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I-G-H-KKM Mappings and Minimax Inequalities

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Abstract—First, based on the notions of I-G-H-KKM mappings and I-G-H-KKM selections, some nonempty intersection theorems are proved, and then the obtained I-G-H-KKM theorems are applied to the theory of a new class of generalized minimax inequalities in a G-H-space setting.

Keywords—I-G-H-KKM mapping, I-G-H-KKM selection, Generalized minimax inequality, G-H-space.

1. INTRODUCTION

The KKM theorem and its several generalizations give rise to minimax theorems and saddle points, which play a pivotal role in the solvability of a wider range of problems, from game theory to mathematical economies and optimization theory. Here we intend to establish a new class of generalized minimax inequalities along with several other special cases in a G-H-space setting. This class of generalized minimax inequalities does have significant applications to some generalized minimax theorems, saddle point existence theorems, and generalized variational inequalities [1–6]. For more details on variational inequalities, we refer to [7–12].

Let X be a topological space, $P(X)$ denote the power set of X, and $\langle X\rangle$ the family of all finite subsets of X. Let Δ^n denote a standard $(n-1)$ simplex $\{e_1, e_2, \dots, e_n\}$ of R^n.

DEFINITION 1.1. *A triple $(X, H, \{p\})$ is called a G-H-space if X is a topological space, and $H : \langle X\rangle \to P(X) \setminus \{\emptyset\}$ a mapping such that:*

(i) *for each $F, G \in \langle X\rangle$, there exists an $F_1 \subset F$ such that $F_1 \subset G \to H(F_1) \subset H(G)$;*
(ii) *for each $F = \{x_1, x_2, \dots, x_n\} \in \langle X\rangle$, there exist $\{x_{i1}, x_{i2}, \dots, x_{ik}\} \subset F$ and a continuous mapping $p : \Delta^n \to H(F)$ such that for $\{i1, i2, \dots, ik\} \subset \{1, 2, \dots, n\}$, we have*

$$p(\{e_{i1}, e_{i2}, \dots, e_{ik}\}) \subset H(\{x_{i1}, x_{i2}, \dots, x_{ik}\}).$$

A subset K of X is said to be G-H-convex if for each $A \in \langle X\rangle$, there exists an $A' \subset A$ such that $A' \subset K$ implies $H(A') \subset K$.

A subset K of X is called compactly closed in X if $K \cap L$ is closed in L for all compact subsets L of X.

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DEFINITION 1.2. *Let $(X, H, \{p\})$ be a G-H-space, Y a topological space, and $M_1, \dots, M_n$ be subsets of Y. Let $V : X \to P(Y)$ be a multivalued mapping. A subset $\{x_1, \dots, x_n\} \in \langle X \rangle$ is said to be an I-G-H-KKM selection for $M_1, \dots, M_n$ if there exists an $\{x_{i1}, x_{i2}, \dots, x_{ik}\} \subset \{x_1, \dots, x_n\}$ such that*

$$V\left(H\left(\{x_{i1}, x_{i2}, \dots, x_{ik}\}\right)\right) \subset \bigcup_{j=1}^{k} M_{ij},$$

where $x_1, \dots, x_n$ are not necessarily distinct.

DEFINITION 1.3. *Let $(X, H, \{p\})$ be a G-H-space, Y a topological space, and $V : X \to P(Y)$ any mapping. A subset K of Y is called I-G-H-closed in Y if $K \cap V(H(A))$ is closed in $V(H(A))$ for all $A \in \langle X \rangle$.*

DEFINITION 1.4. *Let $(X, H, \{p\})$ be a G-H-space, Y a topological space, and $V : X \to P(Y)$ any mapping. A mapping $T : X \to P(Y)$ is called I-G-H-KKM if for each $\{x_1, \dots, x_n\} \in \langle X \rangle$, there exists a subset $\{x_{i1}, \dots, x_{ik}\} \subset \{x_1, \dots, x_n\}$ such that*

$$V\left(H\left(\{x_{i1}, \dots, x_{ik}\}\right)\right) \subset \bigcup_{j=1}^{k} T\left(x_{ij}\right).$$

For $X = Y$ and $V = T$, Definition 1.4 reduces to the following.

DEFINITION 1.5. *Let $(X, H, \{p\})$ be a G-H-space and $T : X \to P(X)$ any mapping. The mapping T is I-G-H-KKM if for each subset $\{x_1, \dots, x_n\} \in \langle X \rangle$, there exists a subset $\{x_{i1}, \dots, x_{ik}\}$ of $\{x_1, \dots, x_n\}$ such that*

$$T\left(H\left(\{x_{i1}, \dots, x_{ik}\}\right)\right) \subset \bigcup_{j=1}^{k} T\left(x_{ij}\right).$$

Next, we give an example [8] of an I-G-H-KKM mapping in an interval space. A topological space X is called an interval space if there exists a mapping $[\ ,\ ] : X \times X \to \{$connected subsets of $X\}$ such that $\{x_1, x_2\} \subset [x_1, x_2] = [x_2, x_1]$ for all $x_1, x_2 \in X$. Among the notable special cases of the interval spaces, we mention Hausdorff topological vector spaces, contractible spaces, and connected spaces.

EXAMPLE 1.1. (See [8].) Let X be an interval space, Y a topological space, and $V : X \to P(Y)$ any mapping. Then a mapping $T : X \to P(Y)$ is I-KKM if

$$V\left([x_1, x_2]\right) \subset \bigcup_{i=1}^{2} T(x_i), \qquad \text{for all } x_1, x_2 \in X.$$

For $V = T$ in Example 1.1, T is called I-KKM in the sense of [2,4].

2. GENERALIZED MINIMAX INEQUALITIES

This section is intended to provide, based on I-G-H-KKM theorems, a new class of generalized minimax inequality theorems in G-H-spaces.

THEOREM 2.1. *Let $(X, H, \{p\})$ be a G-H-space, Y a topological space, and $M_1, \dots, M_n$ be I-G-H-closed subsets of Y. Let $V(H(A))$ be compact for all $A \in \langle X \rangle$, $V : X \to P(Y)$ any mapping, and $q : H(A) \to V(H(A))$ a continuous function. Suppose that $M_1, \dots, M_n$ have an I-G-H-KKM selection. Then we have*

$$\bigcap_{i=1}^{n} M_i \neq \emptyset.$$

PROOF. Since the subsets $M_1, \dots, M_n$ have an I-G-H-KKM selection, there exist an $\{x_{i1}, \dots, x_{ik}\} \subset \{x_1, \dots, x_n\} = A \in \langle X \rangle$ and any mapping $V : X \to P(Y)$ such that

$$V\left(H\left(\{x_{i1}, \dots, x_{ik}\}\right)\right) \subset \bigcup_{j=1}^{k} M_{ij}.$$

Since $(X, H, \{p\})$ is a G-H-space, there exists a continuous function $p : \Delta^n \to H(A)$, where $\Delta^n = \{e_1, \dots, e_n\}$. It follows that $q \circ p : \Delta^n \to V(H(A))$ is a continuous function. Let us set

$$E_i = (q \circ p)^{-1}\left(M_i \cap V(H(A))\right), \qquad \text{for } i = 1, \dots, n.$$

Since each M_i is I-G-H-closed in Y, it suffices to show

$$\operatorname{co}\left(\{e_{i1}, \dots, e_{ik}\}\right) \subset \bigcup_{j=1}^{k} E_{ij}.$$

Assume an element $z \in \operatorname{co}(\{e_{i1}, \dots, e_{ik}\})$. Then we have

$$(q \circ p)(z) \in V\left(H\left(\{x_{i1}, \dots, x_{ik}\}\right)\right) \subset \bigcup_{j=1}^{k} M_{ij}.$$

Therefore, there exists an index m $(1 \le m \le k)$ such that $(q \circ p)(z) \in M_{im}$, so $(q \circ p)(z) \in (M_{im} \cap V(H(A)))$. This implies

$$z \in (q \circ p)^{-1}\left(M_{im} \cap V(H(A))\right) = E_{im}.$$

Finally, by the classical KKM theorem, we have $\bigcap_{i=1}^{n} E_i \neq \emptyset$, and as a result, $\bigcap_{i=1}^{n} M_i \neq \emptyset$.

THEOREM 2.2. *Let $(X, H, \{p\})$ be a G-H-space and $T : X \to P(X)$ an I-G-H-KKM mapping. Let $V(H(A))$ be a compact subset of X and $q : H(A) \to V(H(A))$ a continuous function for all $A \in \langle X \rangle$ and for any mapping $V : X \to P(X)$. Suppose that:*

(i) *for each $x \in X$, $T(x)$ is compactly closed in X;*
(ii) *there exists an $A \in \langle X \rangle$ such that $\bigcap_{x \in A} T(x)$ is a compact subset of X.*

Then $\bigcap_{x \in X} T(x) \neq \emptyset$.

PROOF. Since T is an I-G-H-KKM mapping, it implies, for any $\{x_1, \dots, x_n\} \in \langle X \rangle$, that there is a subset $\{x_{i1}, \dots, x_{ik}\} \subset \{x_1, \dots, x_n\}$ such that for $\{i1, \dots, ik\} \subset \{1, \dots, n\}$ and for any mapping $V : X \to P(X)$, we have

$$V\left(H\left(\{x_{i1}, \dots, x_{ik}\}\right)\right) \subset \bigcup_{j=1}^{k} T(x_{ij}).$$

Since each $T(x)$ is compactly closed in X (and hence I-G-H-closed), by Theorem 2.1, the family $\{T(x) : x \in X\}$ has the finite intersection property. On top of that by (ii), $\{T(x) \cap (\bigcap_{x' \in A} T(x')) : x \in X\}$ is a family of compact subsets of X with the finite intersection property, and as a result, we have $\bigcap_{x \in X} T(x) \neq \emptyset$.

THEOREM 2.3. *Let $X, H, \{p\})$ be a G-H-space, $V : X \to P(X)$ any mapping such that $V(H(A))$ is compact for all $A \in \langle X \rangle$, and $q : H(A) \to V(H(A))$ a continuous function. Suppose that $f, g : X \times X \to R$ and $h : X \to R$ are functions such that:*

(i) *$f(x, y) \le g(x, y)$ for all $(x, y) \in X \times X$;*
(ii) *f is lower semicontinuous in y on compact subsets of X;*
(iii) *h is lower semicontinuous in y on compact subsets of X;*
(iv) *for each $A \in \langle X \rangle$, $\bigcap_{x \in A}\{y \in X : f(x, y) + h(y) - h(x) \le 0\}$ is a compact subset of X;*

(v) *for each* $\{x_1,\dots,x_n\}\in\langle X\rangle$, *there exist some* $\{x_{i1},\dots,x_{ik}\}\subset\{x_1,\dots,x_n\}$ *and an* $x\in H(\{x_{i1},\dots,x_{ik}\})$ *such that*

$$f(x,y)+h(y)-h(x)\geq\min_{1\leq j\leq k}\{f(x_{ij},y)+h(y)-h(x_{ij}),\\ g(x_{ij},y)+h(y)-h(x_{ij})\},\qquad \text{for all } y\in X.$$

Then one of the following statements holds.

(1) *There exists an element* $y'\in X$ *such that* $f(x,y')+h(y')-h(x)\leq 0$ *for all* $x\in X$.
(2) *There is an element* $x'\in X$ *such that* $g(x',x')>0$.

PROOF. Let us define mappings $V,T:X\to P(X)$, respectively, by

$$\begin{aligned} V(x)&=\{y\in X: g(x,y)+h(y)-h(x)\leq 0\} &&\text{and}\\ T(x)&=\{y\in X: f(x,y)+h(y)-h(x)\leq 0\}, &&\text{for all } x\in X.\end{aligned}$$

Assume that (2) does not hold. That means $g(x,x)\leq 0$ for all $x\in X$. Thus, $V(x)$ is nonempty. By (i), $V(x)\subset T(x)$. Next, for each $\{x_1,\dots,x_n\}\in\langle X\rangle$, there exists some $\{x_{i1},\dots,x_{ik}\}\subset\{x_1,\dots,x_n\}$ such that for any $x\in H(\{x_{i1},\dots,x_{ik}\})$ and for any $y\in V(x)\subset T(x)$, we have

$$f(x,y)+h(y)-h(x)\leq 0.$$

Now, applying (v), there exists some index m $(1\leq m\leq k)$ such that $g(x_{im},y)+h(y)-h(x_{im})\leq 0$ or $f(x_{im},y)+h(y)-h(x_{im})\leq 0$. This implies

$$y\in V(x_{im})\subset\bigcup_{j=1}^{k}V(x_{ij})\quad\text{or}\quad y\in T(x_{im})\subset\bigcup_{j=1}^{k}T(x_{ij}),$$

so

$$y\in\left(\bigcup_{j=1}^{k}V(x_{ij})\cup\bigcup_{j=1}^{k}T(x_{ij})\right)\subset\bigcup_{j=1}^{k}T(x_{ij}).$$

Therefore, we have

$$V\left(H\left(\{x_{i1},\dots,x_{ik}\}\right)\right)\subset\bigcup_{j=1}^{k}T(x_{ij}),$$

that is, T is an I-G-H-KKM mapping. Next, by (ii), each $T(x)$ is compactly closed in X. As of now, all the conditions of Theorem 2.2 are met, we have $\bigcap_{x\in X}T(x)\neq\emptyset$, that is, there exists an element $y'\in X$ such that $f(x,y')+h(y')-h(x)\leq 0$ for all $x\in X$. This completes the proof of (1).

For $V=T$ in Theorem 2.3, we have the following.

THEOREM 2.4. *Let* $(X,H,\{p\})$ *be a G-H-space and* $T:X\to P(X)$ *a mapping such that* $T(H(A))$ *is compact for all* $A\in\langle X\rangle$. *Let* $q:H(A)\to T(H(A))$ *be a continuous function. Suppose that* $f,g:X\times X\to R$ *and* $h:X\to R$ *are functions such that:*

(i) $f(x,y)\leq g(x,y)$ *for all* $(x,y)\in X\times X$;
(ii) f *is lower semicontinuous in the second variable* y *on compact subsets of* X;
(iii) h *is lower semicontinuous in* y *on compact subsets of* X;
(iv) *for each* $A\in\langle X\rangle$, $\bigcap_{x\in A}\{y\in X: f(x,y)+h(y)-h(x)\leq 0\}$ *is a compact subset of* X;
(v) *for each* $\{x_1,\dots,x_n\}\in\langle X\rangle$, *there exist a* $\{x_{i1},\dots,x_{ik}\}\subset\{x_1,\dots,x_n\}$ *and an* $x\in H(\{x_{i1},\dots,x_{ik}\})$ *such that* $f(x,y)+h(y)-h(x)\geq\min_{1\leq j\leq k}\{f(x_{ij},y)+h(y)-h(x_{ij}), g(x_{ij},y)+h(y)-h(x_{ij})\}$ *for all* $y\in X$.

Then one of the following statements holds.

(a) *There exists an element* $y_0 \in X$ *such that* $f(x, y_0) + h(y_0) - h(x) \leq 0$ *for all* $x \in X$.

(b) *There is an element* $x_0 \in X$ *such that* $g(x_0, x_0) > 0$.

PROOF. Define mappings $S, T : X \to P(X)$, respectively, by

$$S(x) = \{y \in X : g(x,y) + h(y) - h(x) \leq 0\} \quad \text{and} \quad T(x) = \{y \in X : f(x,y) + h(y) - h(x) \leq 0\}.$$

Then by (i), $S(x) \subset T(x)$ for all $x \in X$. Assume that (b) is false. Then there exists an element $x_0 \in X$ such that $g(x_0, x_0) \leq 0$. This implies that $S(x)$ is nonempty. Before we can apply Theorem 2.2, we need to show that T is an I-G-H-KKM mapping. For each $\{x_1, \dots, x_n\} \in \langle X \rangle$, there exists a subset $\{x_{i1}, \dots, x_{ik}\} \subset \{x_1, \dots, x_n\}$ such that for any $x \in H(\{x_{i1}, \dots, x_{ik}\})$ and for any $y \in T(x)$, we have $f(x,y) + h(y) - h(x) \leq 0$. By (v), there exists an index m $(1 \leq j \leq k)$ such that either $g(x_{im}, y) + h(y) - h(x_{im}) \leq 0$ or $f(x_{im}, y) + h(y) - h(x_{im}) \leq 0$, that is, $y \in S(x_{im}) \subset \bigcup_{j=1}^{k} S(x_{ij})$ or $y \in T(x_{im}) \subset \bigcup_{j=1}^{k} T(x_{ij})$. Therefore, we have

$$y \in \left(\bigcup_{j=1}^{k} S(x_{ij}) \cup \bigcup_{j=1}^{k} T(x_{ij}) \right) \subset \bigcup_{j=1}^{k} T(x_{ij}).$$

Hence, we have

$$T(H(\{x_{i1}, \dots, x_{ik}\})) \subset \bigcup_{j=1}^{k} T(x_{ij}),$$

that is, T is I-G-H-KKM. Now the proof follows from an application of Theorem 2.2.

For X compact in Theorem 2.3, we arrive at the following theorem.

THEOREM 2.5. *Let* $(X, H, \{p\})$ *be a compact G-H-space,* $V : X \to P(X)$ *any mapping such that* $V(H(A))$ *is compact for all* $A \in \langle X \rangle$, *and* $q : H(A) \to V(H(A))$ *be a continuous function. Suppose that* $f, g : X \times X \to R$ *and* $h : X \to R$ *are functions such that:*

(i) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X \times X$*;*

(ii) f *is lower semicontinuous in its second variable* y*;*

(iii) h *is lower semicontinuous in* y*;*

(iv) *for each* $\{x_1, \dots, x_n\} \in \langle X \rangle$, *there exist an* $\{x_{i1}, \dots, x_{ik}\} \subset \{x_1, \dots, x_n\}$ *and an* $x \in H(\{x_{i1}, \dots, x_{ik}\})$ *such that*

$$f(x,y) \geq \min_{1 \leq j \leq k} \{f(x_{ij}, y) + h(y) - h(x_{ij}), g(x_{ij}, y) + h(y) - h(x_{ij})\}, \qquad \text{for all } y \in X.$$

Then there is an element $y_0 \in X$ *such that* $f(x, y_0) + h(y_0) - h(x) \leq 0$ *for all* $x \in X$.

For $f = g$ in Theorem 2.5, we have the following.

THEOREM 2.6. *Let* $(X, H\{p\})$ *be a compact G-H-space,* $V : X \to P(X)$ *any mapping such that* $V(H(A))$ *is compact for all* $A \in \langle X \rangle$, *and* $q : H(A) \to V(H(A))$ *be a continuous function. Let* $f : X \times X \to R$ *and* $h : X \to R$ *be functions such that:*

(i) $y \to f(x,y)$ *is lower semicontinuous;*

(ii) h *is lower semicontinuous in* y*;*

(iii) *for each* $\{x_1, \dots, x_n\} \in \langle X \rangle$, *there exists some* $\{x_{i1}, \dots, x_{ik}\} \subset \{x_1, \dots, x_n\}$ *such that for any* $x \in H(\{x_{i1}, \dots, x_{ik}\})$, *we have*

$$f(x,y) \geq \min_{1 \leq j \leq k} [f(x_{ij}, y) + h(y) - h(x_{ij})], \qquad \text{for all } y \in X.$$

Then there is an element $y_0 \in X$ *such that* $f(x, y_0) + h(y_0) - h(x) \leq 0$ *for all* $x \in X$.

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