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TOPOLOGICAL INTERSECTION THEOREMS FOR TWO SET–VALUED MAPPINGS AND APPLICATIONS TO MINIMAX INEQUALITIES

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ABSTRACT

In this paper, we introduce sufficient conditions for the non-empty intersection of two set-valued mappings in topological spaces. As applications, some topological minimax inequalities for two functions in which one of them is separately lower (or upper) semicontinuous are given. Finally, by employing our topological intersection theorems for two set-valued mappings, some other minimax inequalities have been derived without separately lower (or upper) semicontinuity under but with another condition. These results are topological versions of corresponding minimax inequalities for two functions due to Fan (1964) and Sion (1958) in topological vector spaces.

1. INTRODUCTION

It is well-known that many existence problems in mathematics can be reduced to the following *Intersection Problem*: Let Y be a non-empty set, X an index set and $\{F(x) : x \in X\}$ a family of non-empty subsets of Y. Now the question is when does the family have non-empty intersection, i.e., $\cap_{x\in X}F(x) \neq \emptyset$?

It is convenient to formulate the problem above in terms of correspondences. More precisely, let X and Y be two non-empty sets and $F : X \to 2^Y$ a correspondence with non-empty values. A single-valued mapping $f : X \to Y \setminus \{\emptyset\}$ is said to be a selector for F if $f(x) \in F(x)$ for all $x \in X$. Thus the

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intersection question is when does the correspondence F possess a constant selector ?

In order to facilitate the presentation, we first recall some notations. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and topological spaces are assumed to be Hausdorff unless otherwise specified. Let X and Y be two non-empty sets. Then $\mathcal{F}(X)$ and 2^X denote the family of all non-empty finite subsets of X and the family of all subsets of X, respectively. If A is a subset of a topological space X, then $cl_X A$ denotes the closure of A and $A^C := \{x \in X : x \notin A\}$ denotes the complement of A in X. Let $F : X \to 2^Y$ be a correspondence and A a non-empty subset of Y. Then (1) $F^* : Y \to 2^X$ is defined by $F^*(y) = \{x \in X : y \notin F(x)\}$ for each $y \in Y$ is called the dual of F; (2) F is said to have open inverse values if the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for each $y \in Y$ and (3) $F_A : X \to 2^Y$ is a mapping defined by $F_A(x) = F(x) \cap A$ for each $x \in X$.

It is our purpose in this paper to study some sufficient conditions for the non-empty intersection of two set-valued mappings acting between topological spaces. As applications, we first establish some topological minimax inequalities for two functions in which one of them is separately lower (or upper) semicontinuous. Finally, by employing our topological intersection theorems for two set-valued mappings, we derive some other minimax inequalities without separately lower (or upper) semicontinuity (unfortunately, some other condition is required and we do not know if it is necessary). The idea behind this paper is quite simple, it is, based on the the following property of connectedness (e.g., see Theorem 6.1.1 of Engelking [5, p.352])

Fact A: *Let X be a non-empty connected topological space and let both A and B be non-empty open (respectively, closed) subsets of X such that $X \supset A \cup B$. Then $A \cap B \neq \emptyset$.*

This simple idea was first used by Wu [26] in the study of minimax theory: Suppose X and Y are non-empty sets and $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$. The key problem of minimax theory is to look for sufficient conditions in order that the following equation holds:

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y). \qquad (*)$$

The importance of Wu's connectedness approach finally was recognized more than ten years after its publication by Terklesen [24], Tuy [25], Joó [10] and his Hungarian compatriot Stachò [23] and Komornik [16] (see also Geraghty and Lin [8], Kindler [12-14], Kindler and Trost [15], König [18], Horvath [9] and Simons [19-20]; to mention only a few names). For the historical development of the connectedness approach used in the study of minimax theory, we refer the reader to the excellent survey paper of Simons [21]. Since the necessary and sufficient conditions for the equality (*) can be reduced to the equivalent necessary and sufficient conditions for the non-empty intersection of corresponding set-valued mappings (e.g., see Kindler [12]), it is very important to study the non-empty intersection of set-valued mappings. In particular, our arguments in this note are motivated by the connectedness approach developed in the study of minimax theory and by their applications in game theory and mathematical economics due to Wu [26], Tuy [25], Terkelsen [24], Joó [10], Kindler and Trost [15], Komornik [16], Stachó [23], König [18], Simons [20], Chang et al [3-4], Sion [22] and Fan [6-7].

2. TOPOLOGICAL INTERSECTION THEOREMS FOR TWO MAPPINGS

Recently, a number of topological intersection theorems for one set-valued mappings have been given by Chang et al [3-4], Joó and Stachó [11], Kindler [12-14], Kindler and Trost [15], Komornik [16], Stachó [23], König [18] and Simons [20]. Unfortunately, all these topological intersection theorems concern only one set-valued mapping in topological spaces. Being motivated by minimax inequalities for two or more functions (e.g., see Fan [7], König [18], Simons [21] and references wherein), we shall give some existence results for topological intersection theorems for two set-valued mappings in this section. For convenience, we first state the following lemma which is the Remark 1 of Kindler [12]:

Lemma 1. Let X be a topological space, Y a non-empty set and $F : X \to 2^Y$ a correspondence. Then the following are equivalent:

(a) The set $\cap_{y \in B} F^*(y)$ is connected or empty for each $B \in 2^Y$.

(b) for each $x_1, x_2 \in X$, there exists a connected set $C \supset \{x_1, x_2\}$ such that

$$F(x) \subset F(x_1) \cup F(x_2) \text{ for all } x \in C.$$

(c) for each pair $(x_1, x_2) \in X \times X$, the set $\{x \in X : F(x) \subset F(x_1) \cup F(x_2)\}$ is connected.

We also need the following simple result; for completeness, we include its proof:

Lemma 2. Let X and Y be two topological spaces and A a non-empty subset of Y. Suppose the mapping $F : X \to 2^Y$ has open inverse values. Then $F_A^{-1}(y)$ is open in X for $y \in Y$.

Proof. For each $y \in Y$, let $x \in F_A^{-1}(y)$. Then $y \in F_A(x) = F(x) \cap A$. Let $O_x = F^{-1}(y)$. Note that for each $z \in O_x$, we have $y \in F(z)$ and $y \in A$, so that $y \in F_A(z)$. Hence $x \in O_z \subset F_A^{-1}(y)$. $\square$.

Now we have the following sufficient conditions which guarantee the existence of 2-points intersection for two set-valued mappings in topological spaces.

Theorem 1. Let X and Y be two topological spaces and $F, G : X \to 2^Y$ be both set-valued mappings with non-empty values. Suppose the following conditions are satisfied:

(i) for each $x \in X$, $F(x) \subset G(x)$ and $G(x)$ is closed (respectively, open) and $F(x)$ is connected;

(ii) the set $\{x \in X : G(x) \subset G(x_1) \cup G(x_2)\}$ is connected for each given $x_1, x_2 \in X$;

(iii) the set $F^{-1}(y)$ is open for each $y \in Y$.

Then the family $\{G(x) : x \in X\}$ has the 2-points intersection property, i.e., $G(x_1) \cap G(x_2) \neq \emptyset$ for each $x_1, x_2 \in X$.

Proof. If not, suppose there exist $x_1, x_2 \in X$ such that $G(x_1) \cap G(x_2) = \emptyset$. Let $C = \{x \in X : G(x) \subset G(x_1) \cup G(x_2)\}$. By (ii), C is connected in X. For $i = 1, 2$, let $M_i^F = \{x \in C : F(x) \subset G(x_i)\}$ and $M_i^G = \{x \in C : G(x) \subset G(x_i)\}$. Then $\emptyset \neq M_i^G \subset M_i^F$ for each $i = 1, 2$. Next we show that $C = M_1^F \cup M_2^F$. Note that for each $x \in C$, $F(x) \subset G(x) \subset G(x_1) \cup G(x_2)$, $F(x)$ is non-empty connected and $G(x_1)$ and $G(x_2)$ are both disjoint closed (respectively, open) in Y, so that $F(x) \subset G(x_1)$ or $F(x) \subset G(x_2)$ by the Fact A above. Therefore for each $x \in C$, we have $F(x) \subset G(x_1)$ or $F(x) \subset G(x_2)$. Thus $C = M_1^F \cup M_2^F = M_1^G \cup M_2^G$. Next we claim that both M_i^F for $i = 1, 2$ are closed in C. If not, without loss of generality, we may assume that M_2^F is not closed in C. Then there exist $x_0 \in M_1^F \setminus M_2^F$ and a convergent net $\{x_\alpha\}_{\alpha \in \Gamma}$ in M_2^F such that $x_\alpha \to x_0$. Note that $x_0 \in M_1^F$ and $x_\alpha \in M_2^F$. Take any fixed $y_0 \in F(x_0)$, then $y_0 \notin G(x_\alpha)$ for each $\alpha \in \Gamma$. Thus $x_\alpha \in X \setminus F^{-1}(y_0)$. Since the set $X \setminus F^{-1}(y_0)$ is closed in X by (iii) and $x_\alpha \to x_0$, we have that $x_0 \in X \setminus F^{-1}(y_0)$, i.e., $y_0 \notin F(x_0)$ which is impossible by our choice of that $y_0 \in F(x_0)$. Therefore we have proved that $M_1^F \cup M_2^F = C$, where M_1^F and M_2^F are both non-empty and disjoint closed in X, and C is non-empty connected, which is impossible by Fact A above. This contradiction shows that $G(x_1) \cap G(x_2) \neq \emptyset$. □

As an application of Theorem 1, we have the following topological finite intersection theorem for two set-valued mappings in topological spaces:

Theorem 2. Let X and Y be two topological spaces and $F, G : X \to 2^Y$ be both set-valued mappings with non-empty values. Suppose the following conditions are satisfied:

(1) for each $x \in X$, $F(x) \subset G(x)$ and $\cap_{x \in A} G(x)$ is empty or closed (respectively, open) and the set $\cap_{x \in A} F(x)$ is empty or connected for each $A \in \langle X \rangle$;

(2) $\cap_{y \in B} G^*(y)$ is connected or empty for each $B \in 2^Y$ (or equivalently to say, the set $\{x \in X : G(x) \subset G(x_1) \cup G(x_2)\}$ is empty or connected for each $x_1, x_2 \in X$ by Lemma 1);

(3) the set $F^{-1}(y)$ is open for each $y \in Y$;

(4) for each $A \in \mathcal{F}(X)$, if $\cap_{x\in A} G(x) \neq \emptyset$ implies that $\cap_{x\in A} F(x) \neq \emptyset$.

Then the family $\{G(x) : x \in X\}$ has the finite intersection property, i.e., $\cap_{x\in A} G(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.

Proof. We shall prove the conclusion by induction. By Theorem 1, without loss of generality, we may assume that $\cap_{i=1}^{n} G(x_i) \neq \emptyset$ for each $x_1, x_2, \cdots, x_n \in X$, where $n \geq 2$. Now we shall prove that $\cap_{i=1}^{n+1} G(x_i) \neq \emptyset$ for any $n+1$ elements $x_1, x_2, \cdots, x_{n+1}$ of X. We define $F', G' : X \to 2^Y$ by

$$F'(x) := \cap_{i=1}^{n-1} F(x_i) \cap F(x)$$

and

$$G'(x) := \cap_{i=1}^{n-1} G(x_i) \cap G(x)$$

for each $x \in X$. Then for each $x \in X$, we have

(i) $F'(x)$ is non-empty by (4) and our induction hypothesis above and $F'(x) \subset G'(x)$;

(ii) $G'(x)$ is closed (respectively, open) by (1);

(iii) the set $\{x \in X : G'(x) \subset G'(x_1) \cup G'(x_2)\}$ is connected by (2) and Lemma 1; and

(iv) $(F')^{-1}(y)$ is open in X for each $y \in Y$ by (3) and Lemma 2.

Thus F' and G' satisfy all hypotheses of Theorem 1. By Theorem 1, $G'(x') \cap G'(x'') \neq \emptyset$ for each $x', x'' \in X$. Let $x' = x_n$ and $x'' = x_{n+1}$. Then $G'(x_n) \cap G'(x_{n+1}) \neq \emptyset$, i.e., $\cap_{i=1}^{n+1} G(x_i) \neq \emptyset$. Thus the family $\{G(x) : x \in X\}$ has the finite intersection property. □

We also have the following:

Theorem 2'. Let X and Y be two topological spaces and $F, G : X \to 2^Y$ be both set-valued mappings with non-empty values. Suppose the following conditions are satisfied:

(i) for each $x \in X$, $F(x) \subset G(x)$ and $\cap_{x\in A} G(x)$ is empty or closed (respectively, open) for each $A \in \mathcal{F}(X)$;

(ii) the set $\cap_{x\in A} F(x)$ is empty or connected for each $A \in \mathcal{F}(X)$;

(iii) the set $\{x \in X : F(x) \subset F(x_1) \cup F(x_2)\}$ is connected for each $x_1, x_2 \in X$ (equivalently to say, set $\cap_{y\in B} F^*(y)$ is connected or empty for each $B \in 2^Y$ by Lemma 1);

(iv) the set $F^{-1}(y)$ is open for each $y \in Y$;

(v) $\cap_{x\in A} F(x) \neq \emptyset$ when $\cap_{x\in A} G(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.

Then the family $\{G(x) : x \in X\}$ has the finite intersection property, i.e., $\cap_{x\in A} G(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.

Proof. We shall first prove that $G(x_1) \cap G(x_2) \neq \emptyset$ for each $x_1, x_2 \in X$. If not, suppose there exist $x_1, x_2 \in X$ such that $G(x_1) \cap G(x_2) = \emptyset$. Then we shall derive a contradiction. Let $C := \{x \in X : F(x) \subset F(x_1) \cup F(x_2)\}$. By (iii), C is connected in X. or $i = 1, 2$, let $M_i = \{x \in C : F(x) \subset F(x_i)\}$. Then both M_1 and M_2 are non-empty. Note that for each $x \in C$, $F(x)$ is connected and $F(x) \subset F(x_1) \cup F(x_2) \subset G(x_1) \cup G(x_2)$, and $G(x_1)$ and $G(x_2)$ are non-empty closed (respectively, open) and therefore $F(x) \subset G(x_1)$ or $F(x) \subset G(x_2)$. Without loss of generality, we may assume that $F(x) \subset G(x_1)$. Now we claim that $F(x) \subset F(x_1)$. If not, note that $F(x) \subset F(x_1) \cup F(x_2)$. Then there exists $y_0 \in F(x)$ such that $y_0 \in F(x_2) \subset G(x_2)$. Therefore $y_0 \in F(x) \cap G(x_2) \subset G(x_1) \cap G(x_2)$, which contradicts our assumption that $G(x_1) \cap G(x_2) = \emptyset$. Thus $F(x) \subset F(x_1)$. Therefore we have proved that $F(x) \subset F(x_1)$ or $F(x) \subset F(x_2)$, so that $C = M_1 \cup M_2$. Next we claim that both M_1 and M_2 are closed in C. If not, we may assume that M_2 is not closed in C. Then there exist $x_0 \in M_1 \setminus M_2$ and a net $\{x_\alpha\}_{\alpha \in \Gamma}$ in M_2 such that $x_\alpha \to x_0$. Note that since $x_0 \in M_1$ and $x_\alpha \in M_2$. Take any fixed $y_0 \in F(x_0)$, then $y_0 \notin F(x_\alpha)$ for each $\alpha \in \Gamma$. Thus $x_\alpha \in X \setminus F^{-1}(y_0)$. Note that the set $X \setminus F^{-1}(y_0)$ is closed in X by (iv) and $x_\alpha \to x_0$, we have that $x_0 \in X \setminus f^{-1}(y_0)$, i.e., $y_0 \notin F(x_0)$ which is impossible. Thus both M_1 and M_2 must be closed in C. Therefore we have that $M_1 \cup M_2 = C$, where M_1 and M_2 are both non-empty and closed in C and C is non-empty connected, which is a contradiction, by Fact A. This contradiction shows $G(x_1) \cap G(x_2) \neq \emptyset$.

Secondly, we shall prove by the induction that the family $\{G(x) : x \in X\}$ has the finite intersection property. We may assume that $\cap_{i=1}^{n} G(x_i) \neq \emptyset$ for each $x_1, x_2, \cdots, x_n \in X$, where $n \geq 2$. Now we prove that $\cap_{i=1}^{n+1} G(x_i) \neq \emptyset$ for any $n+1$ elements $x_1, x_2, \cdots, x_{n+1}$ of X. We define $F', G' : X \to 2^Y$ by

$$F'(x) := \cap_{i=1}^{n-1} F(x_i) \cap F(x)$$

and

$$G'(x) := \cap_{i=1}^{n-1} G(x_i) \cap G(x)$$

for each $x \in X$. Then for each $x \in X$, we have

(1) $F'(x) \subset G'(x)$ and $F'(x)$ is non-empty by (v) and our assumption;
(2) $G'(x)$ is closed (respectively, open) by (i);
(3) the set $\{x \in X : G'(x) \subset G'(x_1) \cup G'(x_2)\}$ is connected by (iii) and Lemma 1; and
(4) $(F')^{-1}(y)$ is open in X for each $y \in Y$ by (iv) and Lemma 2.

Thus F' and G' satisfy all hypotheses of Theorem $2'$. By the proof above again, $G'(x') \cap G'(x'') \neq \emptyset$ for each $x', x'' \in X$. Let $x' = x_n$ and $x'' = x_{n+1}$. Then $G'(x_n) \cap G'(x_{n+1}) \neq \emptyset$, i.e., $\cap_{i=1}^{n+1} G(x_i) \neq \emptyset$. Thus the family $\{G(x) : x \in X\}$ has the finite intersection property. □

Remark 1. (a): Recently, Kindler in [12-14] has established a number of topological intersection theorems which involve only one set-valued mapping (see also König [18] and reference wherein), hence their results are not comparable with our Theorems 2 and 2′.

(b): Let X and Y be topological spaces and $F : X \to 2^Y$ a set-valued mapping. Chang et al first introduced the following property (P) for the mapping F in [4, p.756]:

(P): For any $x_0 \in Y$ and any net $\{x_\alpha\}_{\alpha\in\Gamma}$ of X with $x_\alpha \to x_0$, if $y_0 \notin F(x_\alpha)$ for all $\alpha \in \Gamma$, then $y_0 \notin F(x_0)$.

We wish to point out that the property (P) of F is, in fact, equivalent to saying that the mapping F has open inverse values (i.e., the set $F^{-1}(y)$ is open in X for each $y \in Y$). Hence our Theorems 2 and 2′ include Theorem 1 of Chang et al [3-4] as an special case.

The following example shows that the conclusion of Theorems 2 (respectively, Theorem 2′) is not true if we withdraw the condition (4) (respectively, (v)):

One Counterexample: Let $X = [0, 2\pi)$ and $Y = \{e^{i\theta} : 0 \le \theta < 2\pi\}$. Define $F, G : X \to 2^Y$ by

$$F(\theta) = \{e^{i\psi} : \theta - 1 < \psi < \theta + 1\}$$

for each $\theta \in X$ and

$$G(\theta) = \begin{cases} \{e^{i\psi} : \quad -1 \le \psi \le \frac{2\pi}{3} + 1\} & \text{if } \theta \in [0, \frac{2\pi}{3}]; \\ \{e^{i\psi} : \frac{2\pi}{3} - 1 \le \psi \le \frac{4\pi}{3} + 1\} & \text{if } \theta \in (\frac{2\pi}{3}, \frac{4\pi}{3}]; \\ \{e^{i\psi} : \quad \frac{4\pi}{3} - 1 \le \psi \le 1\} & \text{if } \theta \in (\frac{4\pi}{3}, 2\pi) \end{cases}$$

for each $\theta \in X$. Then it is easy to verify that

(1) $F(\theta) \subset G(\theta)$ and $\cap_{\theta\in A} G(\theta)$ is closed and connected or empty for each $A \in \mathcal{F}(X)$;

(2) $F^{-1}(e^{i\psi}) = \{\theta : \psi - 1 < \theta < \psi + 1\}$, which is open in X for each $e^{i\psi} \in Y$;

(3) the set $\{\theta \in X : G(\theta) \subset G(\theta_1) \cup G(\theta_2)\}$ is connected.

Note that for each $\theta_1, \theta_2 \in X$, $G(\theta_1) \cap G(\theta_2) \neq \emptyset$, but $F(\theta_i) \cap F(\theta_j) = \emptyset$ for each $\theta_i \in [0, \frac{2\pi}{3}]$ and $\theta_j \in (\frac{2\pi}{3}, \frac{4\pi}{3}]$ or $\theta_j \in (\frac{3\pi}{3}, 2\pi)$, i.e., the condition (4) (respectively, (v)) of Theorem 2 (respectively, Theorem 2′) does not holds.

Thus the family $\{G(\theta) : \theta \in X\}$ does not have the finite intersection property, e.g., $G(\frac{\pi}{6}) \cap G(\frac{2\pi}{3} + \frac{\pi}{6})) \cap G(\frac{4\pi}{3} + \frac{\pi}{6}) = \emptyset$.

Letting $F = G$ in Theorem 2, we have the following corollary which includes Theorem 1 of Chang et al [3-4]:

Corollary 3. Let X and Y be two topological spaces and $G : X \to 2^Y$ be a set-valued mapping with non-empty values. Suppose the following conditions are satisfied:

(i) the set $\cap_{x\in A} G(x)$ is closed (respectively, open) connected or empty for each $A \in \mathcal{F}(X)$;

(ii) the set $\{x \in X : G(x) \subset G(x_1) \cup G(x_2)\}$ is empty or connected for each $x_1, x_2 \in X$ (or equivalently to say, $\cap_{y\in B} G^*(y)$ is connected or empty for each $B \in 2^Y$);

(iii) the set $G^{-1}(y)$ is open in X for each $y \in Y$.

Then the family $\{G(x) : x \in X\}$ has the finite intersection property.

3. TOPOLOGICAL MINIMAX INEQUALITIES FOR TWO FUNCTIONS

Let X and Y be topological spaces and $a : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function. We set

$$a^* = \inf_{y\in Y} \sup_{x\in X} a(x,y), \qquad a_* = \sup_{x\in X} \inf_{y\in Y} a(x,y)$$

and

$$\hat{a}^* = \sup_{A\in\mathcal{F}(x)} \inf_{y\in Y} \sup_{x\in A} a(x,y).$$

Then it is clear that $a_* \leq \hat{a}^* \leq a^*$. We say that a fulfills

(1) the minimax relation (in short, MM) if $a_* = a^*$;
(2) the preminimax relation (in short, PMM) if $\hat{a}^* = a_*$; and
(3) the minimum minimax relation (in short, MMM) if $\sup_{x\in X} a(x, \hat{y}) = a_*$ for some $\hat{y} \in Y$.

In this section, as applications of the topological intersection theorems above, we shall study topological minimax inequalities for "two-function versions" of minimax theorems which are generalizations of Sion's minimax theorem [22] (see also Fan [7]).

Theorem 4. Let X and Y be topological spaces. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that for each $\lambda > b_*$,

(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;

(2) for each fixed $y \in Y$, $x \mapsto b(x,y)$ is upper semicontinuous and for each fixed $x \in X$, $y \mapsto b(x,y)$ is upper semicontinuous;

(3) for each fixed $x \in X$, $y \mapsto a(x,y)$ is lower semicontinuous;

(4) the set $\cap_{x\in A}\{y \in Y : b(x,y) < \lambda\}$ is empty or connected for each $A \in \mathcal{F}(X)$;

(5) the set $\cap_{y\in B}\{x \in X : b(x,y) \geq \lambda\}$ is empty or connected for each $B \in 2^Y$;

(6) there exist $\lambda_0 > b_*$ and $A_0 \in \mathcal{F}(X)$ such that the set $\cap_{x\in A_0}\{y \in Y : a(x,y) \leq \lambda_0\}$ is empty or compact. Then we have

$$\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq \sup_{x\in X}\inf_{y\in Y} b(x,y).$$

Proof. For each $\lambda > b_*$, we shall prove that the family $\{y \in Y : a(x,y) \leq \lambda\}_{x\in X}$ has the finite intersection property. We define $F_\lambda, G_\lambda : X \to 2^Y$ by

$$F_\lambda(x) := \{y \in Y : b(x,y) < \lambda\} \text{ and } G_\lambda(x) := \{y \in Y : a(x,y) \leq \lambda\}$$

for each $x \in X$. Then we have that

(i) $F_\lambda(x) \neq \emptyset$ and $F_\lambda(x) \subset G_\lambda(x)$ by (1);

(ii) $F_\lambda(x)$ is open for each $x \in X$ and the set $F_\lambda^{-1}(y)$ is open in X for each $y \in Y$ by (2).

(iii) the set $\cap_{x\in A}F_\lambda(x)$ is empty or connected for each $A \in \mathcal{F}(X)$ by (4).

(iv) the set $\{x \in X : F_\lambda(x) \subset F_\lambda(x_1) \cup F_\lambda(x_2)\}$ is empty or connected for each $x_1, x_2 \in X$ by (5) and Lemma 1.

Thus F_λ satisfies all hypotheses of Corollary 3. By Corollary 3, the family $\{F_\lambda(x) : x \in X\}$ has the finite intersection property, hence so does the family $\{G_\lambda(x) : x \in X\}$. Therefore for each $\lambda > b_*$, $\{y \in Y : a(x,y) \leq \lambda\}_{x\in X}$ has the finite intersection property. Note that for each $\lambda \in (b_*, \lambda_0)$, $\{y \in Y : a(x,y) \leq \lambda\} \subset \{y \in Y : a(x,y) \leq \lambda_0\}$ for each $x \in X$ and the set $\cap_{x\in A_0}\{y \in Y : a(x,y) \leq \lambda_0\}$ is non-empty and compact by (6), so that $\cap_{x\in X}\{y \in Y : a(x,y) \leq \lambda\} \neq \emptyset$. For each $\lambda \in (b_*, \lambda_0)$, taking any $y_0 \in \cap_{x\in X}\{y \in Y : a(x,y) \leq \lambda\}$, then $\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq \sup_{x\in X} a(x,y_0) \leq \lambda$. Hence for each $\lambda \in (b_*, \lambda_0)$, $\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq \lambda$. Therefore we must have the following inequality,

$$\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq b_* = \sup_{x\in X}\inf_{Y\in Y} b(x,y) \qquad \square$$

As an immediate consequence of Theorem 4, we have the following corollary which includes and improves the corresponding results of Bardaro and Ceppitelli [1], Chang et al [3-4], Fan [6] and Geraghty and Lin [8].

Corollary 5. Let X be a topological space and Y a compact topological space. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that for each $\lambda > b_*$,

(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;

(2) for each fixed $y \in Y$, $x \mapsto b(x,y)$ is upper semicontinuous and for each fixed $x \in X$, $y \mapsto b(x,y)$ is upper semicontinuous;

(3) for each fixed $y \in Y$, $x \mapsto a(x,y)$ is lower semicontinuous;

(4) the set $\cap_{x \in A}\{y \in Y : b(x,y) < \lambda\}$ is empty or connected for each $A \in \mathcal{F}(X)$;

(5) the set $\cap_{y \in B}\{x \in X : b(x,y) \geq \lambda\}$ is empty or connected for each $B \in 2^Y$.

Then

$$\inf_{y \in Y} \sup_{x \in X} a(x,y) \leq \sup_{x \in X} \inf_{y \in Y} b(x,y)$$

Proof. Since Y is compact, the condition (5) of Theorem 4 is automatically satisfied. Hence the conclusion follows from Theorem 4. □

Let $a(x,y) = b(x,y)$ for each $(x,y) \in X \times Y$ in Corollary 5. Then Corollary 5 implies the MMM result, i.e., there exists $y_0 \in Y$ such that $\sup_{x \in X} a(x, y_0) = \sup_{x \in X} \inf_{y \in Y} a(x,y)$.

By a similar proof to that of Theorem 4, we also have the following minimax inequality:

Theorem 6. Let X and Y be topological spaces. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that for each $\lambda < a^*$,

(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;

(2) for each fixed $y \in Y$, $x \mapsto a(x,y)$ is lower semicontinuous and for each fixed $x \in X$, $y \mapsto a(x,y)$ is lower semicontinuous;

(3) for each fixed $y \in Y$, $x \mapsto b(x,y)$ is upper semicontinuous;

(4) the set $\cap_{y \in A}\{x \in X : a(x,y) > \lambda\}$ is empty or connected for each $A \in \mathcal{F}(Y)$;

(5) the set $\cap_{x \in B}\{y \in Y : a(x,y) \leq \lambda\}$ is empty or connected for each $B \in 2^X$;

(6) there exist $\lambda_0 < a^*$ and $A_0 \in \mathcal{F}(Y)$ such that the set $\cap_{y \in A_0}\{x \in X : b(x,y) \geq \lambda_0\}$ is empty or compact.

Then we have

$$\inf_{y \in Y} \sup_{x \in X} a(x,y) \leq \sup_{x \in X} \inf_{y \in Y} b(x,y).$$

Proof. The idea is similar to that in Theorem 4. For each $\lambda < a^*$, we wish to show that the family $\{x \in X : b(x,y) \geq \lambda\}_{y \in Y}$ has the finite intersection property. Now define $F_\lambda, G_\lambda : Y \multimap 2^X$ by

$$F_\lambda(y) := \{x \in X : a(x,y) > \lambda\} \text{ and } G_\lambda(y) := \{x \in X : b(x,y) \geq \lambda\}$$

for each $y \in Y$. Then it is easy to verify that

(i) $F_\lambda(y) \neq \emptyset$ and $F_\lambda(y) \subset G_\lambda(y)$ by (1);
(ii) $F_\lambda(y)$ is open for each $y \in Y$ and the set $F_\lambda^{-1}(x)$ is open in Y for each $x \in X$ by (2).
(iii) the set $\cap_{y \in A} F_\lambda(y)$ is empty or connected for each $A \in \mathcal{F}(Y)$ by (4).
(iv) the set $\{y \in Y : F_\lambda(y) \subset F_\lambda(y_1) \cup F_\lambda(y_2)\}$ is empty or connected by (5) and Lemma 1 for each $y_1, y_2 \in Y$.

Thus F_λ satisfies all hypotheses of Corollary 3 (by exchanging X and Y). By Corollary 3, the family $\{F_\lambda(y) : y \in Y\}$ has the finite intersection property, hence so does the family $\{G_\lambda(y) : y \in Y\}$. Hence for each $\lambda < a^*$, $\{x \in X : a(x,y) > \lambda\}_{y \in Y}$ has the finite intersection property. Note that for each $\lambda \in (\lambda_0, a^*)$, $\{x \in X : a(x,y) > \lambda\} \subset \{x \in X : b(x,y) \geq \lambda_0\}$ for each $y \in Y$ and the set $\cap_{y \in A_0}\{x \in X : b(x,y) \geq \lambda_0\}$ is non-empty and compact by (6), so that $\cap_{y \in Y}\{x \in X : b(x,y) \geq \lambda\} \neq \emptyset$. For each $\lambda \in (\lambda_0, a^*)$, taking any $x_0 \in \cap_{y \in Y}\{x \in X : b(x,y) \geq \lambda\}$. Then $\sup_{x \in X} \inf_{y \in Y} b(x,y) \geq \inf_{y \in Y} b(x_0, y) \geq \lambda$. Hence for each $\lambda \in (\lambda_0, a^*)$, $\sup_{x \in X} \inf_{y \in Y} b(x,y) \geq \lambda$. Therefore we must have the following inequality,

$$\sup_{x \in X} \inf_{y \in Y} b(x,y) \geq a^* = \inf_{y \in Y} \sup_{x \in X} a(x,y) \qquad \square$$

By Lemma 1 and Theorem 6, it is not difficult to derive the following corollary which includes Theorem 3 of Chang [2] as a special case:

Corollary 7. Let X and Y be topological spaces. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that
(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;
(2) for each fixed $y \in Y$, $x \mapsto a(x,y)$ is lower semicontinuous and for each fixed $x \in X$, $y \mapsto a(x,y)$ is lower semicontinuous;
(3) for each fixed $y \in Y$, $x \mapsto b(x,y)$ is upper semicontinuous;
(4) for each $\lambda < a^*$, the set $\cap_{y \in A}\{x \in X : a(x,y) > \lambda\}$ is empty or connected for each $A \in \mathcal{F}(Y)$;
(5) for each $y_1, y_2 \in Y$, there exists a connected subset C_{y_1,y_2} of Y with $\{y_1, y_2\} \subset C_{\{y_1,y_2\}}$ such that

$$a(x,y) \leq a(x,y_1) \vee a(x,y_2) \text{ for all } (x,y) \in X \times C_{\{y_1,y_2\}},$$

where "$\vee$" stands for "maximum";

(6) there exist $\lambda_0 < a^*$ and $A_0 \in \mathcal{F}(Y)$ such that the set $\cap_{y\in A_0}\{x \in X : b(x,y) \geq \lambda_0\}$ is empty or compact.
Then

$$\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq \sup_{x\in X}\inf_{y\in Y} b(x,y).$$

By Lemma 5 of Geraghty and Lin [8, p.379], it is clear that Corollary 7 also includes Corollary 2 of Chang et al [2] as a special case. We would like to point out that the proof of Theorems 4 and 6, actually produce the following PMM theorems:

Theorem 8. Let X and Y be topological spaces. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that for each $\lambda > b_*$,

(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;

(2) for each fixed $y \in Y$, $x \mapsto b(x,y)$ is upper semicontinuous and for each fixed $x \in X$, $y \mapsto b(x,y)$ is upper semicontinuous;

(3) the set $\cap_{y\in B}\{x \in X : b(x,y) \geq \lambda\}$ is empty or connected for each $B \in 2^Y$;

(4) the set $\cap_{x\in A}\{y \in Y : b(x,y) < \lambda\}$ is empty or connected for each $A \in \mathcal{F}(X)$.

Then we have

$$\sup_{A\in\mathcal{F}(X)}\inf_{y\in Y}\sup_{x\in A} a(x,y) \leq \sup_{x\in X}\inf_{y\in Y} b(x,y) \leq \sup_{A\in\mathcal{F}(X)}\inf_{y\in Y}\sup_{x\in A} b(x,y).$$

Proof. By the proof of Theorem 4, for each $\lambda > b_*$, the family $\{y \in Y : b(x,y) < \lambda\}_{x\in X}$ has the finite intersection property, and hence so does the family $\{y \in Y : a(x,y) < \lambda\}_{x\in X}$. Thus for each $A \in \mathcal{F}(X)$, taking any fixed $y_0 \in \cap_{x\in A}\{y \in Y : a(x,y) < \lambda\}$, we have $\inf_{y\in Y}\sup_{x\in A} a(x,y) \leq \sup_{x\in A} a(x,y_0) < \lambda$. Therefore $\sup_{A\in\mathcal{F}(X)}\inf_{y\in Y}\sup_{x\in A} a(x,y) \leq \lambda$ for each $\lambda > b_*$, so that

$$\sup_{A\in\mathcal{F}(X)}\inf_{y\in Y}\sup_{x\in A} a(x,y) \leq b_* = \sup_{x\in X}\inf_{y\in Y} b(x,y).$$

Note that $\sup_{x\in X}\inf_{y\in Y} b(x,y) \leq \sup_{A\in\mathcal{F}(X)}\inf_{y\in Y}\sup_{x\in A} b(x,y)$. Then the conclusion follows. □

Theorem 9. Let X and Y be topological spaces. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that for each $\lambda < a^*$,

(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;

(2) for each fixed $y \in Y$, $x \mapsto a(x,y)$ is lower semicontinuous and for each fixed $x \in X$, $y \mapsto a(x,y)$ is lower semicontinuous;

(3) the set $\cap_{x\in B}\{y \in Y : a(x,y) \leq \lambda\}$ is empty or connected for each $B \in 2^X$;

(4) the set $\cap_{y\in A}\{x \in X : a(x,y) > \lambda\}$ is empty or connected for each $A \in \mathcal{F}(Y)$.

Then we have

$$\sup_{A\in\mathcal{F}(X)}\inf_{y\in Y}\sup_{x\in A}a(x,y)\le\inf_{y\in Y}\sup_{x\in X}a(x,y)\le\inf_{A\in\mathcal{F}(Y)}\sup_{x\in X}\inf_{y\in A}b(x,y).$$

Proof. From the proof of Theorem 6, for each $\lambda < a^*$, the family $\{x \in X : b(x,y) \ge \lambda\}_{y\in Y}$ has the finite intersection property. For each $A \in \mathcal{F}(Y)$, there exists $x_0 \in \cap_{y\in A}\{x \in X : b(x,y) \ge \lambda\}$ such that $\sup_{x\in X}\inf_{y\in A} b(x,y) \ge \inf_{y\in Y} b(x_0,y) \ge \lambda$. Therefore $\inf_{A\in\mathcal{F}(Y)}\sup_{x\in X}\inf_{y\in A} b(x,y) \ge \lambda$ for each $\lambda > a^*$. Hence

$$\inf_{A\in\mathcal{F}(Y)}\sup_{x\in X}\inf_{y\in A}b(x,y)\ge a^*.$$

Note that

$$\sup_{A\in\mathcal{F}(X)}\inf_{y\in Y}\sup_{x\in A}a(x,y)\le\inf_{y\in Y}\sup_{x\in X}a(x,y).$$

The conclusion follows. □

Before we conclude this section, we recall the following result which is the Consequence 2.4 of König [18] (compare also Lemma 5 of Geraghty and Lin [8, p.379]):

Lemma 3. Let X be a non-empty set, Y a topological space, $f : X\times Y \to \mathbb{R}$ a function and I a non-empty open interval of $\mathbb{R}$. Consider the following properties:

(a) for each $\lambda \in I$, the set $\cap_{x\in A}\{y \in Y : f(x,y) \le \lambda\}$ is empty or connected for each $A \in \mathcal{F}(X)$.

(b) for each $\lambda \in I$, the set $\cap_{x\in A}\{y \in Y : f(x,y) < \lambda\}$ is empty or connected for each $A \in \mathcal{F}(X)$.

Then we have

(1): The property (a) implies property (b) is always true and

(2): If Y is a compact Hausdorff topological space and for each fixed $x \in X$, $y \mapsto f(x,y)$ is lower semicontinuous, then property (b) implies property (a) (i.e., the property (a) is equivalent to (b) in this case).

Remark 2. By Lemma 3, it is clear all conclusions from Theorems and Corollaries 4 to 9 still hold if the " $<$" (respectively, " $>$") is replaced by " $\le$" (or " $\ge$") in each condition 4.

4. TOPOLOGICAL MINIMAX INEQUALITIES WITHOUT SEPARATELY SEMICONTINUITY

In the section 3, as applications of a topological intersection theorem for one set-valued mapping, we have given some minimax inequalities of two functions in which one of them is separately upper or lower semicontinuous.

In this section, as applications of our topological intersection theorems for two set-valued mappings, we shall give some minimax inequalities for two functions without separately semicontinuous conditions but we need some other assumption (i.e., the condition (5) in both Theorems 10 and 11).

As an application of Theorem 2, we have:

Theorem 10. Let X and Y be topological spaces. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that for each $\lambda > b_*$,

(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;

(2) for each fixed $y \in Y$, $x \mapsto b(x,y)$ is upper semicontinuous and for each fixed $x \in X$, $y \mapsto a(x,y)$ is lower semicontinuous;

(3) the set $\cap_{x\in A}\{y \in Y : b(x,y) < \lambda\}$ is empty or connected for each $A \in \mathcal{F}(X)$;

(4) the set $\cap_{y\in B}\{x \in X : a(x,y) > \lambda\}$ is empty or connected for each $B \in 2^Y$;

(5) for each $A \in \mathcal{F}(X)$, if $\cap_{x\in A}\{y \in Y : a(x,y) \leq \lambda\} \neq \emptyset$, then $\cap_{x\in A}\{y \in Y : b(x,y) < \lambda\} \neq \emptyset$.

(6) there exist $\lambda_0 > b_*$ and $A_0 \in \mathcal{F}(X)$ such that the set $\cap_{x\in A_0}\{y \in Y : a(x,y) \leq \lambda_0\}$ is empty or compact. Then we have

$$\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq \sup_{x\in X}\inf_{y\in Y} b(x,y).$$

Proof. Using the same idea as in the proof of Theorem 4, for each $\lambda > b_*$, we shall first prove that the family $\{y \in Y : a(x,y) \leq \lambda\}_{x\in X}$ has the finite intersection property. We define $F_\lambda, G_\lambda : X \to 2^Y$ by

$$F_\lambda(x) := \{y \in Y : b(x,y) < \lambda\} \text{ and } G_\lambda(x) := \{y \in Y : a(x,y) \leq \lambda\}$$

for each $x \in X$. Then it is easy to check that both mapping F_λ and G_λ satisfy all hypotheses of Theorem 2. By Theorem 2, the family $\{G_\lambda(x) : x \in X\}$ has the finite intersection property. Hence each $\lambda > b_*$, $\{y \in Y : a(x,y) \leq \lambda\}_{x\in X}$ has the finite intersection property. Note that for each $\lambda \in (b_*, \lambda_0)$, $\{y \in Y : a(x,y) \leq \lambda\} \subset \{y \in Y : a(x,y) \leq \lambda_0\}$ for each $x \in X$ and the set $\cap_{x\in A_0}\{y \in Y : a(x,y) \leq \lambda_0\}$ is non-empty and compact by (6), so that $\cap_{x\in X}\{y \in Y : a(x,y) \leq \lambda\} \neq \emptyset$. For each $\lambda \in (b_*, \lambda_0)$, taking any $y_0 \in \cap_{x\in X}\{y \in Y : a(x,y) \leq \lambda\}$, then $\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq \sup_{x\in X} a(x,y_0) \leq \lambda$. Hence for each $\lambda \in (b_*, \lambda_0)$, $\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq \lambda$. Therefore we must have the following inequality,

$$\inf_{y\in Y}\sup_{x\in X} a(x,y) \leq b_* = \sup_{x\in X}\inf_{Y\in Y} b(x,y) \qquad \square$$

As an application of Theorem 2′ instead of Theorem 2, and by the same proof as in Theorem 10, we have the following:

Theorem 11. Let X and Y be topological spaces. Let $a, b : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two functions such that for each $\lambda > b_*$,

(1) $a(x,y) \leq b(x,y)$ for each $(x,y) \in X \times Y$;

(2) for each fixed $y \in Y$, $x \mapsto b(x,y)$ is upper semicontinuous and for each fixed $x \in X$, $y \mapsto a(x,y)$ is lower semicontinuous;

(3) the set $\cap_{x \in A}\{y \in Y : b(x,y) < \lambda\}$ is empty or connected for each $A \in \mathcal{F}(X)$;

(4) the set $\cap_{y \in B}\{x \in X : b(x,y) \geq \lambda\}$ is empty or connected for each $B \in 2^Y$;

(5) for each $A \in \mathcal{F}(X)$, if $\cap_{x \in A}\{y \in Y : a(x,y) \leq \lambda\} \neq \emptyset$, then $\cap_{x \in A}\{y \in Y : b(x,y) < \lambda\} \neq \emptyset$.

(6) there exist $\lambda_0 > b_*$ and $A_0 \in \mathcal{F}(X)$ such that the set $\cap_{x \in A_0}\{y \in Y : a(x,y) \leq \lambda_0\}$ is empty or compact. Then we have

$$\inf_{y \in Y} \sup_{x \in X} a(x,y) \leq \sup_{x \in X} \inf_{y \in Y} b(x,y).$$

Proof. For each $\lambda > b_*$, we define $F_\lambda, G_\lambda : X \to 2^Y$ by

$$F_\lambda(x) := \{y \in Y : b(x,y) < \lambda\} \text{ and } G_\lambda(x) := \{y \in Y : a(x,y) \leq \lambda\}$$

for each $x \in X$. By employing Theorem 2′ instead of Theorem 2 and the same proof as in Theorem 10, the conclusion follows. □

Remark 3. We do not know if the conclusions of Theorems 10 and 11 are still valid without the condition (5). Moreover we note that Theorem 10 is a topological version of corresponding minimax inequalities in topological vector spaces given by Fan [7] and Sion [22] except the condition (5). For some other generalizations of minimax inequalities for more than two functions in topological vector spaces, we refer the reader to Ben-El-Mechaiekh et al [2] and references in [21].

Finally, for other work on topological intersection theorems and their applications to minimax inequalities, we refer the reader to Horvath [7], Joó and Stachó [11], Kindler [12-14], Komornik [16], Komiya [17], König [18], Simons [19-21], Stachó [23] and references therein.

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