-invariant subspaces of $\mathbf{d}\mathcal{M}(\Omega) = \mathcal{C}_0(\Omega)'$. In particular, since $\dim(F) = \dim(U^* F)$ for any subspace $F \subset \mathbf{d}\mathcal{M}(\Omega)$, $\widetilde{F}$ is a minimal finite-dimensional $\mathrm{Her}^*(\widetilde{D})$-invariant subspace if and only if $\widetilde{F} = U^* F$ for some minimal finite-dimensional $\mathrm{Her}^*(D)$-invariant subspace. Thus by Proposition 4.3 we have

$$\{U^* \mathbf{d}\mathcal{M}(S) : \ S \in \Pi\} = \{\mathbf{d}\mathcal{M}(\widetilde{S}) : \ \widetilde{S} \in \widetilde{\Pi}\}. \tag{4.8}$$

Then the invertibility of the operator U^* implies the existence of a (unique) bijection $T\colon \Pi \leftrightarrow \widetilde{\Pi}$ such that $U^*\mathbf{d}\mathcal{M}(S) = \mathbf{d}\mathcal{M}(T(S))$, $S \in \Pi$. Consider any partition member $S \in \Pi$ and let $\widetilde{S} := T(S)$. Clearly $\#S = \dim(\mathcal{M}(S)) = \dim(U^*\mathcal{M}(S)) = \#\widetilde{S}$. Given any point $\omega \in S$, $U^*\mathbf{d}\delta_\omega \in \mathbf{d}\mathcal{M}(\widetilde{S})$. Hence, by introducing the $\widetilde{S} \times S$-indexed matrix $u^{(S)} := [u^{(S)}_{\widetilde{\eta}\omega}]_{\widetilde{\eta}\in\widetilde{S},\omega\in S}$ with the entries $u^{(S)}_{\widetilde{\eta}\omega} := \int_{\{\widetilde{\eta}\}} U^*\mathbf{d}\delta_\omega$, we have

$$U\widetilde{f}(\omega) = [U^*\mathbf{d}\delta_\omega]\widetilde{f} = \sum_{\widetilde{\eta}\in\widetilde{S}} u^{(S)}_{\widetilde{\eta}\omega}\widetilde{f}(\widetilde{\eta}), \qquad \widetilde{f} \in \mathcal{C}_0(\widetilde{\Omega}).$$

Thus (1.5) holds with the linear mapping $\mathbf{u}(S)\colon \mathcal{C}(T(S)) \to \mathcal{C}(S)$ defined by $\mathbf{u}(S)\widetilde{\varphi} := \left[S \ni \omega \mapsto \sum_{\widetilde{\eta}\in\widetilde{S}} u^{(S)}_{\widetilde{\eta}\omega}\widetilde{\varphi}(\widetilde{\eta})\right]$ for any function $\widetilde{\varphi}\colon T(S) = \widetilde{S} \to \mathbb{C}$. Since U^* is a $\|.\|_* \to \|.\|^{\sim}_*$ isometry, from (4.4) applied to both of these norms, we see that the mapping $\mathbf{u}(S)$ is a $\langle .|.\rangle^{\sim}_{T(S)} \to \langle .|.\rangle_S$ isometry with respect to the inner products $\langle\widetilde{\varphi}|\widetilde{\psi}\rangle^{\sim}_{T(S)} := \sum_{\widetilde{\omega}\in T(S)}[\|\mathbf{d}\delta_{\widetilde{\omega}}\|^{\sim}_*]^2\widetilde{\varphi}\overline{\widetilde{\psi}}$ and $\langle\varphi|\psi\rangle_S := \sum_{\omega\in S}\|\mathbf{d}\delta_\omega\|^2_*\varphi(\omega)\overline{\psi(\omega)}$, respectively.

■

5. A counterexample to continuous Sunada type theorems

Let Ω and $\widetilde{\Omega}$ be the following compact topological subspaces of $\mathbb{C}^2$:

$$\Omega := \bigcup_{k=1,2}\{\omega_{k,t} :\ 0 \le t < 2\pi\} \ \text{ where } \ \omega_{1,t} := (1, e^{it}),\ \omega_{2,t} := (-1, e^{it})$$
$$\widetilde{\Omega} := \bigcup_{k=1,2}\{\widetilde{\omega}_{k,t} :\ 0 \le t < 2\pi\} \ \text{ where } \ \widetilde{\omega}_{1,t} := (e^{it/2}, e^{it}),\ \widetilde{\omega}_{2,t} := (-e^{it/2}, e^{it}).$$

Consider the following (symmetric) continuous Reinhardt domains

$$D := \{f \in \mathcal{C}(\Omega) :\ |f(\omega_{1,t})|^2 + |f(\omega_{2,t})|^2 < 1,\ 0 \le t < 2\pi\},$$
$$\widetilde{D} := \{\widetilde{f} \in \mathcal{C}(\widetilde{\Omega}) :\ |\widetilde{f}(\widetilde{\omega}_{1,t})|^2 + |\widetilde{f}(\widetilde{\omega}_{2,t})|^2 < 1,\ 0 \le t < 2\pi\}.$$

Notice that $\Omega = \{\pm 1\} \times \mathbb{T}$ is the union of two disjoint circles and can be regarded as the border of a cylindric band with middle circle $\{0\} \times \mathbb{T}$, while $\widetilde{\Omega} = \{(e^{it/2}, e^{it}) : t \in \mathbb{R}\}$ is topologically equivalent to a circle and can be regarded as the border of a Möbius band with the same middle circle $\{0\} \times \mathbb{T}$. Conveniently, we can identify the spaces $\mathcal{C}(\Omega)$ and $\mathcal{C}(\widetilde{\Omega})$ with simple subspaces of couples of continuous functions on the compact interval $[0, 2\pi]$. Namely, given $\varphi_1, \varphi_2 \in \mathcal{C}[0, 2\pi]$, with the functions

$$f_{\varphi_1,\varphi_2}(\omega_{k,t}) := \varphi_k(t), \quad \widetilde{f}_{\varphi_1,\varphi_2}(\widetilde{\omega}_{k,t}) := \varphi_k(t), \quad 0 \le t < 2\pi,$$

we have $\mathcal{C}(\Omega) = \{f_{\varphi_1,\varphi_2} :\ \varphi_1, \varphi_2 \in \mathcal{F}\}$ and $\mathcal{C}(\widetilde{\Omega}) = \{\widetilde{f}_{\varphi_1,\varphi_2} :\ (\varphi_1, \varphi_2) \in \widetilde{\mathcal{F}}\}$ where

$$\mathcal{F} := \{(\varphi_1,\varphi_2) \in (\mathcal{C}[0,2\pi])^2 :\ \varphi_1(0) = \varphi_1(2\pi),\ \varphi_1(0) = \varphi_1(2\pi)\},$$
$$\widetilde{\mathcal{F}} := \{(\varphi_1,\varphi_2) \in (\mathcal{C}[0,2\pi])^2 :\ \varphi_1(0) = \varphi_2(2\pi),\ \varphi_2(0) = \varphi_1(2\pi)\}.$$

Hence the mapping

$$\widetilde{U}_* f_{\varphi_1,\varphi_2} := \widetilde{f}_{\cos(t/2)\varphi_1(t)+\sin(t/2)\varphi_2(t),e^{it/2}[-\sin(t/2)\varphi_1(t)+\cos(t/2)\varphi_2(t)]}, \quad f \in \mathcal{F},$$

is a linear isomorphism $\mathcal{C}(\Omega) \leftrightarrow \mathcal{C}(\widetilde{\Omega})$ and $U_* D = \widetilde{D}$.

5.1 Proof of Theorem 1.8. There is no linear isomorphism $W_*: \mathcal{C}(\Omega) \leftrightarrow \mathcal{C}(\widetilde{\Omega})$ such that $W_* D = \widetilde{D}$ and $W_* \mathrm{Re}\, \mathcal{C}(\Omega) = \mathrm{Re}\, \mathcal{C}(\widetilde{\Omega})$.

Proof. Given a discrete complex measure $\mu := \sum_{n=1}^\infty (a_n \delta_{\omega_{1,t_n}} + b_n \delta_{\omega_{1,t_n}})$ on Ω, we have $|\int f\ d\mu| < 1$ for all $f \in D$ if and only if $\sum_{n=1}^\infty (|a_n|^2 + |b_n|^2)^{1/2} \leq 1$. Hence, with the notation of the previous section, $\Pi_D = \{\{\omega_{1,t}, \omega_{2,t}\} :\ 0 \leq t < 2\pi\}$ for the partition associated with the domain D. Similarly, $\Pi_{\widetilde{D}} = \{\{\widetilde{\omega}_{1,t}, \widetilde{\omega}_{2,t}\} :\ 0 \leq t < 2\pi\}$. Suppose indirectly that $W_*: \mathcal{C}(\Omega) \leftrightarrow \mathcal{C}(\widetilde{\Omega})$ is a linear isomorphism with $W_* D = \widetilde{D}$ and $W_* \mathrm{Re}\, \mathcal{C}(\Omega) = \mathrm{Re}\, \mathcal{C}(\widetilde{\Omega})$. By Theorem 4.8 (applied with the weight functions $m = 1_\Omega$ and $\widetilde{m} = 1_{\widetilde{\Omega}}$), the operator W_* must have the form

$$W_* f(\widetilde{\omega}_{1,t}) = w_{11}(t) f(T\widetilde{\omega}_{1,t}) + w_{12}(t) f(T\widetilde{\omega}_{2,t}),$$
$$W_* f(\widetilde{\omega}_{2,t}) = w_{21}(t) f(T\widetilde{\omega}_{1,t}) + w_{22}(t) f(T\widetilde{\omega}_{2,t}), \quad 0 \leq t < 2\pi,\ f \in \mathcal{C}(\Omega),$$

where $\begin{pmatrix} w_{11} w_{12} \\ w_{21} w_{22} \end{pmatrix}$ is a unitary matrix for any $t \in [0, 2\pi)$ and $T: \widetilde{\Omega} \leftrightarrow \Omega$ is a mapping with the effect $\{\{T\widetilde{\omega}_{1,t}, T\widetilde{\omega}_{2,t}\} :\ 0 \leq t < 2\pi\} = \{T(\widetilde{S}) :\ \widetilde{S} \in \Pi_{\widetilde{D}}\} = \Pi_D = \{\{\omega_1, \omega_2\} :\ 0 \leq t < 2\pi\}$. We can write without loss of generality $T\widetilde{\omega}_{k,t} = \omega_{k,T_\#(t)}$, $k = 1, 2$, $0 \leq t < 2\pi$ with a suitable permutation $T_\#: [0, 2\pi) \leftrightarrow [0, 2\pi)$. Thus given any couple $(\varphi_1, \varphi_2) \in \mathcal{F}$, we have

$$W_* f_{\varphi_1,\varphi_2} = \widetilde{f}_{\psi_1,\psi_2} \ \text{ for some } (\psi_1, \psi_2) \in \widetilde{\mathcal{F}} \text{ with } \psi_k(t) = \textstyle\sum_{\ell=1}^2 w_{k\ell}(t) \varphi_\ell(T_\#(t))$$

if $k = 1, 2$ and $0 \leq t < 2\pi$. The assumption $W_* \mathrm{Re}\, \mathcal{C}(\Omega) = \mathrm{Re}\, \mathcal{C}(\widetilde{\Omega})$ means that the functions ψ_1, ψ_2 are real valued whenever φ_1, φ_2 are real valued in the above formula. By considering the particular cases $(\varphi_1^{(1)}, \varphi_2^{(1)}) := (1_\Omega, 0)$ and $(\varphi_1^{(2)}, \varphi_2^{(2)}) := (0, 1_\Omega)$ we see that there are continuous functions $\psi_{k\ell}: [0, 2\pi] \to \mathbb{R}$ such that

$$\psi_{k\ell}(t) = w_{k\ell}(t), \quad 0 \leq t < 2\pi,\ k, \ell = 1, 2; \quad (\psi_{11}, \psi_{21}), (\psi_{12}, \psi_{22}) \in \widetilde{\mathcal{F}}.$$

This is impossible for the following reasons. The matrices $(\psi_{k\ell}(t))_{k,\ell=1}^2$, $0 \le t \le 2\pi$ are orthogonal with real entries. Thus necessarily

$$\psi_{21}(t) = \varepsilon(t)\psi_{12}(t), \quad \psi_{22}(t) = -\varepsilon(t)\psi_{11}(t), \quad \psi_{11}(t)^2 + \psi_{12}(t)^2 = 1, \quad 0 \le t \le 2\pi,$$

for some function $\varepsilon\colon [0, 2\pi] \to \{\pm 1\}$. The connectedness of the interval $[0, 2\pi]$ along with the continuity of the functions $\psi_{k\ell}$ entails that actually $\varepsilon(t) \equiv$ const, say $\varepsilon(t) \equiv \varepsilon_0$. However, since $(\psi_{11}, \psi_{21}), (\psi_{12}, \psi_{22}) \in \widetilde{\mathcal{F}}$, we also have the boundary conditions

$$\psi_{11}(0) = \psi_{21}(2\pi),\ \psi_{21}(0) = \psi_{11}(2\pi),\ \psi_{12}(0) = \psi_{22}(2\pi),\ \psi_{22}(0) = \psi_{12}(2\pi).$$

Hence $\psi_{11}(0) = \psi_{21}(2\pi) = \varepsilon_0\psi_{12}(2\pi) = \varepsilon_0\psi_{22}(0)$. On the other hand, $\psi_{11}(0) = -\varepsilon_0\psi_{22}(0)$. Thus necessarily $\psi_{22} = \psi_{22}(0) = 0$. Similarly, $\psi_{21}(0) = \psi_{11}(2\pi) = -\varepsilon_0\psi_{22}(2\pi) = -\varepsilon_0\psi_{12}(0)$ and $\psi_{21}(0) = \varepsilon_0\psi_{12}$ implying $\psi_{21}(0) = \psi_{12}(0) = 0$. These conclusions contradict the fact that $(\psi_{k\ell}(0))_{k,\ell=1}^2 \ne 0$.

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L.L. STACHÓ, SZTE Bolyai Institute, Aradi Vértanúk tere 1, 6720 Szeged, Hungary;
e-mail: stacho@math.u-szeged.hu