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EXTENSIONS OF KY FAN SECTION THEOREMS AND MINIMAX INEQUALITY THEOREMS

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Dedicated to Professor Ky Fan on his 80th birthday

1. Introduction

In 1961, Fan [10] proved a generalization of the classical KKM theorem in infinite dimensional Hausdorff topological vector spaces and established an elementary but very basic 'geometric lemma' for multivalued mappings. It was stated as:

THEOREM A. *Let E be a Hausdorff topological vector space, $X \subset E$ be a non-empty compact convex subset and A be a subset of $X \times X$ such that*
(a) *for each $y \in X$, the set $\{x \in X : (x,y) \in A\}$ is closed in X;*
(b) *for each $x \in X$, the set $\{y \in X : (x,y) \notin A\}$ is convex or empty;*
(c) *for each $x \in X$, $(x,x) \in A$.*
Then there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subset A$.

In 1968, Browder [4] gave a fixed point form (using different techniques) of Fan's geometric lemma and it is now called Fan–Browder fixed point theorem. Fan's geometric lemma above was stated by himself [13] in the following equivalent form for convex sets (later it was called the Fan's section theorem):

THEOREM B. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space and let $B \subset X \times X$. Assume*
(a) *for each fixed $x \in X$, the section $\{y \in X : (x,y) \in B\}$ is open in X;*
(b) *for each fixed $y \in X$, the set $\{x \in X : (x,y) \in B\}$ is non-empty and convex.*
Then there exists a point $x_0 \in X$ such that $(x_0, x_0) \in B$.

Fan's theorem above has numerous connections with other areas of mathematics and serves to unify many results in the literature, in particular in the study of minimax inequality theory, fixed point theory for set-valued mappings, mathematical economics, game theory and so on (see e.g. Aubin [1] and references therein). A number of generalizations and applications have been given by Bardaro and Ceppitelli [2], Ben-El-Mechaiekh et al [3], Chang and Yang [6], Degundji and Granas [7], Ding and Tan [8], Fan [14-15], Granas and Liu [18], Ha [19-21], Horvath [22], Park [34], Shih and Tan [35], Tan and Yuan [41], Tarafdar [42] and others. It is our purpose in this paper to give

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some generalizations of Fan's section theorem. Then as applications, some new Ky Fan minimax inequalities and fixed point theorems are given.

Now we explain some definitions and notations. Throughout this paper all topological spaces are assumed to be *Hausdorff* unless otherwise specified. The set of all real numbers is denoted by $\mathbf{R}$ and the set of natural numbers is denoted by $\mathbf{N}$. If X is a set, we shall denote by 2^X the family of all subsets of X. Let A be a subset of a topological space X. We shall denote by $\operatorname{int}_X(A)$ the interior of A in X and by $\operatorname{cl}_X(A)$ the closure of A in X. If A is a subset of a vector space, we shall denote by $\operatorname{co} A$ the convex hull of A. Let X and Y be two non-empty sets and $T : X \to 2^Y$ be a mapping. Then the graph of T, denoted by $\operatorname{Graph} T$ is the set $\{(x,y) \in X \times Y : y \in T(x)\}$. Suppose X is a non-empty convex subset of a topological vector space E and $f : X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is an extended valued function. Then f is said to be quasi-concave (respectively, quasi-convex) if the set $\{x \in X : f(x) > \lambda\}$ (respectively, the set $\{x \in X : f(x) < \lambda\}$) is convex for each $\lambda \in \mathbf{R}$. A topological space X is said to be *contractible* if the identity mapping I_X of X is homotopic to a constant function.

2. Extensions of Ky Fan section theorems

By employing the same arguments of Eilenberg and Montgomery [9, p.106-107] and using Theorem 6.3 of Gòrniewicz [17, p.111] instead of the coincidence theorem used in [9, p.106-107], the following lemma was obtained by Shioji [36, p.188]:

LEMMA 1 (Eilenberg and Montgomery, Gòrniewicz and Shioji). *Let Δ_N be an n-dimensional simplex with the Euclidean topology and Y be a compact topological space. Let $\psi : Y \to \Delta_N$ be a single-valued continuous mapping and $T : \Delta_N \to 2^Y$ be a set-valued upper semicontinuous mapping with non-empty compact contractible values. Then there exists $x_0 \in \Delta_N$ such that $x_0 \in \psi \circ T(x_0)$, where $\psi \circ T$ denotes the composition of the mapping T with ψ.*

As an application of Lemma 1, we have the following section theorem which generalizes Theorem 3 of Ha [19] which in turn improves Fan's section Theorem in [10] and [13].

THEOREM 2. *Let X be a topological space and Y be a non-empty convex subset of a Hausdorff topological vector space F. Suppose that A is a subset of $X \times Y$ and there exist a subset B of A and a non-empty compact subset K of X such that B is closed in $X \times Y$ and*

(a) *for each $y \in Y$, the set $\{x \in X : (x,y) \in A\}$ is closed in X;*

(b) *for each $x \in X$, the set $\{y \in Y : (x,y) \notin A\}$ is convex or empty;*

(c) *for each $y \in Y$, the set $\{x \in K : (x,y) \in B\}$ is non-empty and contractible.*

Then there exists $x_0 \in K$ such that $\{x_0\} \times Y \subset A$.

PROOF. Following Ha [19], for each $y \in Y$, let $A(y) = \{x \in X : (x, y) \notin A\}$. Suppose that the conclusion of Theorem 2 were false. Then for each $x \in K$, there exists $y \in Y$ such that $(x, y) \notin A$, i.e., $x \in A(y)$, so that $K \subset \bigcup_{y \in Y} A(y)$. By (a) and the compactness of K, there exists a finite subset $\{y_1, y_2, \dots, y_n\}$ of Y such that $K \subset \bigcup_{j=1}^{n} A(y_j)$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be non-negative real valued continuous functions on K such that for each $1 \leqq j \leqq n$, λ_j vanishes on $K \setminus A(y_j)$ and $\Sigma_{j=1}^{n} \lambda_j = 1$ for all $x \in K$. Define a function $p : K \to Y$ by

$$p(x) = \Sigma_{j=1}^{n} \lambda_j(x) y_j$$

for each $x \in K$. Note that for each $x \in K$ and $j = 1, 2, \dots, n$, if $\lambda_j(x) > 0$, then $x \in A(y_j)$. By (b), we have that $(x, p(x)) \notin A$. Now let $Z := \mathrm{co}(y_1, \dots, y_n)$ and define a mapping $q : Z \to 2^K$ by

$$q(z) = \{x \in K : (x, z) \in B\}$$

for each $z \in Z$. Then q has non-empty closed values by (c). Note that B is closed in $X \times Y$ so that the graph of q is also closed in $Z \times K$ and q has non-empty compact contractible values. Since p is continuous, by Lemma 1, there exists $z_0 \in Z$ such that $z_0 \in p(q(z_0))$. Let $x_0 \in q(z_0)$ be such that $p(x_0) = z_0$. Then $(x_0, p(x_0)) \in B \subset A$ which contradicts that $(x, p(x)) \notin A$ for all $x \in X$. Thus there must exist $x_0 \in K$ such that $\{x_0\} \times Y \subset A$. □

Theorem 2 improves Theorem 3 of Ha [19] in the following ways: (i) the space X may not have linear structure and (ii) the set $\{x \in K : (x, y) \in B\}$ may not be convex.

As an equivalent form of Theorem 2, we have the following section theorem which improves Fan's section Theorem B:

THEOREM 2′. *Let X be a topological space and Y be a non-empty convex subset of a Hausdorff topological vector space F. Suppose that A_1 is a non-empty subset of $X \times Y$ and there exist a non-empty open subset B_1 of $X \times Y$ and a non-empty compact subset K of X such that*

(a) *for each $y \in Y$, the set $\{x \in X : (x, y) \in A_1\}$ is open in X;*

(b) *for each $x \in X$, the set $\{y \in Y : (x, y) \in A_1\}$ is convex and for each $x \in K$, the set $\{y \in Y : (x, y) \in A_1\} \neq \emptyset$;*

(c) *for each $y \in Y$, the set $\{x \in K : (x, y) \notin B_1\}$ is non-empty and contractible.*

Then there exists $(x_0, y_0) \in A_1$ such that $(x_0, y_0) \notin B_1$.

PROOF. Let $A := X \times Y \setminus A_1$ and $B := X \times Y \setminus B_1$. By condition (b), the conclusion of Theorem 2 does not hold. Note that A and B satisfy all the conditions (a) and (b) of Theorem 2. Thus B must not be a subset of A, so that $B_1 \not\supset A_1$. Therefore there exists $(x_0, y_0) \in A_1$ such that $(x_0, y_0) \notin B_1$. □

It is clear that Theorems 2 and 2′ include Theorems A and A′ as special cases. Let $X := Y := K$; $B := \Delta$ and $X := Y := K$ in Theorem 2, where $\Delta = \{(x,x) \in X \times X : x \in X\}$. Then Theorem 2 reduces to Theorem A. Let $X := Y := K$ and $B_1 := X \times X \setminus \Delta$ in Theorem 2′. By Theorem 2′, there exists $(x_0, y_0) \in A_1$ with $(x_0, y_0) \notin B_1$, i.e., $(x_0, y_0) \in \Delta$ so that $(x_0, x_0) \in A_1$ and Theorem B follows.

3. Ky Fan minimax inequalities

As an analytical form of Theorem 2, we have the following minimax inequality which is a generalization of Theorem 1 of Ha [21] which in turn improves the well-known Ky Fan minimax inequality in [13]:

THEOREM 3. *Let X be a non-empty convex subset of a Hausdorff topological vector space E and Y be a compact topological space. Suppose $f : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that*

(a) *for each $y \in Y$, $x \mapsto f(x,y)$ is quasi-convex in X;*

(b) *for each $x \in X$, $y \mapsto f(x,y)$ is upper semicontinuous;*

(c) *suppose $T : X \to 2^Y$ is a set-valued mapping with non-empty contractible compact values.*

Then there exists $y_0 \in Y$ such that

$$\inf_{x \in X, y \in T(x)} f(x,y) \leqq \inf_{x \in X} f(x, y_0) \leqq \max_{y \in Y} \inf_{x \in X} f(x,y).$$

PROOF. In order to apply Theorem 2, let $\lambda := \inf_{x \in X, y \in T(x)} f(x,y)$, $A := \{(x,y) \in X \times Y : f(x,y) \geqq \lambda\}$ and $B := \{(x,y) \in X \times Y : y \in T(x)\}$. Then we have:

(a′) for each $y \in Y$, the set $\{x \in X : (x,y) \notin A\} = \{x \in X : f(x,y) < \lambda\}$ is convex or empty by (a);

(b′) for each $x \in X$, the set $\{y \in Y : (x,y) \in A\} = \{y \in Y : f(x,y) \geqq \lambda\}$ is closed in X by (b);

(c′) note that T is upper semicontinuous with closed values so that B is closed in $X \times Y$;

(d′) for each $x \in X$, the set $\{y \in Y : (x,y) \in B\} = \{y \in Y : y \in T(x)\} = T(x)$ which is non-empty and contractible.

It is clear that $B = \operatorname{Graph} T \subset A$. Now let $K := Y$ and exchange both X and Y in Theorem 2. By Theorem 2, there exists $y_0 \in Y$ such that $X \times \{y_0\} \subset A$. From the definition of A, we have

$$\inf_{x \in X, y \in T(x)} f(x,y) \leqq \inf_{x \in X} f(x, y_0) \leqq \max_{y \in Y} \inf_{x \in X} f(x,y). \qquad \square$$

As an application of Theorem 3, we have the following best approximation theorem which generalizes Theorem 2 of Ha in [21] which in fact can be used to improve Theorem 1 of Fan [12] by using his argument hence we omit it here.

THEOREM 4. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E and Y be a compact topological space. Suppose $T : X \to 2^Y$ is upper semicontinuous with non-empty contractible compact values and the extended real-valued function $g : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that*

(a) *g is (jointly) lower semicontinuous and for each fixed $x \in X$, $y \mapsto g(x,y)$ is upper semicontonus (hence, $y \mapsto g(x,y)$ is continuous for each fixed $x \in X$);*

(b) *$x \mapsto g(x,y)$ is quasi-convex for each fixed $y \in Y$.*

Then there exists $x_0 \in X$ and $y_0 \in T(x_0)$ such that

$$g(x_0, y_0) \leqq \inf_{x \in X} g(x, y_0).$$

PROOF. Define a mapping $f : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ by

$$f(x,y) = g(x,y) - \inf_{z \in X} g(z,y)$$

for each $(x,y) \in X \times Y$. Clearly f satisfies condition (a) of Theorem 3. Note that $y \mapsto g(x,y)$ is continuous for each fixed $x \in X$, therefore f is lower semicontinuous by (a). Hence by Theorem 3, there exists $y_0 \in Y$ such that

$$(1) \qquad \inf_{x \in X, y \in T(x)} f(x,y) \leqq \inf_{x \in X} f(x, y_0) = \inf_{x \in X} \left[g(x, y_0) - \inf_{z \in X} g(z, y_0) \right] = 0.$$

Note that T has a closed graph which is compact in $X \times Y$, and $x \mapsto f(x,y)$ is lower semicontinuous for each fixed $y \in Y$, so that the infimum on the left side of (1) above is attained. Thus there exists $y_0 \in T(x_0)$ such that $g(x_0, y_0) \leqq \inf_{x \in X} g(x, y_0)$. □

As another application of Theorem 2, we have the following minimax inequality:

THEOREM 5. *Let X be a topological space and Y be a non-empty convex subset of a Hausdorff topological vector space F. Suppose $f : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ is lower semicontinuous such that*

(a) *for each $x \in X$, $y \mapsto f(x,y)$ is quasi-concave;*

(b) *for each non-empty compact subset K of X, the set $\{x \in K : f(x,y) \leqq t\}$ is contractible for each $t \in \mathbf{R}$.*

Then

$$(2) \qquad \inf_{x \in X} \sup_{y \in Y} f(x,y) = \inf_{K \subset X} \sup_{y \in Y} \min_{x \in K} f(x,y)$$

where the infimum on the right side of (2) *is taken over all non-empty compact subsets K of X. If in addition, X is compact, then*

$$\min_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\min_{x\in X} f(x,y).$$

PROOF. It is clear that $\inf_{x\in X}\sup_{y\in Y} f(x,y) \leqq \sup_{y\in Y}\inf_{x\in X} f(x,y)$. To prove (2), we can assume that $\inf_{K\subset X}\sup_{y\in Y}\inf_{x\in K} f(x,y) < +\infty$. Choose $t \in \mathbf{R}$ with $t > \inf_{K\subset X}\sup_{y\in Y}\inf_{x\in K} f(x,y)$ and let $A := B := \{(x,y) \in X \times Y : f(x,y) \leqq t\}$. It is clear that A satisfies conditions (a) and (b) of Theorem 2. Let K be a non-empty compact subset of X such that $t > \sup_{y\in Y}\inf_{x\in K} f(x,y)$. Then for each $y \in Y$, the set $\{x \in K : f(x,y) \leqq t\}$ is contractible and non-empty by the choice of K. By Theorem 2, there exists $x_0 \in K$ such that $\{x_0\} \times Y \subset A$, that is, $f(x_0,y) \leqq t$ for all $y \in Y$. Thus $\inf_{x\in X}\sup_{y\in Y} f(x,y) \leqq t$ and hence (2) holds. In addition, if X is compact, then

$$\min_{x\in X}\sup_{y\in Y} f(x,y) = \inf_{K\subset X}\sup_{y\in Y}\min_{x\in K} f(x,y) = \sup_{y\in Y}\min_{x\in X} f(x,y). \qquad \square$$

As an immediate consequence of Theorem 5, we have the following corollary which is Theorem 4 of Ha in [19]:

COROLLARY (Ha [19]). *Let X be a non-empty convex subset of a topological space E and Y be a non-empty convex subset of a Hausdorff topological vector space F. Suppose $f : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ is lower semicontinuous such that*

(a) *for each $x \in X$, $y \mapsto f(x,y)$ is quasi-concave;*
(b) *for each $y \in Y$, $x \mapsto f(x,y)$ is quasi-convex.*

Then

$$\inf_{x\in X}\sup_{y\in Y} f(x,y) = \inf_{K\subset X}\sup_{y\in Y}\min_{x\in K} f(x,y) \tag{3}$$

where the infimum on the right side of (3) *is taken over all non-empty compact convex subsets K of X. If in addition, X is compact, then*

$$\inf_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\min_{x\in X} f(x,y).$$

PROOF. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the family of all non-empty compact convex subsets of X and non-empty compact subsets of X respectively. Clearly, $\mathcal{F}_1 \subset \mathcal{F}_2$. Note that

$$\inf_{x\in X}\sup_{y\in Y} f(x,y) \geqq \inf_{K\in\mathcal{F}_1}\sup_{y\in Y}\min_{x\in K} f(x,y)$$

and

$$\inf_{Z\in\mathcal{F}_2}\sup_{y\in Y}\min_{z\in Z} f(x,y) \leqq \inf_{K\in\mathcal{F}_1}\sup_{y\in Y}\min_{x\in K} f(x,y).$$

Hence the conclusion follows by (2) of Theorem 5. □

4. Extensions of Ky Fan's fixed point theorems

Finally as applications of Lemma 1, we shall give some extensions of Ky Fan's fixed point theorems in [12]. We first have:

THEOREM 6. *Let X be a non-empty compact and convex subset of a Hausdorff topological vector space E and F be a Hausdorff topological vector space with sufficient continuous linear functionals. Suppose $T: X \to 2^F$ is upper semicontinuous with non-empty closed convex values and $g: X \to F$ is a (single-valued) continuous mapping such that*

(a) *$T(x) \cap g(X) \neq \emptyset$ for each $x \in X$;*

(b) *for a given continuous seminorm p on F and $x \in X$, the set $\{z \in Y : p(g(z) - g(x)) \leqq \delta\}$ is convex, where $\delta \in \mathbf{R}$, and $g^{-1}(T(z))$ is contractible for each $z \in X$.*

Then there exists $x_0 \in X$ such that $g(x_0) \in T(x_0)$.

PROOF. Following the argument of Ha [20, pp.13–14], suppose the conclusion were false. Then for each $x \in X$, $0 \notin g(x) - T(x)$ and so there exists $\delta_x > 0$ and a continuous seminorm p_x on F such that $p_x(g(x) - u) > 2\delta_x$ for all $u \in T(x)$. By the upper semicontinuity of T and the continuity of g, there exists an open neighborhood $N(x)$ of x in X such that $p_x(g(z) - v) > \delta_x$ for all $z \in N(x)$ and $v \in T(x)$. Since the family $\{N(x) : x \in X\}$ is an open cover of the compact set X, there exists a finite subset $\{x_1, \dots, x_m\}$ of X such that $\cup\{N(x_i) : i = 1, 2, \dots, m\} \supset X$. Now define a seminorm $p: F \to \mathbf{R}$ by $p(z) = \max_{i=1,2,\dots,m}\{p_{x_i}(z)\}$ for each $z \in F$. Let $\delta = \min_{i=1,2,\dots,m}\{\delta_{x_i}\}$. Then p is a continuous seminorm on F and

$$p(g(x) - u) > \delta \tag{4}$$

for all $x \in X$ and $u \in T(x)$. Let $B(z) = \{x \in X : p(g(z) - g(x)) < \delta\}$ for each $z \in X$. Then the family $\{B(z) : z \in X\}$ is an open cover of the compact set X so that there exists a finite subset $\{z_1, \dots, z_n\}$ of X such that $\cup_{j=1}^n B(z_j) \supset X$. Let $\{\lambda_i : i = 1, 2, \dots, n\}$ be a continuous partition of unity on X subordinated to the cover $\{B(z_j) : j = 1, 2, \dots, n\}$, that is $\lambda_1, \dots, \lambda_n$ are non-negative real valued continuous functions on X such that $\lambda_j(x) = 0$

for each $x \in X \setminus B(z_j)$ and $\Sigma_{i=1}^{n} \lambda_j(x) = 1$ for all $x \in X$. Let $Z = \text{co}(z_j : j = 1, 2, \ldots, n)$. Then $Z \subset X$. Define a function $\Phi : X \to Z$ by

$$\Phi(x) = \Sigma_{j=1}^{n} \lambda_j(x) z_j$$

for each $x \in X$. Note that for each $x \in X$, and $j = 1, 2, \ldots, n$, if $\lambda_j(x) \neq 0$, then $x \in B(z_j)$ and so that $p(g(z_j) - g(x)) < \delta$. For each $x \in X$, let $C_x = \{ z \in Y : p(z - g(x)) \leqq \delta \}$. Then C is a non-empty closed subset of C and $g(z_j) \in C$ if $\lambda_j(x) \neq 0$. Since $g^{-1}(C)$ is convex by (c), so that $g(\Phi(x)) \in C$, i.e., $p(g(\Phi(x)) - g(x)) \leqq \delta$. Now define $h : Z \to 2^X$ by

$$h(z) = \{ x \in X : g(x) \in T(z) \}$$

for each $z \in Z$. Then h is a set-valued mapping with non-empty and contractible values by (a) and (b). Moreover it is easy to verify that the graph of h is closed in $Z \times X$ and so h is upper semicontinuous on Z. By Lemma 1, there exist $z_0 \in Z \subset X$ and $x_0 \in X$ such that $z_0 = \Phi(x_0)$ and $x_0 \in h(z_0)$. This means that $g(x_0) \in T(z_0)$. Note that $p(g(\Phi(x)) - g(x))) \leqq \delta$ for all $x \in X$. Thus $p(g(z_0) - g(x_0)) \leqq \delta$, which contradicts (1). Therefore there must exist $x_0 \in X$ such that $g(x_0) \in T(x_0)$. □

REMARK 1. The condition (b) of Theorem 6 is satisfied if $g^{-1}(C)$ is convex (maybe empty) for each closed convex C of F. Thus Theorem 6 improves Theorem 2 of Ha [20] in the sense that $g^{-1}(C)$ may be not convex for each closed convex subset C of F.

THEOREM 7. *Let X be a non-empty compact and convex subset of a Hausdorff topological vector space E and F be a Hausdorff topological vector space with sufficient continuous linear functionals. Suppose $f, g : X \to F$ are two (single-valued) continuous mappings such that $g(X)$ is convex and $g^{-1}(y)$ is contractible for each $y \in g(X)$.*

Then one of the following must hold:

(a) *there exists a point $x_0 \in X$ such that $g(x_0) = f(x_0)$, or*

(b) *there exists a point $\hat{x} \in X$ such that*

$$0 < p(g(\hat{x}) - f(\hat{x})) \leqq p(y - f(\hat{x})). \quad (5)$$

PROOF. Assume that the conclusion were false. Then $g(x) \neq f(x)$ for all $x \in X$. By using similar arguments as in the proof of Theorem 6 above, there exist $\delta > 0$ and a continuous seminorm p on F such that $p(g(x) - f(x)) > 0$ for all $x \in X$. Moreover for each $x \in X$, there is $y \in g(X)$ such that $p(g(x) - f(x)) > p(y - f(x))$ (since we assume that (b) does not hold). Now for each $y \in g(X)$, let $A(y) = \{ x \in X : p(g(x) - f(x)) > p(y - f(x)) \}$. Then $\{ A(y) : y \in g(X) \}$ is an open cover of the compact set X. Thus there exists a finite subset $\{y_1,\dots,y_m\}$ in $g(X)$ such that $\cup_{i=1}^m A(y_i) \supset X$. Let $\{\lambda_i,\dots,\lambda_m\}$ be a continuous partition of unity on X subordinated to $\{A(y_i) : i=1,2,\dots,m\}$ and $Z := \mathrm{co}(y_i : i=1,2,\dots,m)$. Define a mapping $\Psi : X \to Z$ by

$$\Psi(x) = \Sigma_{i=1}^m \lambda_i(x) y_i$$

for each $x \in X$. Now for each $x \in X$, if $\lambda_i(x) \neq \emptyset$ for some $i = 1,2,\dots,m$, then $x \in A(y_i)$ so that $p(g(x) - f(x)) > p(y_i - f(x))$. Hence $p(g(x) - f(x)) > p(\Psi(x) - f(x))$ for all $x \in X$. For each $z \in Z$, let $h(z) = g^{-1}(z)$. Then clearly h is an upper semicontinuous set-valued mapping on Z with non-empty contractible values. By Lemma 1, there exist $z_0 \in Z \subset g(X)$ (since $g(X)$ is convex) and $x_0 \in X$ such that $z_0 = \Psi(x_0)$ and $x_0 \in h(z_0)$, i.e., $z_0 = g(x_0)$. Note that since $p(g(x) - f(x)) > p(\Psi(x) - f(x))$ for all $x \in X$, we must have that

$$p(g(x_0) - f(x_0) > p(z_0 - f(x_0)) = p(g(x_0) - f(x_0))$$

which is impossible. Therefore the conclusion must hold. □

REMARK 2. Theorem 7 shows that Theorem 3 of Ha [20] is still true in the case the set $g^{-1}(y)$ may not be convex. When $E := F$ and $g : X \to X$ is the identity mapping, Theorem 7 reduces to Theorem 2 of Fan in [12].

Finally as an immediate consequence of Theorem 7, we have:

THEOREM 8. *Let X be a non-empty compact and convex subset of a Hausdorff topological vector space E and F be a Hausdorff topological vector space with sufficient continuous linear functionals. Suppose $f, g : X \to F$ are two (single-valued) continuous mappings such that*

(a) *$g(X)$ is convex and $g^{-1}(y)$ is contractible for each $y \in g(X)$;*

(b) *for each $x \in X$, there exists a number λ (real or complex, depending on whether the vector space F is real or complex) such that $|\lambda| < 1$ and $\lambda g(x) + (1-\lambda) f(x) \in g(X)$.*

Then there exists $x_0 \in X$ such that $f(x_0) = g(x_0)$.

PROOF. Suppose that $g(x) \neq f(x)$ for all $x \in X$. By Theorem 7, there is a point $x_0 \in X$ and a continuous seminorm p on F which satisfies (5). Now there exists a number λ with $|\lambda| < 1$ and $y = \lambda g(x_0) + (1-\lambda) f(x_0) \in g(X)$. Substituting this y in (5), it follows that

$$0 < p(g(x_0) - f(x_0)) \leqq |\lambda| p(g(x_0) - f(x_0)),$$

which contradicts $|\lambda| < 1$.

REMARK 3. Let X and Y be non-empty sets and $f : X \times Y \to \mathbf{R}$ be a function. A minimax problem is to find certain conditions such that the following holds:

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

It is well-known that this minimax problem could be solved by fixed point theorems under certain circumstances. But it was Wu in [45] who first recognized the importance of the *connectedness* in the study of minimax theory. This idea then was picked by Terkerson in [43]. By a refined method, it is Tuy who derived a generalized version of Sion's classical minimax theorem in [44] (see also Geraghty and Lin [16]). Independently, inspired by Joó's paper [24], the *method of level sets* was developed by Joó and his Hungarian compatriot Stachó [40] and Komornik [31]. For example, by introducing the concept of the *interval space,* it was Stachó [40], who established an intersection theorem which was used by Komornik [31] to derive a generalization of Ha's minimax theorem in [19]. All these results were unified by Kindler and Trost [29]. Following this line, many minimax theorems which only involve *connectedness* instead of convexity were obtained by Chang et al [5], Horvath [22-23], Komiya [30], Lin and Quan [33], Kindler [26-28], König [32], Simons [37-38], Thompson and Yuan [46] et al.

Finally we would like to note that our minimax inequalities and fixed point theorems in this paper are proved by section theorems of Ky Fan type. Therefore our results here are independent of those given by authors mentioned above.

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