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NORM OF SUMS OF PEIRCE PROJECTIONS

Borut Zalar

University of Maribor, Faculty of Civil Engineering
Dept. of Basic Sciences, Smetanova 17, 2000 Maribor
Slovenija

Peirce projections P_1, $P_{1/2}$ and P_0 are one of the fundamental technical tools in the theory of JB*-triples. It is well known that all three of them are contractive. We show that the sum of two Peirce projections need not be contractive. We also give the upper estimate for the norm of such a sum, valid in all JC*-triples, and present a conjecture of the exact value of this norm, based on some numerical experiments.

Keywords: Peirce projections; JB*-triple.

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1. Introduction

Historically JB*-triples arose in the study of complex holomorphy. A bounded symmetric domain is biholomorphic to the open unit ball of a JB*-triple. From the viewpoint of operator theory, results about the structure of JB*-triples and the methods used for the study of their properties are closely connected with the theory of C*-algebras. In fact every C*-algebra can be viewed as a JB*-triple with the same norm and ternary product $\{abc\} = 1/2(ab^*c + cb^*a)$. Abstractly, a JB*-triple is a complex Banach space W together with a ternary product $\{...\} : W^3 \to W$ such that

- $\{abc\} = \{cba\}$ and the product is linear in outer variables and conjugate linear in the middle one;
- $\|\{abc\}\| \leq \|a\| \cdot \|b\| \cdot \|c\|$ for all $a, b, c \in W$;
- $\|\{aaa\}\| = \|a\|^3$ for all $a \in W$;
- the operator $x \longmapsto \{aax\}$ is positive hermitian for all $a \in W$;
- the Jordan identity $\{xy\{abc\}\} = \{\{xya\}bc\} + \{ab\{xyc\}\} - \{a\{yxb\}c\}$ holds for all $a, b, c, x, y \in W$.

A basic example of JB*-triple, which cannot be given a C*-algebra structure, is the space of all bounded linear operators $B(H, K)$ between two complex Hilbert spaces H and K, where the same formula $\{abc\} = 1/2(ab^*c + cb^*a)$ as above still makes sense and satisfies all required properties.

It is possible to develop a theory of JBW*-factors, which is somewhat richer than the theory of von Neumann algebras. There exist 6 families of type I factors. Four of them are infinite. Factors of type $B(H,K)$ are called rectangular. Factors of type $\{x \in B(H) : x^T = \pm x\}$ are called Hermitian and symplectic respectively. Every Hilbert space with dimension ≥ 3 gives rise to a construction of a spin factor, but we do not need to go into details here. JB*-triples for which all type I representations are of one of the above four types, are called JC*-triples. Every JC*-triple can be imbedded into $B(H)$ for a suitable Hilbert space H. There exist also two exceptional type I factors, constructed from octonion matrices. They are both finite dimensional, their dimensions being 16 and 27, and are therefore more interesting from algebraic viewpoint than a functional-analytic. It is known that neither of them is embeddable into $B(H)$.

An element $u \in W$ is called a tripotent if $\{uuu\} = u$. These elements play a similar role in JB*-theory as orthogonal projections play in C*-theory. In the case of the triple $B(H,K)$ it is not difficult to see that tripotents are precisely partial isometries from H into K. If we consider the operator $T(x) = \{uux\}$, it can be shown that the Jordan identity implies $T(2T-1)(T-1) = 0$ which shows that W decomposes as $W_1 \oplus W_{1/2} \oplus W_0$ where $W_i = \{x \in W : \{uux\} = ix\}$. Of course this decomposition, which is called Peirce decomposition, depends on the choice of u. The corresponding Peirce projections $P_i : W \to W$, whose ranges are W_i, can all be given with an explicit algebraic formula

$$P_1(x) = \{u\{uxu\}u\}, \qquad P_{1/2}(x) = 2(\{uux\} - \{u\{uxu\}u\},$$
$$P_0(x) = x - 2\{uux\} + \{u\{uxu\}u\}.$$

Peirce projections are one of the most widely used technical tools in research about JB*-triples. It is known that all three Peirce projections are contractive. Moreover $P_1 + P_0$ is also contractive. The purpose of this paper is to show that the remaining combinations $P_1 + P_{1/2}$ and $P_{1/2} + P_0$ are in general not contractive and to give a reasonable estimate for their norms in JC* case.

The reader can find proofs of the above explained facts in classical papers [3, 9, 10, 11, 12, 13, 16] and surveys [7, 18, 19, 23]. For those interested in various modern trends in JB*-theory, a starting sample is [1, 2, 4, 5, 6, 8, 14, 17, 20, 21, 22].

2. Example of a lower estimate

In this section we consider the special situation when $W = B(H,K)$ is a JB*-triple of bounded linear operators between two complex Hilbert spaces H and K. We use a rather standard notation $\beta \otimes \alpha$, where $\alpha \in H$ and $\beta \in K$, for a rank one operator given by $(\beta \otimes \alpha)\xi = \langle \xi, \alpha \rangle \beta$ for all $\xi \in H$. It is easy to see that $\beta \otimes \alpha \in B(H,K)$; more precisely $\|\beta \otimes \alpha\| = \|\alpha\| \cdot \|\beta\|$ holds.

It is well known that tripotents in the case of a JB*-triple $B(H,K)$ are precisely the partial isometries. Let $u \in W$ be a nonzero partial isometry and $P_1, P_{1/2}, P_1 : W \to W$ Peirce projections associated with u. Our aim in this section is to show that in triples of the above form operators $P_1 + P_{1/2}$ and $P_{1/2} + P_0$ are not contractive; in fact we show that their norms, except in a trivial case where $P_0 = 0$, $P_1 + P_{1/2} = 1$, are at least $1.154\,7$.

At present it seems that the logic behind this example cannot easily be generalized to arbitrary JB*-triples, so it may still be true that in some classes of JB*-triples operators $P_1 + P_{1/2}$ and $P_{1/2} + P_0$ are contractive.

Lemma 2.1. *Let W be a JB*-triple of the form $B(H,K)$ and $u \in W$ a partial isometry. If $P_0 \neq 0$, then $\operatorname{Ker} u \neq 0$ and $(\operatorname{Im} u)^{\perp} \neq 0$.*

Proof. Let $x \in \operatorname{Im} P_0$ be a nonzero operator. This means that $0 = 2\{xuu\} = xu^*u + uu^*x$ and, taking $uu^*u = u$ into account,

$$0 = u^*\left(xu^*u + uu^*x\right)u^* = 2u^*xu^*.$$

Therefore

$$\begin{aligned}\langle \operatorname{Im} u, \operatorname{Im} x\rangle &= \langle uu^*u(H), x(H)\rangle = \langle u(H), uu^*x(H)\rangle \\ &= -\langle u(H), xu^*u(H)\rangle = -\langle H, (u^*xu^*)u(H)\rangle = 0\end{aligned}$$

implies $0 \neq \operatorname{Im} x \subset (\operatorname{Im} u)^{\perp}$. On the other hand

$$\begin{aligned}ux^* &= uu^*ux^* = u(xu^*u)^* \\ &= -u(uu^*x)^* = -(ux^*u)u^* = 0\end{aligned}$$

means that $0 \neq \operatorname{Im} x^* \subset \operatorname{Ker} u$. □

With the aid of a numerical experiment, using Maple software, we found a numerical example, which serves as the basis for the following result.

Proposition 2.1. *Let W be a JB*-triple of the form $B(H,K)$ and $u \in W$ a partial isometry. If $P_0 \neq 0$, then $\operatorname{Ker} u \neq 0$ and $(\operatorname{Im} u)^{\perp} \neq 0$. If $P_0 \neq 0$, then the estimates*

$$\frac{2}{\sqrt{3}} \leq \left\|P_0 + P_{1/2}\right\|, \frac{2}{\sqrt{3}} \leq \left\|P_{1/2} + P_1\right\|$$

hold.

Proof. From the above lemma we know that we may choose four unit vectors

$$\begin{aligned}&\xi_1 \in \operatorname{Ker} u, \xi_2 \in (\operatorname{Ker} u)^{\perp} = \operatorname{Im} u^* = \operatorname{Im} u^*u, \\ &\eta_2 \in \operatorname{Im} u = \operatorname{Im} uu^*, \eta_1 = (\operatorname{Im} u)^{\perp} = \operatorname{Ker} u^*.\end{aligned}$$

Now we define a rank 2 operator $x : X \to Y$ by

$$x = \frac{1}{\sqrt{3}}\eta_1 \otimes \xi_1 + \frac{\sqrt{2}}{\sqrt{3}}\eta_2 \otimes \xi_1 + \frac{\sqrt{2}}{\sqrt{3}}\eta_1 \otimes \xi_2 - \frac{1}{\sqrt{3}}\eta_2 \otimes \xi_2.$$

Since $\langle \eta_1, \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle = 0$, we have, using the general multiplication formula $(\alpha \otimes \beta)(\gamma \otimes \delta) = \langle \gamma, \beta \rangle \, \alpha \otimes \delta$ for rank 1 operators,

$$\begin{aligned} x^* x = {} & \left(\frac{1}{\sqrt{3}} \xi_1 \otimes \eta_1 + \frac{\sqrt{2}}{\sqrt{3}} \xi_2 \otimes \eta_1 + \frac{\sqrt{2}}{\sqrt{3}} \xi_1 \otimes \eta_2 - \frac{1}{\sqrt{3}} \xi_2 \otimes \eta_2 \right) \cdot \\ & \left(\frac{1}{\sqrt{3}} \eta_1 \otimes \xi_1 + \frac{\sqrt{2}}{\sqrt{3}} \eta_2 \otimes \xi_1 + \frac{\sqrt{2}}{\sqrt{3}} \eta_1 \otimes \xi_2 - \frac{1}{\sqrt{3}} \eta_2 \otimes \xi_2 \right) \\ & = \frac{1}{3} \xi_1 \otimes \xi_1 + \frac{\sqrt{2}}{3} \xi_1 \otimes \xi_2 + \frac{2}{3} \xi_1 \otimes \xi_1 - \frac{\sqrt{2}}{3} \xi_1 \otimes \xi_2 \\ & + \frac{\sqrt{2}}{3} \xi_2 \otimes \xi_1 + \frac{2}{3} \xi_2 \otimes \xi_2 - \frac{\sqrt{2}}{3} \xi_2 \otimes \xi_1 + \frac{1}{3} \xi_2 \otimes \xi_2 \\ & = \xi_1 \otimes \xi_1 + \xi_2 \otimes \xi_2. \end{aligned}$$

The orthogonality of ξ_1 and ξ_2 implies that the last operator is an orthogonal projection. Its norm is thus 1, which in turn gives $\|x\| = \sqrt{\|x^* x\|} = 1$.

Next we compute $(P_1 + P_{1/2})(x)$. We know that $u^* u \xi_2 = \xi_2$, $u^* \eta_1 = 0$, $u \xi_1 = 0$ and $u u^* \eta_2 = \eta_2$. Using a general rule $a(\alpha \otimes \beta) = (a\alpha \otimes \beta)$ for a multiplication between a linear operator a and a rank one operator, we obtain

$$\begin{aligned} L(u,u)(\eta_2 \otimes \xi_2) &= \frac{1}{2} \left(u u^* (\eta_2 \otimes \xi_2) + (\eta_2 \otimes \xi_2) u^* u \right) \\ &= \frac{1}{2} \left((u u^* \eta_2) \otimes \xi_2 + \eta_2 \otimes (u^* u \xi_2) \right) \\ &= \frac{1}{2} \left(\eta_2 \otimes \xi_2 + \eta_2 \otimes \xi_2 \right) = \eta_2 \otimes \xi_2 \end{aligned}$$

which shows $\eta_2 \otimes \xi_2 \in \operatorname{Im} P_1$. Similarly

$$\begin{aligned} L(u,u)(\eta_2 \otimes \xi_1 + \eta_1 \otimes \xi_2) &= \frac{1}{2} \left(u u^* (\eta_2 \otimes \xi_1 + \eta_1 \otimes \xi_2) + (\eta_2 \otimes \xi_1 + \eta_1 \otimes \xi_2) u^* u \right) \\ &= \frac{1}{2} \left(u u^* \eta_2 \right) \otimes \xi_1 + \frac{1}{2} \eta_2 \otimes (u^* u \xi_1) + \frac{1}{2} \left(u u^* \eta_1 \right) \otimes \xi_2 \\ &+ \frac{1}{2} \eta_1 \otimes (u^* u \xi_2) \\ &= \frac{1}{2} \eta_2 \otimes \xi_1 + 0 + 0 + \frac{1}{2} \eta_1 \otimes \xi_2 = \frac{1}{2} (\eta_2 \otimes \xi_1 + \eta_1 \otimes \xi_2) \end{aligned}$$

shows $\eta_2 \otimes \xi_1 + \eta_1 \otimes \xi_2 \in \operatorname{Im} P_{1/2}$ while

$$\begin{aligned} L(u,u)(\eta_1 \otimes \xi_1) &= \frac{1}{2} \left(u u^* (\eta_1 \otimes \xi_1) + (\eta_1 \otimes \xi_1) u^* u \right) \\ &= \frac{1}{2} \left((u u^* \eta_1) \otimes \xi_1 + \eta_1 \otimes (u^* u \xi_1) \right) \\ &= 0 + 0 \end{aligned}$$

shows $\eta_1 \otimes \xi_1 \in \operatorname{Im} P_0$. Therefore

$$z = (P_1 + P_{1/2}) x = \frac{\sqrt{2}}{\sqrt{3}} \eta_2 \otimes \xi_1 + \frac{\sqrt{2}}{\sqrt{3}} \eta_1 \otimes \xi_2 - \frac{1}{\sqrt{3}} \eta_2 \otimes \xi_2.$$

We compute the norm of z using

$$\|z\|^2 = \|zz^*\| = \max\{\lambda : \lambda \in \mathrm{sp}(zz^*)\}.$$

The operator zz^* has rank 2, so its spectrum consists of 0 and two additional positive eigenvalues. Computation

$$\begin{aligned} zz^* &= \left(\frac{\sqrt{2}}{\sqrt{3}}\eta_2 \otimes \xi_1 + \frac{\sqrt{2}}{\sqrt{3}}\eta_1 \otimes \xi_2 - \frac{1}{\sqrt{3}}\eta_2 \otimes \xi_2\right) \\ &\quad \cdot \left(\frac{\sqrt{2}}{\sqrt{3}}\xi_1 \otimes \eta_2 + \frac{\sqrt{2}}{\sqrt{3}}\xi_2 \otimes \eta_1 - \frac{1}{\sqrt{3}}\xi_2 \otimes \eta_2\right) \\ &= \eta_2 \otimes \eta_2 + \frac{\sqrt{2}}{3}\eta_2 \otimes \eta_1 + \frac{\sqrt{2}}{3}\eta_1 \otimes \eta_2 + \frac{2}{3}\eta_1 \otimes \eta_1 \end{aligned}$$

shows that zz^* has the same positive eigenvalues as the complex matrix

$$\begin{bmatrix} 1 & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{bmatrix}.$$

Those are easily seen to be $1/3$ and $4/3$ respectively, which means $\|zz^*\| = 4/3$ and so

$$\left\|P_1 + P_{1/2}\right\| \geq \frac{\left\|(P_1 + P_{1/2})x\right\|}{\|x\|} = \frac{\sqrt{4/3}}{1} = \frac{2}{\sqrt{3}}.$$

In a similar way we can also prove that $\left\|(P_0 + P_{1/2})x\right\| \geq 2/\sqrt{3}$. □

3. Universal upper estimate for JC*-triples

In this section we first give an upper estimate for the special case of the C*-algebra $B(H)$, regarded as a JB*-triple. As in the previous section, let $u : H \to H$ be a partial isometry and $P_1, P_{1/2}, P_0$ the associated Peirce projections. If we have two orthogonal decompositions of the space H as $H_1 \oplus H_2$ and $K_1 \oplus K_2$, we can decompose every operator $v \in B(H)$ as a block operator $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a : H_1 \to K_1$, $b : H_2 \to K_1$, $c : H_1 \to K_2$ and $d : H_2 \to K_2$. In order to simplify further notation we avoid the index notation 1_{K_1} and $0_{H_1,H_2}$ for the identity operator on the space K_1 and the zero operator from H_1 into H_2 respectively, as the context will be clear in all cases so that both 1_{K_1}, 1_{K_2} will be denoted by 1. Our estimate will be based on the following fundamental result from [15].

Theorem 3.1. *Let A be a complex unital C*-algebra and $x \in A$ an element whose norm is less than 1. Then there exists a subset $v_1, ..., v_n$ of unitary elements such that*

$$x = \frac{1}{n}\sum_{i=1}^{n} v_i.$$

Note that n depends on x. From this result it follows that it is sufficient to establish an upper bound for $\left\|(P_1 + P_{1/2})v\right\|$ where v is an unitary operator. We consider decompositions $H_1 = (\operatorname{Ker} u)^\perp$, $H_2 = \operatorname{Ker} u$ and $K_1 = \operatorname{Im} u$, $K_2 = (\operatorname{Im} u)^\perp$. If we represent the unitary operator v as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ according to the above decomposition, we have the following:

Lemma 3.1. *Operators a, b, c, d in the above decomposition satisfy the following relations:*

$$aa^* + bb^* = 1, cc^* + dd^* = 1, ac^* + bd^* = 0$$
$$a^*a + c^*c = 1, d^*d + b^*b = 1, a^*b + c^*d = 0.$$

Their norms are all less than or equal to 1.

Proof. Let p, q be projections onto $(\operatorname{Ker} u)^\perp$ and $\operatorname{Im} u$ respectively. Let $\overline{p}$, $\overline{q}$ be their complementary projections. Then the above operators act on the following spaces:

$$\begin{aligned} a &: \operatorname{Im} p \to \operatorname{Im} q, \\ b &: \operatorname{Im} \overline{p} \to \operatorname{Im} q, \\ c &: \operatorname{Im} p \to \operatorname{Im} \overline{q}, \\ d &: \operatorname{Im} \overline{p} \to \operatorname{Im} \overline{q}. \end{aligned}$$

From this it follows that $aa^* + bb^*$ is a well-defined operator acting on $\operatorname{Im} q$. If h, k denote arbitrary vectors from H, we can write explicit formulae for a, a^*, b and b^*, via $B(H)$, as

$$a(ph) = qvph, a^*(qk) = pv^*qk,$$
$$b(\overline{p}h) = qv\overline{p}h, b^*(qk) = \overline{p}v^*qk,$$

which imply

$$\begin{aligned} (aa^* + bb^*)(qk) &= aa^*(qk) + bb^*(qk) \\ &= a(p(v^*qk)) + b(\overline{p}(v^*qk)) \\ &= qvpv^*qk + qv\overline{p}v^*qk \\ &= qv(p + \overline{p})v^*qk \\ &= qvv^*qk = qk. \end{aligned}$$

The last equality follows from the fact that v is unitary, while q is a projection. The above computation confirms that the operator $aa^* + bb^*$ acts as the identity operator on the space $\operatorname{Im} q$. Other equalities can be proved in a similar way. □

Lemma 3.2. *In the same situation as above, the following norm estimates hold:*

$$\|ac^*\|, \|ca^*\|, \|bd^*\|, \|db^*\|, \|a^*b\|, \|b^*a\|, \|c^*d\|, \|d^*c\| \leq \frac{1}{2}.$$

This estimate is in general the best possible, as the example of a unitary 2×2 matrix

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

shows.

Proof. From $c^*c = 1 - a^*a$ we have

$$\|ac^*\|^2 = \|ac^*ca^*\| = \|a(1-a^*a)a^*\| = \left\|aa^* - (aa^*)^2\right\|.$$

Since $\|aa^*\| \le 1$, the estimate

$$\begin{aligned} \left\|aa^* - (aa^*)^2\right\| &= \max\left\{\lambda - \lambda^2 : \lambda \in \mathrm{sp}(aa^*)\right\} \\ &\le \max\left\{\lambda - \lambda^2 : 0 \le \lambda \le 1\right\} = \frac{1}{4} \end{aligned}$$

implies $\|ac^*\| \le 1/2$. Other estimates can be proved in a similar way. □

In course of the proof we shall use another numerical result

Lemma 3.3. *The maximal value of the real function*

$$f(\varphi) = A\cos^2\varphi + B\cos\varphi\sin\varphi,$$

where A, B are positive constants, is

$$\frac{1}{2} \cdot \frac{B^2}{\sqrt{A^2+B^2} - A}.$$

This result is just an interesting exercise for a little more advanced students of calculus, so we omit the proof.

Proposition 3.1. *Let $W = B(H)$ be a JB*-triple, $u \in W$ a tripotent and $P_1, P_{1/2}, P_0$ Peirce projections corresponding to u. Then the estimates*

$$\left\|P_1 + P_{1/2}\right\|, \left\|P_{1/2} + P_0\right\| \le 1.1775$$

hold.

Proof. We already know that it is enough to consider $\left\|(P_1 + P_{1/2})v\right\|$ where $v \in B(H)$ is unitary. Consider the decomposition of v into components a, b, c, d with respect to the same H_1, H_2, K_1, K_2 as above. In a similar way as in Proposition 2, we can compute that the operator $q = (P_1 + P_{1/2})v$ is represented by $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$. Using Lemma 4 we can see that $(qq^*)^2$ is represented by

$$\begin{bmatrix} 1 + ac^*ca^* & a(1+c^*c)c^* \\ c(1+c^*c)a^* & cc^* \end{bmatrix}.$$

We already know that $\|cc^*\| \le 1$. The estimate

$$\|1 + ac^*ca^*\| \le 1 + \|ac^*\| \cdot \|ca^*\| \le 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{4}$$

follows from Lemma 5. In order to estimate off-diagonal elements, we use the identity $a^*a = 1 - c^*c$, which implies

$$\begin{aligned}\|a(1+c^*c)c^*\|^2 &= \|a(1+c^*c)c^*c(1+c^*c)a^*\| \\ &= \|c(1+c^*c)(1-c^*c)(1+c^*c)c^*\| \\ &= \left\|(cc^*) + (cc^*)^2 - (cc^*)^3 - (cc^*)^4\right\|.\end{aligned}$$

Since cc^* is a positive operator on a subspace K_2 with the spectrum inside the interval $[0,1]$, the last norm in the above computation is less than or equal to

$$\max\left\{\lambda + \lambda^2 - \lambda^3 - \lambda^4 : 0 \le \lambda \le 1\right\} = \frac{107 + 51\sqrt{17}}{512} \approx 0.61968$$

which implies

$$\|a(1+c^*c)c^*\| \le \frac{\sqrt{214 + 102\sqrt{17}}}{32} \approx 0.7872.$$

If $\xi \in H$ is a unit vector, we decompose it into $\xi = \alpha + \beta$ with respect to the decomposition $K_1 \oplus K_2$ so that $\|\alpha\|^2 + \|\beta\|^2 = 1$. Then, using the above estimates,

$$\begin{aligned}\left\|(qq^*)^2\xi\right\|^2 &= \left\| \begin{bmatrix} 1 + ac^*ca^* & a(1+c^*c)c^* \\ c(1+c^*c)a^* & cc^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2 \\ &= \left\| \begin{matrix} (1 + ac^*ca^*)\alpha + a(1+c^*c)c^*\beta \\ c(1+c^*c)a^*\alpha + cc^*\beta \end{matrix} \right\|^2 \\ &= \|(1 + ac^*ca^*)\alpha + a(1+c^*c)c^*\beta\|^2 + \|c(1+c^*c)a^*\alpha + cc^*\beta\|^2 \\ &\le \frac{25}{16}\|\alpha\|^2 + \frac{107 + 51\sqrt{17}}{512}\|\beta\|^2 + 2 \cdot \frac{5}{4} \cdot \frac{\sqrt{214 + 102\sqrt{17}}}{32}\|\alpha\| \cdot \|\beta\| \\ &+ \frac{107 + 51\sqrt{17}}{512}\|\alpha\|^2 + \|\beta\|^2 + 2 \cdot \frac{\sqrt{214 + 102\sqrt{17}}}{32}\|\alpha\| \cdot \|\beta\| \\ &= 1 + \frac{9}{16}\|\alpha\|^2 + \frac{107 + 51\sqrt{17}}{512} + \frac{9\sqrt{214 + 102\sqrt{17}}}{64}\|\alpha\| \cdot \|\beta\|.\end{aligned}$$

We can denote $\|\alpha\| = \cos\varphi$, $\|\beta\| = \sin\varphi$ which bring us to the situation of Lemma 6. Its application gives us

$$\begin{aligned}\left\|(qq^*)^2\right\|^2 &\le 1 + \frac{107 + 51\sqrt{17}}{512} + \frac{1}{2} \cdot \frac{\frac{81\cdot(214 + 102\sqrt{17})}{64^2}}{\sqrt{\frac{9^2}{16^2} + \frac{81\cdot(214 + 102\sqrt{17})}{64^2}} - \frac{9}{16}} \\ &= \frac{619\sqrt{230 + 102\sqrt{17}} + 5228 + 51\sqrt{17}\sqrt{230 + 102\sqrt{17}} + 3468\sqrt{17}}{512\left(\sqrt{230 + 102\sqrt{17}} - 4\right)}\end{aligned}$$

and thus

$$\begin{aligned}
&\left\|(P_1+P_{1/2})v\right\| \\
&\quad\le \sqrt[8]{\frac{619\sqrt{230+102\sqrt{17}}+5228+51\sqrt{17}\sqrt{230+102\sqrt{17}}+3468\sqrt{17}}{512\left(\sqrt{230+102\sqrt{17}}-4\right)}} \\
&\quad< 1.1775
\end{aligned}$$

□

Theorem 3.2. *Let W be a JC*-triple and $u \in W$ a nonzero tripotent. If P_1, $P_{1/2}$ and P_0 denote the Peirce projections with respect to u, then*

$$\left\|P_1 + P_{1/2}\right\|, \left\|P_{1/2} + P_0\right\| \le 1.1775.$$

Proof. There exists an injective triple homomorphisms $\phi : W \to B(H)$ for a suitable Hilbert space H. Obviously $\phi(u)$ is a nonzero tripotent (i.e. partial isometry) in $B(H)$. Let $\widetilde{P_i}$ denote Peirce projections, acting on $B(H)$, with respect to $\phi(u)$. Since Peirce projections can be given in terms of triple product, for example

$$(P_1 + P_{1/2})x = 2\{uux\} - \{u\{uxu\}u\},$$

it is obvious that $\phi(P_1 + P_{1/2}) = (\widetilde{P_1} + \widetilde{P_{1/2}})\phi$. It is well known that injective triple homomorphisms are isometric, so, for all $w \in W$, we have

$$\begin{aligned}
\left\|(P_1 + P_{1/2})w\right\| = \left\|\phi(P_1 + P_{1/2})w\right\| &= \left\|(\widetilde{P_1} + \widetilde{P_{1/2}})\phi w\right\| \\
&\le 1.1775 \left\|\phi w\right\| = 1.7775 \left\|w\right\|.
\end{aligned}$$

□

Conjecture 3.3. From the above proof it seems obvious that perhaps the given upper bound could be improved by considering further similar computation for $(qq^*)^4$, $(qq^*)^8$ and so on. At present, however, it seems that the polynomials arising in this way become too complicated and their maximal values don't seem to be explicitly computable as was the case with

$$p(\lambda) = \lambda + \lambda^2 - \lambda^3 - \lambda^4.$$

Even so, it seems improbable, that the given upper estimate 1.1775 is the best possible one. Complex square matrices of dimension n can be viewed as a special case of $B(H)$. The operator $P_1 + P_{1/2}$ with respect to partial isometries of the type $u = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ can be easily computed so it is possible to use various computer software in order to form a conjecture. Our experiment consisted of generating matrices M of various dimensions with the aid of a random number generator and computing the quotient

$$\frac{\left\|\left(P_1 + P_{1/2}\right)(M)\right\|_{op}}{\|M\|_{op}}$$

where the approximation for the operator norm of M is computed numerically as the square root of the largest eigenvalue for M^*M. Such program, once we fix dimension and u, can run random generation for many hours, and quotients can be stored in separate file. In no such attempt were we able to obtain a number greater than $2/\sqrt{3} \approx 1.1547$, so we believe that the following is true. If u is a nonzero tripotent of a JC*-triple, then either we have a trivial case, $P_0 = 0$ or $P_{1/2} = 0$, when $\|P_1 + P_{1/2}\| = 1$ or if all three projections are nonzero then $\|P_1 + P_{1/2}\| = 2/\sqrt{3}$.

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