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Norms of certain Jordan elementary operators ☆

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ABSTRACT

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{B}(\mathcal{H})$, the Jordan elementary operator $U_{A,B}$ is defined by $U_{A,B}(X) = AXB + BXA$, $\forall X \in \mathcal{B}(\mathcal{H})$. In this short note, we discuss the norm of $U_{A,B}$. We show that if $\dim \mathcal{H} = 2$ and $\|U_{A,B}\| = \|A\|\|B\|$, then either AB^* or B^*A is 0. We give some examples of Jordan elementary operators $U_{A,B}$ such that $\|U_{A,B}\| = \|A\|\|B\|$ but $AB^* \neq 0$ and $B^*A \neq 0$, which answer negatively a question posed by M. Boumazgour in [M. Boumazgour, Norm inequalities for sums of two basic elementary operators, J. Math. Anal. Appl. 342 (2008) 386–393].

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in B(\mathcal{H})$, we define the Jordan elementary operator $U_{A,B}$ on $B(\mathcal{H})$ by

$$U_{A,B}(X) = AXB + BXA \quad \big(\forall X \in \mathcal{B}(\mathcal{H})\big).$$

The lower bound of $\|U_{A,B}\|$ was studied by many authors, see for instance [1,2,4,7]. In [1], it is shown that $\|U_{A,B}\| \geqslant \|A\|\|B\|$. This lower bound is the best known result to date. In [2] and in [8], M. Boumazgour get this lower bound. He proved that if $AB^* = B^*A = 0$, then $\|U_{A,B}\| = \|A\|\|B\|$. Conversely, if $\|U_{A,B}\| = \|A\|\|B\|$, does it follow that $AB^* = B^*A = 0$? This question was posed by the author in [2, Question 4.3(1)]. In this note, we prove that the converse does not hold in general. On the other hand, M. Boumazgour also considered some additional necessary conditions for $\|U_{A,B}\|$ to be $\|A\|\|B\|$ by use of numerical range in [2] (cf. Proposition 2.8). We recall that for $A, B \in \mathcal{B}(\mathcal{H})$, the numerical range $W_B(A^*B)$ of A^*B relative to B is defined to be the set $W_B(A^*B) = \{\lambda \in \mathbb{C}\colon$ there exists $\{x_n\} \subseteq \mathcal{H}$, $\|x_n\| = 1$ such that $\lim_{n\to\infty}\langle A^*Bx_n, x_n\rangle = \lambda$ and $\lim_{n\to\infty}\|Bx_n\| = \|B\|\}$.

It is known that $W_B(A^*B)$ is a closed convex subset of the complex plane $\mathbb{C}$ for each pair $A, B \in \mathcal{B}(\mathcal{H})$. Some exceptional properties are listed in [3]. In [2], M. Boumazgour proved that $0 \in W_B(A^*B) \cup W_A(B^*A)$ if $\|U_{A,B}\| = \|A\|\|B\|$ for some special pairs A, B and he asked whether this holds for any pairs A, B such that $\|U_{A,B}\| = \|A\|\|B\|$ (Question 4.3(2) in [2]). We also consider this problem and give some partial results.

2. Main results

Let $\mathcal{H}$ be a Hilbert space. We denote by $N(\mathcal{H})$ and $B_2(\mathcal{H})$ respectively the algebras of nuclear (trace-class) operators and Hilbert–Schmidt operators on $\mathcal{H}$. The nuclear (respectively Hilbert–Schmidt) norm of a nuclear (respectively Hilbert–

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Schmidt) operator T will be denoted by $\|T\|_N$ (respectively $s_2(T)$). Recall that for a nuclear (respectively Hilbert–Schmidt) operator T, we have $\|T\|_N = \sum_i \sigma_i(T)$ (respectively $s_2(T) = (\sum_i \sigma_i^2(T))^{1/2}$), where $\sigma_i(T)$ denotes the sequence of singular values of T. We refer readers to see [1] for details.

We firstly consider two dimensional Hilbert space case, that is $\mathcal{H} = \mathbb{C}^2$. We identify $\mathcal{B}(\mathcal{H})$ with 2×2 complex matrices M_2. The idea of the following proof comes from [1].

Theorem 1. *Suppose* $\dim \mathcal{H} = 2$. *If* $\|U_{A,B}\| = \|A\|\|B\|$, *then either* $AB^* = 0$ *or* $B^*A = 0$.

Proof. We can assume that $\|A\| = \|B\| = 1$. Note that $\|U_{A,B}\| = \|U_{WAV,WBV}\|$ for any unitary matrices $W, V \in M_2$. It is clear that $WAV(WBV)^* = WAB^*W^*$ and $(WBV)^*WAV = V^*B^*AV$. Hence from the proof of Proposition 3.6 in [1, p. 485], we may chose an orthonormal basis $\{e_1, e_2\}$ of $\mathcal{H}$ such that A has the representation $\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$, where $\mu \in \mathbb{C}$ with $|\mu| \leqslant 1$, and B has the representation $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$, with w, x and z real, non-negative and $x \geqslant |y|$. From Remark 7 in [8], we know that $\|U_{A,B}\| \geqslant s_2(A)s_2(B)$. Since $s_2(A) \geqslant \|A\| = 1$ and $s_2(B) \geqslant \|B\| = 1$, $s_2(A) = s_2(B) = 1$. From $s_2^2(A) = 1 + |\mu|^2 = 1$, we get $\mu = 0$. We similarly have that B is of rank-one. If $w = x = 0$, then we easily have that $B^*A = 0$. Thus we may assume that $y = \lambda w$, $z = \lambda x$ for some constants $\lambda \in \mathbb{C}$. That is, $B = \begin{pmatrix} w & x \\ \lambda w & \lambda x \end{pmatrix}$, where $w \geqslant 0$, $x \geqslant 0$, $\lambda x \geqslant 0$, $x \geqslant |\lambda w|$.

If $\lambda = 0$, then $B = \begin{pmatrix} w & x \\ 0 & 0 \end{pmatrix}$. In this case we have $s_2^2(B) = w^2 + x^2 = 1$. From Lemma 3.2(iii) and Proposition 2.1 in [1], we get

$$\|U_{A,B}\|^2 \geqslant \left\| U_{A,B}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \right\|_N^2 = 4w^2 + x^2 = 3w^2 + 1.$$

It follows that $w = 0$, which implies that $AB^* = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} = 0$.

We now assume that $\lambda \neq 0$. If $x = 0$, then $B = \begin{pmatrix} w & 0 \\ \lambda w & 0 \end{pmatrix}$ and $s_2^2(B) = (1 + |\lambda|^2)w^2 = 1$. From Lemma 3.2(iii) in [1] again, we have $\|U_{A,B}\|^2 \geqslant \|U_{A,B}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\|_N^2 = 4w^2 + |\lambda|^2 w^2 = 3w^2 + 1$, so $w = 0$. This is a contradiction since $\|B\| = 1$. Hence $x > 0$. Note that $w \geqslant 0$, $\lambda x > 0$, $\lambda w \geqslant 0$ and $x \geqslant \lambda w$. It is known that

$$s_2^2(B) = w^2 + x^2 + \lambda^2 w^2 + \lambda^2 x^2 = 1. \tag{1}$$

From Lemma 3.2(iii) in [1], we get $\|U_{A,B}(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\|_N^2 = 4w^2 + (x + \lambda w)^2$. Thus

$$4w^2 + (x + \lambda w)^2 \leqslant 1. \tag{2}$$

By (1) and (2), we obtain

$$w^2 \leqslant \frac{1}{3}\lambda^2 x^2. \tag{3}$$

From the proof of Proposition 3.6 in [1, p. 486], we have

$$\left\| U_{A,B}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \right\|_N^2 \geqslant 1 + (\lambda x + w)(x + \lambda w) - \frac{1}{2}(x - \lambda w)^2.$$

It now follows that $(\lambda x + w)(x + \lambda w) - \frac{1}{2}(x - \lambda w)^2 \leqslant 0$, which implies that

$$0 < \lambda x + w \leqslant \frac{1}{2}\frac{(x - \lambda w)^2}{x + \lambda w}. \tag{4}$$

Similarly, we can get

$$\left\| U_{A,B}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right) \right\|_N^2 \geqslant 1 + (\lambda x - w)(x - \lambda w) - \frac{1}{2}(x + \lambda w)^2$$

and thus

$$\lambda x - w \leqslant \frac{1}{2}\frac{(x + \lambda w)^2}{x - \lambda w}. \tag{5}$$

Multiplying together (4) and (5), we obtain

$$\lambda^2 x^2 - w^2 \leqslant \frac{1}{4}\left(x^2 - \lambda^2 w^2\right). \tag{6}$$

Combined (2) with (6), we get

$$\lambda^2 x^2 \leqslant \frac{1}{4}\left(x^2 - \lambda^2 w^2\right) + \frac{1}{4}\left[1 - (x + \lambda w)^2\right] = \frac{1}{4} - \frac{1}{2}\lambda w x - \frac{1}{2}\lambda^2 w^2 \leqslant \frac{1}{4}. \tag{7}$$

From (2), we know that

$$x + \lambda w \leqslant 1. \tag{8}$$

Since $x \geqslant \lambda w$, it follows from (8) that

$$\lambda w \leqslant \frac{1}{2}. \tag{9}$$

By (3) and (7), we get

$$w^2 + \lambda^2 x^2 \leqslant \frac{4}{3}\lambda^2 x^2 \leqslant \frac{1}{3}. \tag{10}$$

Taking into account (1), we conclude from the last inequality that

$$x^2 + \lambda^2 w^2 \geqslant \frac{2}{3}. \tag{11}$$

By (9) and (11), we get

$$x^2 \geqslant \frac{5}{12}. \tag{12}$$

Combining (7) with (12), we get $\frac{5}{12}\lambda^2 \leqslant \lambda^2 x^2 \leqslant \frac{1}{4}$, so $\lambda^2 \leqslant \frac{3}{5} < 1$.

Since $0 < \lambda < 1$, we know that $w^2 \geqslant \lambda^2 w^2$ and $\lambda x^2 \geqslant \lambda^2 x^2$. By the proof of Proposition 3.6 in [1, p. 488], we have

$$\begin{aligned}\left\| U_{A,B}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right)\right\|_N^2 &\geqslant \frac{1}{2}\bigl[(2w+x)^2 + x^2 + 2x(\lambda w + \lambda x)\bigr] = \frac{1}{2}\bigl(4w^2 + 4wx + 2x^2 + 2\lambda wx + 2\lambda x^2\bigr)\\ &= 2w^2 + 2wx + x^2 + \lambda wx + \lambda x^2 = w^2 + w^2 + x^2 + \lambda x^2 + 2wx + \lambda wx\\ &\geqslant w^2 + \lambda^2 w^2 + x^2 + \lambda^2 x^2 + 2wx + \lambda wx = 1 + 2wx + \lambda wx.\end{aligned}$$

Since $\lambda > 0$, $x > 0$ and $\|U_{A,B}\| = 1$, we get $w = 0$. Hence $AB^* = \left(\begin{smallmatrix} w & \lambda w\\ 0 & 0\end{smallmatrix}\right) = 0$.

We have thus shown that either $AB^* = 0$ or $B^*A = 0$. The proof is complete. □

Corollary 2. *Assume that* $\dim\mathcal{H} = 2$. *If* $\|U_{A,B}\| = \|A\|\|B\|$, *then* $AB^* = B^*A = 0$ *if one of the following conditions is satisfied*:

(1) $B = A^*$,
(2) *both A and B are self-adjoint.*

Proof. This is obvious from Theorem 1. □

However, in general we cannot get both AB^* and B^*A are 0 even for two dimensional Hilbert spaces.

Example 3. Let $A = \left(\begin{smallmatrix} 1 & 0\\ 0 & 0\end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)$, then $\|U_{A,B}\| = \|A\|\|B\|$, but $B^*A \neq 0$.

If we let $B = A^*$, then U_{A,A^*} is a positive linear map on $\mathcal{B}(\mathcal{H})$. By the Russo–Dye theorem (cf. Corollary 2.9 in [5]), we knew that $\|U_{A,A^*}\| = \|AA^* + A^*A\|$. By Corollary 2, we know that for the positive Jordan elementary operator U_{A,A^*}, the condition that $\|U_{A,A^*}\| = \|A\|\|A^*\|$ does imply that $AB^* = B^*A = A^2 = 0$ if $\dim\mathcal{H} = 2$. However if $\dim\mathcal{H} \geqslant 3$, this does not hold in general.

Example 4. Let $\dim\mathcal{H} = 3$ and $A = \begin{pmatrix} 0 & 0 & 1\\ 0 & \alpha & 0\\ 0 & 0 & 0\end{pmatrix} \in M_3$, where $0 < \alpha \leqslant \frac{1}{\sqrt{2}}$. Then $\|A\| = 1$ and $\|AA^* + A^*A\| = 1$, but $A^2 \neq 0$.

We next consider Question 4.3(2) of [2]. We first note that the answer is positive if $\dim\mathcal{H} = 2$ by Theorem 1.

Corollary 5. *Suppose* $\dim\mathcal{H} = 2$. *Then either* $W_B(A^*B)$ *or* $W_A(B^*A)$ *is* $\{0\}$ *if* $\|U_{A,B}\| = \|A\|\|B\|$.

To show Proposition 7, we need the following lemma proved in [6].

Lemma 6. *(See Theorem 5 in [6].) If* $A, B \in B(\mathcal{H})$ *are not zero, then we have*

$$\|U_{A,B}\| \geqslant \sup\left\{\left|\|A\|\|B\| + \frac{\lambda\mu}{\|A\|\|B\|}\right|,\ \lambda \in W_B\bigl(A^*B\bigr),\ \mu \in W_A\bigl(B^*A\bigr)\right\}.$$

Proposition 7. *Let $A, B \in B(\mathcal{H})$ such that $\|U_{A,B}\| = \|A\|\|B\|$.*

(1) *If $B = A^*$, then $W_B(A^*B) = W_A(B^*A) = \{0\}$.*
(2) *If $\|A\|^2 B^*B \leqslant \|B\|^2 A^*A$ (respectively $\|B\|^2 A^*A \leqslant \|A\|^2 B^*B$), then $W_B(A^*B) = \{0\}$ (respectively $W_A(B^*A) = \{0\}$).*

Proof. (1) If $B = A^*$, then U_{A,A^*} is a positive map on $\mathcal{B}(\mathcal{H})$ and thus $\|U_{A,A^*}\| = \|AA^* + A^*A\| = \|A\|^2$ by the Russo–Dye theorem (cf. Corollary 2.9 in [5]). Let $\{x_n\} \subseteq \mathcal{H}$ be a sequence of unit vectors such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty} \langle A^2x_n, x_n\rangle = \lambda$. Then $\langle (AA^* + A^*A)x_n, x_n\rangle \leqslant \|AA^* + A^*A\| = \|A\|^2$, which implies that $\lim_{n\to\infty} \|A^*x_n\| = 0$. Note that $\lim_{n\to\infty} \langle A^2x_n, x_n\rangle = \lim_{n\to\infty} \langle Ax_n, A^*x_n\rangle = 0$. Then $\lambda = 0$ and thus $W_A(A^2) = \{0\}$. We similarly get $W_{A^*}((A^*)^2) = \{0\}$.

(2) We can assume that $\|A\| = \|B\| = 1$. If x is a unit vector in $\mathcal{H}$, then $\|U_{A,B}\| \geqslant \|U_{A,B}(x \otimes Bx)(x)\| \geqslant |\|Ax\|^2\|Bx\|^2 + \langle B^*Ax, x\rangle\langle A^*Bx, x\rangle| = \|Ax\|^2\|Bx\|^2 + |\langle B^*Ax, x\rangle|^2$. If $B^*B \leqslant A^*A$, we have $\|Ax\| \geqslant \|Bx\|$. For any $\lambda \in W_B(A^*B)$, there exists a sequence of unit vectors $\{x_n\} \subseteq \mathcal{H}$ such that $\lim_{n\to\infty} \|Bx_n\| = \|B\| = 1$ and $\lim_{n\to\infty} \langle A^*Bx_n, x_n\rangle = \lambda$. Then $\lim_{n\to\infty} \langle B^*Ax_n, x_n\rangle = \bar{\lambda}$. Since $\|Ax_n\| \geqslant \|Bx_n\|$, we have $\lim_{n\to\infty} \|Ax_n\| = \|A\| = 1$. It now follows that $\bar{\lambda} \in W_A(B^*A)$. We deduce from Lemma 6 that $1 = \|U_{A,B}\| \geqslant 1 + |\lambda|^2$, which implies that $\lambda = 0$. Therefore $W_B(A^*B) = \{0\}$. The proof is complete. □

We note that if either A or B is an isometry, then the condition (2) of Proposition 7 is satisfied.

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References

[1] A. Blanco, M. Boumazgour, T.J. Ransford, On the norm of elementary operators, J. London Math. Soc. (2) 70 (2004) 479–498.
[2] M. Boumazgour, Norm inequalities for sums of two basic elementary operators, J. Math. Anal. Appl. 342 (2008) 386–393.
[3] B. Magajna, On the distance to finite-dimensional subspaces in operator algebras, J. London Math. Soc. (2) 47 (1993) 516–532.
[4] M. Mathieu, More properties of the product of two derivations of a C^*-algebras, Bull. Austral. Math. Soc. 42 (1990) 115–120.
[5] V. Paulsen, Completely Bounded Maps and Operator Algebras, Cambridge University Press, 2002.
[6] A. Seddik, On the numerical range and norm of elementary operator algebras, Linear Multilinear Algebra 52 (2004) 293–302.
[7] L.L. Stachó, B. Zalar, On the norm of Jordan elementary operators in standard operator algebras, Publ. Math. Debrecen 49 (1996) 127–134.
[8] R.M. Timoney, Norms and CB norms of Jordan elementary operators, Bull. Sci. Math. 127 (2003) 597–609.