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TECHNICAL NOTE

New Saddle Point Theorem Beyond Topological Vector Spaces[1]

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Abstract. The purpose of this note is to prove a new topological saddle-point theorem, which in turn includes classical saddle-point theorems such as the Sion minimax theorem and others in topological vector spaces as special cases.

Key Words. Two-person zero-sum games, saddle points, minimax theorems, connectedness.

1. Introduction

Let X and Y be nonempty sets, and let $f: X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function. A point $(x^*, y^*) \in X \times Y$ is said to be a saddle point of f in $X \times Y$ if

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y),$$

for all $(x, y) \in X \times Y$.

Now, consider a two-person zero-sum game G_f generated by the function f. This means that the first player selects a point x from X and the second player selects a point y from Y. As a result of this choice, the second player pays the first one the amount $f(x, y)$. A point $(x^*, y^*) \in X \times Y$ is said

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to be a solution of the game G_f if and only if it is a saddle point of f in $X \times Y$.

The first saddle-point theorem was proved by Von Neumann (Ref. 1); Sion (Ref. 2) proved a very general saddle-point theorem for a function which is quasiconcave and upper semicontinuous in its first variable and quasiconvex and lower semicontinuous in its second variable in topological vector spaces.

In this note, we shall prove some new saddle-point theorems without any linear structure by an elementary proof which depends mainly on the connectedness of topological spaces. This method has been used by many authors; see also Refs. 3–16. Thus, our result includes the Sion saddle point theorem and related minimax results in topological vector spaces as special cases.

2. Preliminaries

Let X and Y be two Hausdorff topological spaces. Let f be a real-valued function defined in $X \times Y$ such that, for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semicontinuous and, for each fixed $y \in Y$, $x \mapsto f(x, y)$ is upper semicontinuous. If X and Y are both compact, we know that $\min_{y \in Y} \max_{x \in X} f(x, y)$ and $\max_{x \in X} \min_{y \in Y} (x, y)$ both exist. Denote

$$\underline{v} := \max_{x \in X} \min_{y \in Y} f(x, y), \qquad \bar{v} := \min_{y \in Y} \max_{x \in X} f(x, y);$$

then, it is clear that $\underline{v} \leq \bar{v}$. Moreover it is well known and easily proven that $\underline{v} = \bar{v}$ if and only if f has a saddle point.

3. Main Results

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers. We then have the following topological saddle-point theorem by an elementary proof which depends mainly on the connectedness of topological spaces.

Theorem 3.1. Let X and Y be compact Hausdorff topological spaces. Then, the function $f: X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ has a saddle point if the following conditions are satisfied:

(A1) for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semicontinuous;
(A2) for each fixed $y \in Y$, $x \mapsto f(x, y)$ is upper semicontinuous;

(A3) for each $c \in \mathbb{R}$, $m \in \mathbb{N}$, and $y_i \in Y$, $i = 1, \ldots, m$, the set

$$\bigcap_{i=1}^{m} \{x \in X : f(x, y_i) \geq c\}$$

is either connected or empty;

(A4) for each $y_1, y_2 \in Y$, there exists a function $S_{\{y_1,y_2\}}: [0, 1] \to Y$ such that

$$S_{\{y_1,y_2\}}(0) = y_1, \qquad S_{\{y_1,y_2\}}(1) = y_2,$$

and for each $x \in X$, $n \in \mathbb{N}$, $j = 1, \ldots, 2^{n-1}$,

$$\begin{aligned} &f(x, S_{\{y_1,y_2\}}((2j-1)/(2^n))) \\ &\leq \max\{f(x, S_{\{y_1,y_2\}}((j-1)/(2^{n-1}))), \\ &\qquad f(x, S_{\{y_1,y_2\}}(j/(2^{n-1})))\} \end{aligned}$$

and for any $\xi \in [0, 1]$,

$$\liminf_{\lambda \in T, \lambda \to \xi} f(x, S_{\{y_1,y_2\}}(\lambda)) \geq f(x, S_{\{y_1,y_2\}}(\xi)),$$

where $T := \{j/(2^n): n \in \mathbb{N}; j = 0, 1, \ldots, 2^n\}$.

Proof. It suffices to prove that $\underline{v} = \bar{v}$. We set

$$\mathscr{F} := \{F: F = \{x \in X : f(x, y) \geq c\}, \text{ where } y \in Y \text{ and } c < \bar{v}\}.$$

Let us show that the family $\mathscr{F}$ has the finite intersection property, i.e.,

$$\bigcap_{i=1}^{n} \{x \in X : f(x, y_i) \geq c_i\} \neq \varnothing, \qquad \text{for each } n \in \mathbb{N},$$

where $y_i \in Y$, $c_i < \bar{v}$, and $i = 1, \ldots, n$. We prove (1) by induction on $n \in \mathbb{N}$.

For $n = 1$, if there exist $y_0 \in Y$ and $c_0 < \bar{v}$ such that

$$\{x \in X : f(x, y_0) \geq c_0\} = \varnothing,$$

then

$$f(x, y_0) < c_0, \qquad \text{for any } x \in X,$$

and

$$\bar{v} = \min_{y \in Y} \max_{x \in X} f(x, y) \leq \max_{x \in X} f(x, y_0) \leq c_0 < \bar{v},$$

which is a contradiction.

For a given $n\in\mathbb{N}$, we suppose that, for any $y_i\in Y$ and $c_i<\bar{v}$, $i=1,\ldots,n$, it holds that

$$\bigcap_{i=1}^{n}\{x\in X: f(x,y_i)\geq c_i\}\neq\varnothing;$$

but by contradiction we suppose that there exist $y_i\in Y$, $i=1,\ldots,n+1$, such that

$$\bigcap_{i=1}^{n+1}\{x\in X: f(x,y_i)\geq c_i\}=\varnothing.$$

Without loss of generality, we may assume that

$$c_1\geq c_2\geq\cdots\geq c_{n+1}.$$

Then,

$$\bigcap_{i=1}^{n+1}\{x\in X: f(x,y_i)\geq c_1\}\subset\bigcap_{i=1}^{n+1}\{x\in X: f(x,y_i)\geq c_i\}=\varnothing.$$

By (A4), there exists a function $S_{\{y_1,y_2\}}:[0,1]\to Y$ such that

$$S_{\{y_1,y_2\}}(0)=y_1,\qquad S_{\{y_1,y_2\}}(1)=y_2.$$

For any $\lambda\in[0,1]$, define

$$I(\lambda):=\bigcap_{i=3}^{n+1}\{x\in X: f(x,y_i)\geq c_1\}\cap\{x\in X: f(x,S_{\{y_1,y_2\}}(\lambda))\geq c_1\}.$$

In the previous definition, it is understood that

$$\bigcap_{i=3}^{n+1}\{x\in X: f(x,y_i)\geq c_1\}=X,\qquad \text{if } n=1.$$

By the induction assumption, $I(\lambda)$ is nonempty and

$$I(0)\cap I(1)=\bigcap_{i=1}^{n+1}\{x\in X: f(x,y_i)\geq c_1\}=\varnothing.$$

On the other hand, from conditions (A2) and (A3), $I(\lambda)$ is closed (hence, compact) and connected. Now, we construct two sequences $\{u_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in\mathbb{N}}$ in T as follows. Let $u_1:=0$ and $w_1:=1$. Suppose that we have defined $u_h\in T$, $w_h\in T$, with $1\leq h\leq k$, such that

$$u_h<w_h,\qquad u_h\in T_h:=\{(i-1)/(2^{h-1}): i=1,\ldots,2^{h-1}\},$$

$$w_h=u_h+1/(2^{h-1}),\qquad u_h\leq u_{h+1},\qquad w_{h+1}\leq w_h,$$

$$I(u_h)\cap I(w_h)=\varnothing,\qquad I((u_h+w_h)/2)\subset I(u_h)\cup I(w_h).$$

Now, we define u_{k+1} and w_{k+1} as follows: by the induction hypotheses and connectedness properties, we have

$$\text{either } I((u_k+w_k)/2)\subset I(u_k), \qquad \text{or } I((u_k+w_k)/2)\subset I(w_k).$$

If $I((u_k+w_k)/2)\subset I(u_k)$, let

$$u_{k+1} := (u_k+w_k)/2\in T \quad \text{and} \quad w_{k+1} := w_k.$$

Otherwise, set

$$u_{k+1} := u_k \quad \text{and} \quad w_{k+1} := (u_k+w_k)/2\in T.$$

Repeating this procedure, we obtain two sequences $\{u_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in N}$ such that

$$I(u_1)\supset I(u_2)\supset I(u_3)\supset\cdots,$$

$$I(w_1)\supset I(w_2)\supset I(w_3)\supset\cdots,$$

with $u_k\to\xi$ and $w_k\to\xi\in[0, 1]$. Denote

$$y^* := S_{\{y_1,y_2\}}(\xi)\in Y,$$

and select $\bar{v}>c^*>c_1$. By the induction assumption, there exists x^* such that

$$x^*\in\bigcap_{i=3}^{n+1}\{x\in X:f(x, y_i)\geq c_1\}\cap\{x\in X:f(x, y^*)\geq c^*\}.$$

Then,

$$f(x^*, y^*)\geq c^*>c_1, \quad f(x^*, y_i)\geq c_1, i=3, \ldots, n+1.$$

From condition (A4), we have

$$\liminf_{n\to\infty} f(x^*, S_{\{y_1,y_2\}}(u_k))\geq f(x^*, S_{\{y_1,y_2\}}(\xi))=f(x^*, y^*)>c_1,$$

$$\liminf_{n\to\infty} f(x^*, S_{\{y_1,y_2\}}(w_k))>c_1.$$

Therefore, there exist u_p and w_q in Y such that

$$f(x^*, S_{\{y_1,y_2\}}(u_p))>c_1, \qquad f(x^*, S_{\{y_1,y_2\}}(w_q))>c_1,$$

so that

$$x^*\in I(u_p)\cap I(w_q).$$

We may assume that $p\leq q$. Since $I(w_p)\supset I(w_q)$, so that $x^*\in I(w_p)$, this contradicts $I(u_p)\cap I(w_p)=\varnothing$. Therefore, $\mathscr{F}$ has the finite intersection property and $\bigcap_{F\in\mathscr{F}} F\neq\varnothing$ as F is compact for each $F\in\mathscr{F}$.

If $\underline{v} < \bar{v}$, there exists $c \in \mathbb{R}$ such that $\underline{v} < c < \bar{v}$. Select $x_0 \in \bigcap_{F \in \mathcal{F}} F$. Then,

$$f(x_0, y) \geq c, \qquad \text{for all } y \in Y,$$

and we have

$$\underline{v} = \max_{x \in X} \min_{y \in Y} f(x, y) \geq \min_{y \in Y} f(x_0, y) \geq c > \underline{v},$$

which is a contradiction; hence, $\underline{v} = \bar{v}$ and we complete the proof. □

Remark 3.1. The proof of Theorem 3.1 actually tells us that the following minimax theorem holds. For the convenience of our readers, we state it as follows.

Corollary 3.1. Let X be a compact Hausdorff topological space, and let Y be a Hausdorff topological space, not necessarily compact. Suppose that the function $f: X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ satisfies conditions (A1)–(A4) of Theorem 3.1. Then, we have

$$\max_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \max_{x \in X} f(x, y).$$

In particular, we have the following minimax theorem as an application of Theorem 3.1.

Theorem 3.2. Let X be a compact Hausdorff topological space, and let Y be a nonempty convex subset of a Hausdorff topological vector space F. Suppose that $f: X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a function such that the following conditions are satisfied:

(B1) for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semicontinuous;
(B2) for each fixed $y \in Y$, $x \mapsto f(x, y)$ is upper semicontinuous;
(B3) for each $c \in \mathbb{R}$, $m \in \mathbb{N}$, and $y_i \in \mathrm{Y}$, $\mathrm{i} = 1, \ldots, \mathrm{m}$, the set

$$\bigcap_{i=1}^{m} \{x \in X: f(x, y_i) \geq c\}$$

is either connected or empty;
(B4) for each fixed $x \in X$, $y \mapsto f(x, y)$ is quasiconvex.

Then, we have

$$\max_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \max_{x \in X} f(x, y).$$

In particular, if Y is compact, then f has a saddle point in $X \times Y$.

Proof. It suffices to verify that f satisfies all the hypotheses of Theorem 3.1. For each $y_1, y_2 \in Y$, define $S_{\{y_1, y_2\}}: [0, 1] \to Y$ by

$$S_{\{y_1, y_2\}}(\lambda) = (1 - \lambda) y_1 + \lambda y_2.$$

For each $x \in X$, $n \in \mathbb{N}$, $j = 1, \ldots, 2^{n-1}$, by (4) we have

$$\begin{aligned} &f(x, (1-(2j-1)/(2^n))y_1 + (2j-1)/(2^n)y_2)) \\ &\leq \max\{f(x, (1-(j-1)/(2^{n-1}))y_1 + (j-1)/(2^{n-1})y_2), \\ &\qquad f(x, (1-j/(2^{n-1}))y_1 + j/(2^{n-1})y_2)\}. \end{aligned}$$

By (B1) and since $S_{\{y_1, y_2\}}$ is continuous, for each $\xi \in [0, 1]$ we also have

$$\liminf_{\lambda \in T \text{ and } \lambda \to \xi} f(x, S_{\{y_1, y_2\}}(\lambda)) \geq f(x, S_{\{y_1, y_2\}}(\xi)).$$

Thus, all the hypotheses of Theorem 3.1, except the compactness of Y, are satisfied. By Theorem 3.1 and Remark 3.1, it follows that

$$\max_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \max_{x \in X} f(x, y),$$

and the proof is completed. □

Remark 3.2. As the minimax equality plays a very important role in the study of optimization and other subjects, the existence of equalities in the topological minimax equality theory has been studied extensively by many authors in recent years; e.g., see Refs. 3–6, 9, 11. However, the topological minimax theorems from Refs. 3–6, 9, 11 are not comparable with our Theorems 3.1 and 3.2. Moreover, Theorems 3.1 and 3.2 include corresponding results of Refs. 1–3, 7–8, 12, 14–16 as special cases.

As an immediate consequence of Theorem 3.2, we have the following saddle-point theorem in topological vector spaces, which includes the Sion classical minimax inequality in Ref. 2 as a special case.

Corollary 3.2. Let X be a nonempty compact and convex subset of a Hausdorff topological vector space E, and let Y be a nonempty convex subset of a Hausdorff topological space F, respectively. Suppose that $f: X \times Y \to \mathbb{R}$ satisfies the conditions below:

(C1) for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semicontinuous and quasiconvex;

(C2) for each fixed $y \in Y$, $x \mapsto f(x, y)$ is upper semicontinuous and quasiconcave.

Then, we have

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Moreover, if Y is compact, then f has a saddle point in $X \times Y$.

Proof. For each fixed $m \in \mathbb{N}$, each $y_i \in Y$, $i = 1, \ldots, m$, and for each fixed $c \in \mathbb{R}$, if

$$\bigcap_{i=1}^{m} \{x \in X : f(x, y_i) \geq c\} \neq \varnothing,$$

then by (C2), the set

$$\{x \in X : f(x, y_i) \geq c\}$$

is convex for $i = 1, \ldots, m$. Therefore,

$$\bigcap_{i=1}^{m} \{x \in X : f(x, y_i) \geq c\}$$

is nonempty convex, and hence nonempty connected. Thus, all the hypotheses of Theorem 3.2 are satisfied. By Theorem 3.2, the conclusion follows. □

4. Conclusions

In this note, we have established a topological minimax result (i.e., Theorem 3.1), which allows us to consider the existence of saddle points without the traditional linear structure. Also, our results are new and independent from previous results in the literature (Refs. 3–16).

Finally, we note that, for the application of the saddle-point theorem to mathematical economics and game theory, the interested reader can find more details in Aubin and Ekeland (Ref. 17), Border (Ref. 18), Ichiishi (Ref. 19), and references therein. Furthermore, a comprehensive bibliography for the study of topological versions of saddle points and minimax theorems can be found in Simons (Ref. 12).

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