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On the outer Minkowski content of sets

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Abstract We provide general conditions, stable under finite unions, ensuring the existence of the outer Minkowski content of Borel subsets of $\mathbb{R}^d$. Such conditions turn out to be the same which guarantee the existence of the $(d-1)$-dimensional Minkowski content of the boundary of the involved sets. Moreover, our results also apply to the study of the differentiability of the volume function of bounded sets, extending some known results in literature.

Keywords Outer Minkowski content · Sets of finite perimeter

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1 Introduction and main result

Denoted by $E_{\oplus r}$ the parallel set of a subset E of $\mathbb{R}^d$ at distance r and by $\mathcal{H}^n$ the n-dimensional Hausdorff measure in $\mathbb{R}^d$, the *outer Minkowski content* $\mathcal{SM}(E)$ of E is the quantity so defined

$$\mathcal{SM}(E) := \lim_{r\downarrow 0} \frac{\mathcal{H}^d(E_{\oplus r}\setminus E)}{r},$$

provided that the limit exists finite, and it is of interest in many problems arising from real applications (see [1] and reference therein). While quite general results on the existence of the n-dimensional Minkowski content of compact subsets of $\mathbb{R}^d$ are available in literature, only partial results on the outer Minkowski content are known. The most recent paper on this subject until now [1] provides a class of sets stable under finite unions for which the outer Minkowski content exists and equals the perimeter (in the sense of geometric measure theory) of the involved sets, containing, for instance, all sets with Lipschitz boundary and a type of sets with positive reach. Simple examples show that the outer Minkowski content of a set can be greater than its perimeter, but general results about its value are not available in

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the literature yet. This is the main goal of the present paper. Improving some techniques in [1], we prove here that the existence of the outer Minkowski content of a subset E of $\mathbb{R}^d$ is ensured by the same well-known conditions which guarantee the existence of the $(d-1)$-dimensional Minkowski content of its boundary. Namely, referring to the next section for formal definitions and notation, our main theorem (see Sect. 3) states that

$$\mathcal{SM}(E) = P(E) + 2\mathcal{H}^{d-1}(\partial E \cap E^0)$$

(here $P(E)$ denotes the perimeter of E, and E^0 is the set of points where E has null density) for any subset E of $\mathbb{R}^d$ which belongs to the following class of sets, stable under finite unions.

Definition 1.1 (The class $\mathcal{O}$) Let $\mathcal{O}$ be the class of Borel sets E of $\mathbb{R}^d$ such that

(i) ∂E is a countably $\mathcal{H}^{d-1}$-rectifiable bounded set;
(ii) there exist $\gamma > 0$ and a probability measure η in $\mathbb{R}^d$ absolutely continuous with respect to $\mathcal{H}^{d-1}$ such that

$$\eta(B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial E, \ \forall r \in (0,1).$$

As it will emerge in the sequel, the density of E at its boundary points plays a central role in the determination of the value of $\mathcal{SM}(E)$.

In the last section a series of further results, which follow as applications of the main theorem, are provided. For instance, we observe that the proof of the above-mentioned formula for $\mathcal{SM}(E)$ also applies to Borel sets with $(d-1)$-rectifiable boundary, and so to unions of compact sets with positive reach. In particular, we show that the same conclusions stated for the class $\mathcal{O}$, still hold for another class of sets, defined similarly to the class $\mathcal{O}$, by replacing the condition of absolute continuity of η with the assumption that ∂E admits $(d-1)$-dimensional Minkowski content. Finally, we study the differentiability of the so-called *volume function*

$$V_E(r) := \mathcal{H}^d(E_{\oplus r}), \qquad r \geq 0 \tag{1.1}$$

of a given bounded subset E of $\mathbb{R}^d$, at $r > 0$ (clearly, the existence of the right derivative in $r = 0$ corresponds to the existence of $\mathcal{SM}(E)$), improving, in particular, a recent result of Hug et al. in [2].

2 Basic notation and preliminaries

Throughout the paper $\mathcal{H}^n$ is the n-dimensional Hausdorff measure, $\mathcal{B}_{\mathbb{R}^d}$ is the Borel σ-algebra of $\mathbb{R}^d$ and $\mathcal{H}^n_{|A}$ denotes the restriction of $\mathcal{H}^n$ to a $\mathcal{H}^n$-measurable set $A \subset \mathbb{R}^d$ (i.e. $\mathcal{H}^n_{|A}(E) = \mathcal{H}^n(A \cap E)$ for all $E \in \mathcal{B}_{\mathbb{R}^d}$). We shall mainly follow the notation used in [1], to which we refer for a more detailed presentation of some common definitions and results introduced in the present section. For $r \geq 0$ and $x \in \mathbb{R}^d$, $B_r(x)$ is the closed ball with center x and radius r, while for every integer n we denote by b_n the volume of the unit ball in $\mathbb{R}^n$ (for $n = 0$, we set $b_0 := 1$).

Given a subset E of $\mathbb{R}^d$, ∂E will be its (topological) boundary, E^c the complement set of E, int$E$ and cl$E$ the interior and the closure of E, respectively. We denote by $E_{\oplus r} := \{x \in \mathbb{R}^d : \text{dist}(x, E) \leq r\}$ (where "dist" is the usual distance function) the parallel set of E at distance r, and by $d_E : \mathbb{R}^d \to \mathbb{R}$ the *signed distance function* from E, defined as follows

$$d_E(x) := \text{dist}(x, E) - \text{dist}(x, E^c).$$

The *upper* and *lower* outer Minkowski content of E are defined, respectively, as

$$\mathcal{SM}^*(E) := \limsup_{r\downarrow 0} \frac{\mathcal{H}^d(E_r\backslash E)}{r}, \qquad \mathcal{SM}_*(E) := \liminf_{r\downarrow 0} \frac{\mathcal{H}^d(E_r\backslash E)}{r}.$$

We say that a compact set $S \subset \mathbb{R}^d$ is n-rectifiable if it is representable as the image of a compact set $K \subset \mathbb{R}^n$, with $f : \mathbb{R}^n \to \mathbb{R}^d$ Lipschitz, and we recall that, given a subset S of $\mathbb{R}^d$ and an integer n with $0 \le n \le d$, the *n-dimensional Minkowski content* of S is defined by

$$\mathcal{M}^n(S) := \lim_{r\downarrow 0} \frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}},$$

whenever the limit exists finite. The following theorem is proved in [3, p. 275].

Theorem 2.1 (Federer) $\mathcal{M}^n(S) = \mathcal{H}^n(S)$ *for any compact n-rectifiable set* $S \subset \mathbb{R}^d$.

We also remind that a set $S \subset \mathbb{R}^d$ is said to be countably $\mathcal{H}^n$-rectifiable if there exist countably many n-dimensional Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$ such that $S\backslash \bigcup_i f_i(\mathbb{R}^n)$ is $\mathcal{H}^n$-negligible; if, furthermore, $\mathcal{H}^n(S) < \infty$, then S is said $\mathcal{H}^n$-rectifiable. An extension of Federer's theorem to this more general class of sets is given by the following theorem, proved in [4, p. 110].

Theorem 2.2 *Let* $S \subset \mathbb{R}^d$ *be a countably* $\mathcal{H}^n$*-rectifiable compact set and assume that*

$$\eta(B_r(x)) \ge \gamma r^n \qquad \forall x \in S,\ \forall r \in (0,1) \tag{2.1}$$

holds for some $\gamma > 0$ *and some Radon measure* η *in* $\mathbb{R}^d$ *absolutely continuous with respect to* $\mathcal{H}^n$. *Then* $\mathcal{M}^n(S) = \mathcal{H}^n(S)$.

Remark 2.3 If a Radon measure η as in Theorem 2.2 exists, then it can be assumed to be a probability measure without loss of generality. Indeed, it is sufficient to consider the measure $\tilde{\eta}(\cdot) := \eta(W \cap \cdot)/\eta(W)$, where W is a compact subset of $\mathbb{R}^d$ such that $S_{\oplus 1} \subset W$.

The two aforementioned theorems provide the more general sufficient conditions available in the literature concerning the existence of the n-dimensional Minkowski content of compact sets. We shall show that the same conditions on the boundary of a subset E of $\mathbb{R}^d$ guarantee the existence of its outer Minkowski content. To this end, we briefly recall now some definitions and results from geometric measure theory, that will be used in the next sections. (We refer to [4] for further details and remarks.)

If μ is a positive Radon measure in $\mathbb{R}^d$ and $n \in \{1, \ldots, d\}$, then the following implication holds:

$$B \in \mathcal{B}_{\mathbb{R}^d},\ \mu(B) = 0 \quad \Longrightarrow \quad \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{b_n r^n} = 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in B. \tag{2.2}$$

Considering now the restriction of $\mathcal{H}^n$ to a $\mathcal{H}^n$-measurable subset of $\mathbb{R}^d$, the next definition is given.

Definition 2.4 (n-dimensional densities) For any $\mathcal{H}^n$-measurable set $S \subset \mathbb{R}^d$, the *n-dimensional density of S at x* is defined by

$$\Theta_n(S, x) := \lim_{r\downarrow 0} \frac{\mathcal{H}^n(S \cap B_r(x))}{b_n r^n},$$

provided that the limit exists.

It is clear that, since open balls can be approximated from inside by closed balls and closed balls can be approximated from outside by open balls, the limits above do not change if we replace closed balls by open balls.

A classical rectifiability criterium, relying on density properties of the set, tells us that a Borel set $S \subset \mathbb{R}^d$ with $\mathcal{H}^n(S) < \infty$ is $\mathcal{H}^n$-rectifiable if and only if

$$\Theta_n(S, x) = 1 \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in S. \tag{2.3}$$

Let E be a $\mathcal{H}^d$-measurable subset of $\mathbb{R}^d$. It is clear that $\Theta_d(E, x)$ equals 1 for all $x \in \text{int}E$, and 0 for all $x \in \text{int}(E^c)$, while different values can be assumed at the boundary points of E. The set of points where the density is neither 0 nor 1 is called essential boundary and its $\mathcal{H}^{d-1}$ measure is closely related to the notion of *perimeter*.

Definition 2.5 For every $t \in [0, 1]$ and every $\mathcal{H}^d$-measurable set $E \subset \mathbb{R}^d$ let

$$E^t := \{x \in \mathbb{R}^d : \Theta_d(E, x) = t\}.$$

All the sets E^t are Borel sets; in particular, the set $\partial^* E := \mathbb{R}^d \backslash (E^0 \cup E^1)$ is called *essential boundary* of E.

Let χ_E be the characteristic function of E and $B \subseteq \mathbb{R}^d$ be an open set. The *perimeter of E in B* is defined as the total variation $|D\chi_E|$ in B (see also [1]). More generally, if E has finite perimeter in B, we define

$$P(E, A) := |D\chi_E|(A)$$

for any Borel set $A \subseteq B$. In the sequel we will write $P(E)$ instead of $P(E, \mathbb{R}^d)$. E is said to have finite perimeter in A if $P(E, A) < \infty$. General theorems on sets with finite perimeter (see [4, §3.5]) guarantee that if E has finite perimeter in an open set $B \subset \mathbb{R}^d$, then the measures $|D\chi_E|$ and $\mathcal{H}^{d-1}_{|\partial^* E}$ coincide on the Borel subsets of B; as a consequence, the perimeter measure can be computed in terms of the $\mathcal{H}^{d-1}$ measure, and in particular the following equalities can be proved

$$P(E, A) = \mathcal{H}^{d-1}(\partial^* E \cap A) = \mathcal{H}^{d-1}(E^{1/2} \cap A). \tag{2.4}$$

Finally, noticing that $\partial^* E \subseteq \partial E$, it holds $P(E) \leq \mathcal{H}^{d-1}(\partial E)$.

3 Proof of the main result

Theorem 3.1 (Main result) *The class $\mathcal{O}$ is stable under finite unions and any $E \in \mathcal{O}$ admits outer Minkowski content, given by*

$$\mathcal{SM}(E) = P(E) + 2\mathcal{H}^{d-1}(\partial E \cap E^0). \tag{3.1}$$

Before entering into the technical details of the proof, let us say a few words about the idea. It is based on the different roles played by the different densities of E at its boundary points. Intuitively, a small neighbourhood of a point $x \in E^1 \cap \partial E$ is "almost all contained" in E, so that it gives no contribution to the volume of $E_{\oplus r} \backslash E$; thus, roughly speaking, we may say that x has negligible weight in the computing of the outer Minkowski content of E. Conversely, if E has null density in $x \in \partial E$, then, in a small neighbourhood of x, $E_{\oplus r} \backslash E$ "almost all coincides" with the Minkowski enlargement of ∂E, so that, roughly speaking, we may say that the weight of x in the computing of the outer Minkowski content of E is twice the weight of a point $y \in E^{1/2}$. This gives an intuitive explanation of the formula (3.1).

It is easy to prove the following lemma, which will be applied in the proof of Theorem 3.1 to make use of the symmetric role of E and E^c, noticing that $\Theta_d(E^c, x) = 1 - \Theta_d(E, x)$ and $\mathcal{H}^d((\partial E)_{\oplus r}) = \mathcal{H}^d(E_{\oplus r} \setminus E) + \mathcal{H}^d((E^c)_{\oplus r} \setminus E^c)$.

Lemma 3.2 *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences in* $\mathbb{R}$. *If*

$$\limsup_{n\to\infty} (a_n + b_n) \le (a + b), \quad \liminf_{n\to\infty} a_n \ge a, \quad \liminf_{n\to\infty} b_n \ge b,$$

with a *and* b *finite, then* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$.

The next statement is proved in [1].

Lemma 3.3 *Let* $G \subset \mathbb{R}^d$ *be a Borel set and assume that there exist* $\gamma > 0$ *and a probability measure* η *such that*

$$\eta(B_r(x)) \ge \gamma r^{d-1} \qquad \forall x \in \partial G, \ \forall r \in (0, 1).$$

Then

$$\limsup_{r\downarrow 0} \frac{\mathcal{H}^d((\partial G)_{\oplus r} \cap G \cap B_\rho(x))}{r} = o(\rho^{d-1})$$

for $\mathcal{H}^{d-1}$*-a.e.* $x \in G^0 \cap \partial G$.

Let us observe that condition (ii) in Definition 1.1 implies that $\mathcal{H}^{d-1}(\partial E) < \infty$ (e.g. see [1, Remark 2]), and denote by $\mathcal{SM}_*(E, A)$ and $\mathcal{SM}^*(E, A)$ the *lower* and *upper outer Minkowski content of* E *in* $A \in \mathcal{B}_{\mathbb{R}^d}$, respectively, i.e.

$$\mathcal{SM}_*(E, A) := \liminf_{r\downarrow 0} \frac{\mathcal{H}^d((E_{\oplus r} \setminus E) \cap A)}{r}, \quad \mathcal{SM}^*(E, A) := \limsup_{r\downarrow 0} \frac{\mathcal{H}^d((E_{\oplus r} \setminus E) \cap A)}{r}.$$

Lemma 3.4 *For any* $E \in \mathcal{O}$ *it holds*

(a) $\mathcal{SM}_*(E, B_\rho(x)) = o(\rho^{d-1})$ *for* $\mathcal{H}^{d-1}$*-a.e.* $x \in E^1 \cap \partial E$,
(b) $\mathcal{SM}_*(E, B_\rho(x)) \ge \mathcal{H}^{d-1}(E^{1/2} \cap \mathrm{int} B_\rho(x))$ *for* $\mathcal{H}^{d-1}$*-a.e.* $x \in E^{1/2}$,
(c) $\mathcal{SM}_*(E, B_\rho(x)) \ge 2\mathcal{H}^{d-1}(E^0 \cap \partial E \cap \mathrm{int} B_\rho(x)) + o(\rho^{d-1})$ *for* $\mathcal{H}^{d-1}$*-a.e.* $x \in E^0 \cap \partial E$.

Proof Equality (a) follows directly by Lemma 3.3 with $G = E^c$.

From the co-area formula and the fact that ∂E is $\mathcal{H}^d$-negligible it can be proved (see [1]) that $\mathcal{SM}_*(E, A) \ge P(E, A)$ for any open set A in $\mathbb{R}^d$. Then, taking into account (2.4), we get

$$\mathcal{SM}_*(E, B_\rho(x)) \ge \mathcal{H}^{d-1}(E^{1/2} \cap \mathrm{int} B_\rho(x)),$$

and so (b) in particular.

For any closed set C well contained in $B_\rho(x)$ (i.e. the Hausdorff distance between C and the complement of $B_\rho(x)$ is greater than 0) there exists $\tilde{r} > 0$ such that $C_{\oplus r} \subset B_\rho(x)$, $\forall r < \tilde{r}$. So, noticing that $\partial E \cap C$ satisfies the assumptions of Theorem 2.2, we get that

$$2\mathcal{H}^{d-1}(\partial E \cap C) = \liminf_{r\downarrow 0} \frac{\mathcal{H}^d((\partial E \cap C)_{\oplus r})}{r} \le \liminf_{r\downarrow 0} \frac{\mathcal{H}^d((\partial E)_{\oplus r} \cap C_{\oplus r})}{r}$$
$$\le \liminf_{r\downarrow 0} \frac{\mathcal{H}^d((\partial E)_{\oplus r} \cap B_\rho(x))}{r}.$$

Let $\{C_n\}_{n\in\mathbb{N}}$ be an increasing sequence of closed sets well contained in $B_\rho(x)$ such that $C_n \nearrow \mathrm{int}B_\rho(x)$. By taking the limit as n tends to ∞, we obtain that

$$\liminf_{r\downarrow 0}\frac{\mathcal{H}^d((\partial E)_{\oplus r}\cap B_\rho(x))}{r}\geq \lim_{n\to\infty}2\mathcal{H}^{d-1}(\partial E\cap C_n)=2\mathcal{H}^{d-1}(\partial E\cap \mathrm{int}B_\rho(x)). \tag{3.2}$$

Finally, from $\mathcal{H}^d((\partial E)_{\oplus r}\cap E^c\cap B_\rho(x))=\mathcal{H}^d(E_{\oplus r}\cap E^c\cap B_\rho(x))$ we have that

$$\begin{aligned}\mathcal{SM}_*(E,B_\rho(x)) &= \liminf_{r\downarrow 0}\frac{\mathcal{H}^d((\partial E)_{\oplus r}\cap B_\rho(x))-\mathcal{H}^d((\partial E)_{\oplus r}\cap E\cap B_\rho(x))}{r}\\ &\geq \liminf_{r\downarrow 0}\frac{\mathcal{H}^d((\partial E)_{\oplus r}\cap B_\rho(x))}{r}-\limsup_{r\downarrow 0}\frac{\mathcal{H}^d((\partial E)_{\oplus r}\cap E\cap B_\rho(x))}{r}.\end{aligned}$$

Then (c) follows by (3.2) and Lemma 3.3. □

Proof of Theorem 3.1 We have observed that $\mathcal{H}^{d-1}(\partial E)<\infty$, and so E has finite perimeter in $\mathbb{R}^d$. Theorem 3.61 in [4] states that any subset of $\mathbb{R}^d$ of finite perimeter has density either 0 or 1 or 1/2 at $\mathcal{H}^{d-1}$-almost every point of its boundary; therefore,

$$\mathcal{H}^{d-1}(\partial E)=\mathcal{H}^{d-1}(\partial E\cap E^0)+\mathcal{H}^{d-1}(E^{1/2})+\mathcal{H}^{d-1}(\partial E\cap E^1). \tag{3.3}$$

Note also that ∂E satisfies the assumptions of Theorem 2.2, hence

$$\mathcal{M}^{d-1}(\partial E)=\mathcal{H}^{d-1}(\partial E). \tag{3.4}$$

Let us show that the following lower bound for $\mathcal{SM}_*(E)$ holds:

$$\mathcal{SM}_*(E)\geq \mathcal{H}^{d-1}(E^{1/2})+2\mathcal{H}^{d-1}(\partial E\cap E^0). \tag{3.5}$$

Let μ be the measure in $\mathbb{R}^d$ so defined

$$\mu(\cdot):=2\mathcal{H}^{d-1}(E^0\cap\partial E\cap\cdot)+\mathcal{H}^{d-1}(E^{1/2}\cap\cdot).$$

Taking into account [by (2.3) and (2.2)] that the limit

$$\lim_{\rho\downarrow 0}\frac{\mathcal{H}^{d-1}(E^0\cap\partial E\cap B_\rho(x))}{\rho^{d-1}}$$

equals b_{d-1} for $\mathcal{H}^{d-1}$-a.e. $x\in E^0\cap\partial E$, and 0 for $\mathcal{H}^{d-1}$-a.e. $x\in(E^0\cap\partial E)^c$, and analogous conclusions hold for the limits

$$\lim_{\rho\downarrow 0}\frac{\mathcal{H}^{d-1}(E^1\cap\partial E\cap B_\rho(x))}{\rho^{d-1}}\quad\text{and}\quad \lim_{\rho\downarrow 0}\frac{\mathcal{H}^{d-1}(E^{1/2}\cap B_\rho(x))}{\rho^{d-1}},$$

from Lemma 3.4 we get that, for any $\varepsilon>0$,

$$\liminf_{\rho\downarrow 0}\frac{\mathcal{SM}_*(E,B_\rho(x))+\varepsilon\mathcal{H}^{d-1}(E^1\cap\partial E\cap B_\rho(x))}{\mu(\mathrm{int}B_\rho(x))}\geq 1\quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x\in\partial E \tag{3.6}$$

(Note that the term $\varepsilon\mathcal{H}^{d-1}(E^1\cap\partial E\cap B_\rho(x))$ in the above-mentioned fraction is to avoid an indetermination of type 0/0 at points $x\in E^1\cap\partial E$.) Since the family of closed balls $B_\rho(x)$ with $\mu(\partial B_\rho(x))=0$ is a fine cover of ∂E, by Vitali-Besicovitch covering theorem (e.g. see [4, p. 52]), for any $\delta > 0$ there exist finitely many disjoint closed balls $C_1, \dots, C_N$ with $\mu(\partial C_i) = 0$ such that

$$\mu(\partial E \setminus \cup_i C_i) < \delta.$$

The balls C_i can be chosen with centers in ∂E and such that

$$\mathcal{SM}_*(E, C_i) + \varepsilon \mathcal{H}^{d-1}(E^1 \cap \partial E \cap C_i) \overset{(3.6)}{\geq} (1-\delta)\mu(C_i), \qquad i = 1, \dots, N.$$

Note also that $\mu(\mathbb{R}^d \setminus \cup_i C_i) = \mu(\partial E \setminus \cup_i C_i)$, being the support of μ contained in ∂E. Thus the following chain of inequalities holds:

$$\begin{aligned}
\mathcal{SM}_*(E) + \varepsilon \mathcal{H}^{d-1}(E^1 \cap \partial E) &\geq \mathcal{SM}_*\left(E, \bigcup_{i=1}^N C_i\right) + \varepsilon \mathcal{H}^{d-1}\left(E^1 \cap \partial E \cap \bigcup_{i=1}^N C_i\right) \\
&\geq \sum_{i=1}^N \Big(\mathcal{SM}_*(E, C_i) + \varepsilon \mathcal{H}^{d-1}(E^1 \cap \partial E \cap C_i)\Big) \\
&\geq (1-\delta) \sum_{i=1}^N \mu(C_i) = (1-\delta)\mu\left(\bigcup_{i=1}^N C_i\right) \\
&= (1-\delta)\left(\mu(\mathbb{R}^d) - \mu\left(\mathbb{R}^d \setminus \bigcup_{i=1}^N C_i\right)\right) \\
&\geq (1-\delta)\Big(2\mathcal{H}^{d-1}\big(E^0 \cap \partial E\big) + \mathcal{H}^{d-1}(E^{1/2}) - \delta\Big).
\end{aligned}$$

By taking now the limit as $\delta \downarrow 0$, and then as $\varepsilon \downarrow 0$, we obtain the inequality (3.5).

Observing that E^c belongs to $\mathcal{O}$ too, we can also claim that

$$\mathcal{SM}_*(E^c) \geq \mathcal{H}^{d-1}((E^c)^{1/2}) + 2\mathcal{H}^{d-1}(\partial E^c \cap (E^c)^0). \tag{3.7}$$

By the co-area formula it can be shown (see [1]) that

$$\mathcal{H}^d((\partial E)_{\oplus r}) = \int_{-r}^{r} \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t,$$

and, similarly,

$$\mathcal{H}^d(E_{\oplus r} \setminus E) = \int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t.$$

As $d_{E^c}(x) = -d_E(x)$, from the above-mentioned equation we also have

$$\mathcal{H}^d((E^c)_{\oplus r} \setminus E^c) = \int_{-r}^{0} \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t. \tag{3.8}$$

Let

$$a_r := \frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t, \quad b_r := \frac{1}{r} \int_{-r}^{0} \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t,$$

and observe that

$$\liminf_{r\downarrow 0} a_r = \mathcal{SM}_*(E) \overset{(3.5)}{\geq} \mathcal{H}^{d-1}(E^{1/2}) + 2\mathcal{H}^{d-1}(\partial E \cap E^0) =: a,$$

$$\liminf_{r\downarrow 0} b_r = \mathcal{SM}_*(E^c) \overset{(3.7)}{\geq} \mathcal{H}^{d-1}(E^{1/2}) + 2\mathcal{H}^{d-1}(\partial E \cap E^1) =: b,$$

and

$$\limsup_{r\downarrow 0}(a_r + b_r) = \limsup_{r\downarrow 0} \frac{\mathcal{H}^d((\partial E)_{\oplus r})}{r} = 2\mathcal{M}^{d-1}(\partial E) \overset{(3.4)}{=} 2\mathcal{H}^{d-1}(\partial E) \overset{(3.3)}{=} a + b.$$

By applying now Lemma 3.2 we obtain that

$$\mathcal{SM}(E) = P(E) + 2\mathcal{H}^{d-1}(\partial E \cap E^0).$$

Finally, it is easy to check that the class $\mathcal{O}$ is stable under finite unions. □

4 Further results and remarks

In this section we show how applications of Theorem 3.1 provide a series of general results which turn out to be in accordance with available results in literature for some classes of sets.

We know (see [1, Remark 1]) that for any n-rectifiable compact set $S \subset \mathbb{R}^d$ there exist $\gamma > 0$ and a probability measure η in $\mathbb{R}^d$ satisfying (2.1). We also point out that the further assumption of absolute continuity of η with respect to $\mathcal{H}^{d-1}$ in the definition of the class $\mathcal{O}$ is used in the proof of the formula (3.1) in order to apply Theorem 2.2 to guarantee that $\mathcal{M}^{d-1}(\partial E \cap C) = \mathcal{H}^{d-1}(\partial E \cap C)$ for any closed set $C \subset \mathbb{R}^d$ (which leads to the proof of (*c*) in Lemma 3.4 and to the existence of the $(d-1)$-dimensional Minkowski content of ∂E at the very beginning of the proof of Theorem 3.1); therefore, by Federer's Theorem 2.1, we can claim that the proof of formula (3.1) still works for sets whose boundary is $(d-1)$-rectifiable and bounded. This proves the following statement.

Proposition 4.1 $\mathcal{SM}(E) = P(E) + 2\mathcal{H}^{d-1}(\partial E \cap E^0)$ *for any Borel set* $E \subset \mathbb{R}^d$ *such that* ∂E *is* $(d-1)$*-rectifiable and bounded.*

Remark 4.2 Both in Theorem 3.1 and in the proposition stated above, E is a Borel set whose boundary satisfies the assumptions of Theorem 2.2 (for countably $\mathcal{H}^{d-1}$-rectifiable sets) and Federer's Theorem 2.1 (for $(d-1)$-rectifiable sets), respectively, which guarantee that $\mathcal{M}^{d-1}(\partial E)$ exists equal to $\mathcal{H}^{d-1}(\partial E)$. Hence, in relation to the outer Minkowski content of sets, taking into account Remark 2.3, Proposition 4.1 and Theorem 3.1 can be regarded as the corresponding results to Theorem 2.1 and Theorem 2.2, respectively.

Furthermore, since $\mathcal{H}^d(E) = \mathcal{H}^d(\mathrm{cl}E)$, we can state that for any Borel subset E of $\mathbb{R}^d$ with empty interior and such that its boundary is $(d-1)$-rectifiable and bounded, or more in general, satisfies the conditions (i) and (ii) of Definition 1.1, it holds

$$\mathcal{SM}(E) = 2\mathcal{M}^{d-1}(E) = 2\mathcal{H}^{d-1}(\partial E).$$

Note that the class of compact sets with $(d-1)$-rectifiable boundary is stable under finite unions (because $\partial(\cup_{i=1}^N E_i) \subseteq \cup_{i=1}^N \partial E_i$); then, reminding that the boundary of a compact set with positive reach is $(d-1)$-rectifiable (e.g. see [1, Proposition 3]), as a direct application of Proposition 4.1, we get the following general result for unions of sets with positive reach. (For an exhaustive treatment of sets with positive reach see [5].)

Corollary 4.3 *Let $E \subset \mathbb{R}^d$ be a finite union of compact sets with positive reach; then E admits outer Minkowski content, given by* (3.1).

Remark 4.4 Available results in current literature concerning the existence of the outer Minkowski content for unions of sets with positive reach require additional regularity assumptions on the involved sets; we mention the paper [6], where the existence of the outer Minkowski content is proved for unions $\cup_i A_i$ of sets A_i with positive reach such that all possible finite intersections of the A_i's have positive reach as well, and the paper [1], where a certain condition on the normal cone characterizes, among all finite unions of sets with positive reach, those for which the outer Minkowski content coincides with the $\mathcal{H}^{d-1}$ measure of the boundary. Hence, Corollary 4.3 extends such results, and we may also notice that it is in accordance with Example 1 and Example 2 in [1].

In [1] the following class of sets stable under finite unions has been introduced. (To be precise, in the definition given in [1], compact sets are considered, having in mind real applications concerning closed sets as discussed in such paper, but it is easy to see that the same results concerning compact sets in $\mathcal{S}$ still holds for Borel sets with bounded boundary.)

Definition 4.5 (The class $\mathcal{S}$) Let $\mathcal{S}$ be the class of Borel sets E of $\mathbb{R}^d$ with bounded boundary such that

- there exist $\gamma > 0$ and a probability measure η such that

$$\eta(B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial E,\ \forall r \in (0,1);$$

- E admits outer Minkowski content and $\mathcal{SM}(E) = P(E)$.

Note that, in the above-mentioned definition, the absolute continuity of η with respect to $\mathcal{H}^{d-1}$ is not required; on the other hand, the outer Minkowski content of the involved sets is assumed to exist equal to the perimeter. In [1] classes of sets satisfying such condition are provided. Similarly, starting from the definition of the class $\mathcal{O}$, we introduce now a new class of Borel sets E of $\mathbb{R}^d$, replacing the condition of absolute continuity of η with the assumption that ∂E admits $(d-1)$-dimensional Minkowski content.

Definition 4.6 [The class $\mathcal{O}'$] Let $\mathcal{O}'$ be the class of Borel sets E of $\mathbb{R}^d$ such that

(i') ∂E is a countably $\mathcal{H}^{d-1}$-rectifiable bounded set and $\mathcal{M}^{d-1}(\partial E) = \mathcal{H}^{d-1}(\partial E)$;
(ii') there exist $\gamma > 0$ and a probability measure η in $\mathbb{R}^d$ such that

$$\eta(B_r(x)) \geq \gamma r^{d-1} \qquad \forall x \in \partial E,\ \forall r \in (0,1).$$

In order to state for $\mathcal{O}'$ an analogous result to Theorem 3.1, let us prove that the class of countably $\mathcal{H}^k$-rectifiable closed sets which admit Minkowski content is stable under finite unions.

Proposition 4.7 *Let A_1 and A_2 be countably $\mathcal{H}^k$-rectifiable closed sets such that $\mathcal{M}^k(A_i) = \mathcal{H}^k(A_i)$ for $i = 1, 2$. Then the union set $A_1 \cup A_2$ admits k-dimensional Minkowski content $\mathcal{M}^k(A_1 \cup A_2) = \mathcal{H}^k(A_1 \cup A_2)$, as well.*

Proof We first prove that if $E \subset \mathbb{R}^d$ is a countably $\mathcal{H}^k$-rectifiable closed set such that $\mathcal{M}^k(E) = \mathcal{H}^k(E)$, then

$$\mathcal{M}^k(E \cap C) = \mathcal{H}^k(E \cap C) \tag{4.1}$$

for any closed set $C \subset \mathbb{R}^d$.

Reminding that $\mathcal{M}_*^k(S) \geq \mathcal{H}^k(S)$ for any countably $\mathcal{H}^k$-rectifiable closed set S (see [4, Proposition 2.101]), we have that $\mathcal{M}_*^k(E \cap C) \geq \mathcal{H}^k(E \cap C)$. Let us show that the opposite inequality holds for $\mathcal{M}^{*k}(E \cap C)$. Consider the sequence $\{C_n\}_{n\in\mathbb{N}}$ of closed sets $C_n := \{x \in C^c \, : \, \mathrm{dist}(x, \partial C) \geq 1/n\}$, and note that $C_n \nearrow C^c$ as n goes to infinity. Clearly, $E_{\oplus r} \supseteq (E \cap C)_{\oplus r} \cup (E \cap C_n)_{\oplus r}$, and $(E \cap C)_{\oplus r} \cap (E \cap C_n)_{\oplus r} = \emptyset$ for all $r < 1/n$. Hence

$$\mathcal{M}^{*k}(E \cap C) \leq \mathcal{M}^{*k}(E) - \mathcal{M}_*^k(E \cap C_n) \leq \mathcal{H}^k(E) - \mathcal{H}^k(E \cap C_n)$$

for all $n \in \mathbb{N}$. Taking now the limit for n tending to infinity, we get that $\mathcal{M}^{*k}(E \cap C) \leq \mathcal{H}^k(E \cap C)$, and so the equality (4.1).

Consider now $A_1 \cup A_2$; since it is closed and countably $\mathcal{H}^k$-rectifiable, we know that $\mathcal{M}_*^k(A_1 \cup A_2) \geq \mathcal{H}^k(A_1 \cup A_2)$. Observing that $A_1 \cap A_2$ satisfies (4.1) and $(A_1 \cap A_2)_{\oplus r} \subseteq A_{1_{\oplus r}} \cap A_{2_{\oplus r}}$, we obtain

$$\begin{aligned}\mathcal{M}^{*k}(A_1 \cup A_2) &\leq \mathcal{M}^{*k}(A_1) + \mathcal{M}^{*k}(A_2) - \mathcal{M}_*^k(A_1 \cap A_2) \\ &= \mathcal{H}^k(A_1) + \mathcal{H}^k(A_2) - \mathcal{H}^k(A_1 \cap A_2) = \mathcal{H}^k(A_1 \cup A_2).\end{aligned}$$

□

The above-mentioned proposition tells us that the class $\mathcal{O}'$ is stable under finite unions and that, given $E \in \mathcal{O}'$, $\mathcal{M}^{d-1}(C \cap \partial E) = \mathcal{H}^{d-1}(C \cap \partial E)$ for any closed set C. Therefore, the following statement, which extends Proposition 4.1, holds.

Proposition 4.8 *The class $\mathcal{O}'$ is stable under finite unions and any $E \in \mathcal{O}'$ admits outer Minkowski content, given by $\mathcal{SM}(E) = P(E) + 2\mathcal{H}^{d-1}(\partial E \cap E^0)$.*

Remark 4.9 It is easy to see that $\mathcal{H}^{d-1}(\partial E \cap E^0) = 0$ is stable under finite unions; so Theorem 3.1 (and Proposition 4.8) implies that any finite union E of sets $E_i \in \mathcal{O}$ (resp., $\mathcal{O}'$) with $\mathcal{SM}(E_i) = P(E_i)$ admits outer Minkowski content $\mathcal{SM}(E) = P(E)$. This gives an alternative proof of the fact, proved in [1], that the outer Minkowski content of finite unions of compact sets with Lipschitz boundary exists and equals the perimeter. (It is sufficient to observe that if E is a compact set with Lipschitz boundary then E belongs to the class $\mathcal{O}$ with η equal to a suitable multiple of $\mathcal{H}^{d-1}_{|\partial E}$, and $\mathcal{H}^{d-1}(\partial E \cap E^0) = 0$ [4, Proposition 3.62].) More in general, any set E in $\mathcal{O}$ (or $\mathcal{O}'$) with $\mathcal{H}^{d-1}(E^0 \cap \partial E) = 0$ belongs to the class $\mathcal{S}$.

Let us consider now enlarged sets and their respective volume function. It is well known that the boundary of an enlarged set $E_{\oplus r}$ can be much more regular than the boundary of E. In particular, a simple application of Proposition 4.1 permits us to claim that the outer Minkowski content of an enlarged set equals the perimeter; more precisely,

Proposition 4.10 *For any bounded subset E of $\mathbb{R}^d$, the enlarged set $E_{\oplus r}$ belongs to the class $\mathcal{S}$ for all $r > 0$.*

Proof In [7, Proposition 1] Rataj proves that for any bounded subset E of $\mathbb{R}^d$, the set $\partial E_{\oplus r}$ is $(d-1)$-rectifiable for all $r > 0$. Then $E_{\oplus r}$ belongs to the class $\mathcal{O}'$ for all $r > 0$ and, noticing that $\partial E_{\oplus r} \cap E^0_{\oplus r} = \emptyset$ (because if $x \in \partial E_{\oplus r}$, then $x \in \partial B_r(y)$ for some $y \in \partial E$, and so the density of $E_{\oplus r}$ at x is greater than or equal to $1/2$), we conclude that $E_{\oplus r} \in \mathcal{S}$. □

Let V_E be the volume function of E, defined in (1.1). (For a more complete treatment of V_E we refer to [2,7,8] and references therein.) From now on we denote by $V_E'^{(-)}(r)$ and

$V_E'^{(+)}(r)$ the left and right derivative of $V_E(r)$, respectively, at $r > 0$. Clearly, the right derivative of V_E at $r = 0$, whenever it exists, is just the outer Minkowski content of E. As a direct consequence of the above-mentioned proposition, we have that for any bounded set $E \subset \mathbb{R}^d$

$$V_E'^{(+)}(r) = P(E_{\oplus r}) \qquad \forall r > 0.$$

A recent result in this direction for compact sets in $\mathbb{R}^d$ is provided in [2, Corollary 4.6], where the authors, by introducing support measures of arbitrary closed sets, prove that $V_E'^{(+)}(r) = \mathcal{H}^{d-1}(\partial^+ E_{\oplus r})$ for any compact set $E \subset \mathbb{R}^d$ and $r > 0$, having denoted by $\partial^+ A$ the so-called *positive boundary of* A, defined as the set of all boundary points x of A such that there exists a point $y \in A^c$ with $d_A(y) = |y - x|$. In [2, Corollary 4.5], the same authors provide a necessary and sufficient condition for the differentiability of V_E at $r > 0$, which involves the support measures of E (signed measures on the normal bundle of E).

Moreover, a well-known result by Stachó [8] tells us that for all bounded subsets E of $\mathbb{R}^d$, $V_E'^{(-)}(r)$ and $V_E'^{(+)}(r)$ exist at any $r > 0$ and are equal with the possible exception of countably many r's, and that

$$\mathcal{M}^{d-1}(\partial E_{\oplus r}) = \frac{1}{2}\left(V_E'^{(+)}(r) + V_E'^{(-)}(r)\right) \qquad \forall r > 0; \tag{4.2}$$

besides, as observed in [7],

$$V_E'^{(+)}(r) \le \mathcal{H}^{d-1}(\partial E_{\oplus r}) \le V_E'^{-}(r), \tag{4.3}$$

and, in general, none of the inequalities above can be replaced by equality.

The above-quoted results on the differentiability of V_E can be improved by a simple application of the previous arguments, as follows.

Proposition 4.11 *For any bounded subset E of $\mathbb{R}^d$, the function $V_E(r)$ is differentiable at $r > 0$ if and only if*

$$\mathcal{H}^{d-1}(\partial E_{\oplus r} \cap E^1_{\oplus r}) = 0; \tag{4.4}$$

in particular $V_E'(r) = P(E_{\oplus r}) = \mathcal{H}^{d-1}(\partial E_{\oplus r})$.

Proof We have observed in the proof of Proposition 4.10 that $E_{\oplus r} \in \mathcal{O}'$ and $\partial E_{\oplus r} \cap E^0_{\oplus r} = \emptyset$ for all $r > 0$. Since $(E^c)_{\oplus r} \in \mathcal{O}'$ as well,

$$\mathcal{SM}((E_{\oplus r})^c) = P(E_{\oplus r}) + 2\mathcal{H}^{d-1}(\partial E_{\oplus r} \cap E^1_{\oplus r}).$$

Noticing that

$$\begin{aligned} V_E'^{(-)}(r) &= \lim_{h \downarrow 0} \frac{\mathcal{H}^d(E_{\oplus r} \setminus E_{\oplus r-h})}{h} \\ &= \lim_{h \downarrow 0} \frac{1}{h} \int_{r-h}^{r} \mathcal{H}^{d-1}(\{x : d_E(x) = t\}) \mathrm{d}t \\ &= \lim_{h \downarrow 0} \frac{1}{h} \int_{-h}^{0} \mathcal{H}^{d-1}(\{x : d_{E_{\oplus r}}(x) = t\}) \mathrm{d}t \\ &\overset{(3.8)}{=} \lim_{h \downarrow 0} \frac{\mathcal{H}^d(((E_{\oplus r})^c)_{\oplus h} \setminus (E_{\oplus r})^c)}{h} = \mathcal{SM}((E_{\oplus r})^c), \end{aligned}$$

since $V_E'^{(+)}(r) = P(E_{\oplus r})$ for all $r > 0$, it follows that V_E is differentiable at $r > 0$ if and only if condition (4.4) is fulfilled; in such case $V_E'(r) = P(E_{\oplus r})$ and, by (3.3), we also have that $\mathcal{H}^{d-1}(\partial E_{\oplus r}) = P(E_{\oplus r})$. □

Remark 4.12 1. As a corollary of the above proposition, it follows that the set of $r > 0$ such that $\mathcal{H}^{d-1}(\partial E_{\oplus r} \cap E^1_{\oplus r}) > 0$ is at most countable.
Moreover, we know (see [1, Theorem 9]) that $P(A) = \mathcal{H}^{d-1}(\partial A)$ for any compact set $A \subset \mathbb{R}^d$ with positive reach belonging to the class $\mathcal{S}$; thus, accordingly with [2, p. 256], condition (4.4) is satisfied for all $r > 0$ if E is convex, and at least for $r \in (0, \text{reach}(E))$ if E is a compact set with positive reach.

2. We have shown that $V_E'^{(-)}(r) = P(E_{\oplus r}) + 2\mathcal{H}^{d-1}(\partial E_{\oplus r} \cap E^1_{\oplus r})$ and $V_E'^{(+)}(r) = P(E_{\oplus r})$ for all $r > 0$; besides, $\mathcal{M}^{d-1}(\partial E_{\oplus r}) = \mathcal{H}^{d-1}(\partial E_{\oplus r})$ (being $\partial E_{\oplus r}$ $(d-1)$-rectifiable) and $\mathcal{H}^{d-1}(\partial E_{\oplus r}) = P(E_{\oplus r}) + \mathcal{H}^{d-1}(\partial E_{\oplus r} \cap E^1_{\oplus r})$ for all $r > 0$ (being $\partial E_{\oplus r} \cap E^0_{\oplus r} = \emptyset$). This proves the inequalities in (4.3) and it is in accordance with the result (4.2) of Stachó.

In [1, Definition 7] the class $\mathcal{S}_{\text{loc}}$, corresponding to the sets that locally coincide with sets in $\mathcal{S}$, has been introduced in order to get local results for locally finite unions of sets in $\mathcal{S}$. Similarly, let us denote by $\mathcal{O}_{\text{loc}}$ the class of sets E such that for any $R > 0$ there exists $F \in \mathcal{O}$ with $(E \Delta F) \cap B_R(0) = \emptyset$, where Δ is the symmetric difference of sets. By proceeding along the same line of Theorem 3.1, it is easy to obtain the following local version of (3.1) (and, similarly, analogous local versions of the subsequent results).

Proposition 4.13 *If $E \in \mathcal{O}_{\text{loc}}$, then*

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^d((E_{\oplus r} \setminus E) \cap A)}{r} = P(E, A) + 2\mathcal{H}^{d-1}(\partial E \cap E^0 \cap A)$$

for any Borel set $A \subset \mathbb{R}^d$ with $\mathcal{H}^{d-1}(\partial E \cap \partial A) = 0$.

Finally, we conclude this section with an open question. From the aforementioned arguments it emerged that the existence of the outer Minkowski content of a set E in $\mathbb{R}^d$ is closely related to the existence of the $(d-1)$-dimensional Minkowski content of its boundary (see, in particular, Remark 4.2 and Proposition 4.8). Thus, we are led to conjecture that the existence of the outer Minkowski content of a set $E \subset \mathbb{R}^d$ might be equivalent to the existence of $\mathcal{M}^{d-1}(\partial E)$.

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