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TOPOLOGICAL INTERSECTION THEOREMS OF SET-VALUED MAPPINGS WITHOUT CLOSED GRAPHS AND THEIR APPLICATIONS

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1. Introduction

Let X and Y be non-empty sets and $f : X \times Y \to \mathbf{R}$ be a function. A minimax problem is to find certain conditions such that the following holds: $\inf_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y)$. The original motivation for the study of minimax theorems was, of course, von Neumann's work on the game theory of strategies in 1928. After a lapse of nearly ten years, generalization of von Nemuann's original results for matrices started appearing. As time went on, these generalizations became progressively more remote from game theory, and minimax theory started becoming object of study in their own right. The importance of *connectedness* in the study of minimax theory was first recognized by Wu [26]. Then this idea was picked by Terkerson [23]. By a refined method, it is Tuy, who derived a generalized version of Sion's classical minimax theorem in [24] (see also Geraghty and Lin [2])). Independently, inspired by Joó's paper [5], the *method of level sets* was developed by Joó and his Hungarian compatriots Stachó [22] and Komornik [14]. For example, by introducing the concept of the *interval space,* it was Stachó [22], who established an intersection theorem which was used by Komornik [14] to derive a generalization of Ha's minimax theorem [3]. All these results were unified by Kindler–Trost [12]. Following this line, many minimax theorems which only involve the *connectedness* instead of convexity were obtained by Komiya [13], Horvath [4], Lin and Quan [16], Kindler [9–11], König [15], Simons [18–19] et al. We only mention a few names here; for the historical trace of the development of minimax theory starting from von Neumann's work, we refer the reader to Simons' recent survey paper [20]. There have been various types of minimax theorems as mentioned by Professor Simon, for example, such as (i) '*topological minimax theorems*' in which various connectedness hypotheses are assumed for X, Y and the function f; (ii) '*quantitative minimax theorems*' in which no special properties are assumed for X and Y, but various quantitative properties are assumed for f; (iii) '*mixed minimax theorems*' in which the quantitative and the topo-

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logical properties are mixed; and (iv) '*unified metaminimax theorems*' which include the minimax theorems of types (i)–(iii) above as special cases. Using the idea that the minimax theorems can be reduced to the equivalent existence problems of non-empty intersection of a set-valued mappings (e.g., see Remark 5 of Kindler [9, p.1008]), it is Kindler, who is the first person to give a number of topological (resp., abstract set theoretical) characterization of the existence of non-empty intersection theorems [9–11], by unifying ideas of Wu [26], Terkerson [23], Tuy [24], Joó [5–6], Joó and Stachó [7], Komyia [13], Komornik [14], König [15], Simons [18–19], Horvath [4] and so on. As applications of his intersection theorems for set-valued mappings, many minimax theorems are derived in [11]. In particular, Kindler proved the following topological existence theorem of constant selector for set-valued mappings (e.g., see Corollary 1 of Kindler [9], and see also Proposition 2 of Stachó [22]):

THEOREM A. *Let X and Y be topological spaces and $F : X \to 2^Y$ be a set-valued mapping with non-empty values such that*

(i) *the graph of F is closed in $X \times Y$;*

(ii) *Y is compact;*

(iii) *for each $A \in \mathcal{F}(X)$, if the set $\cap_{x\in A}F(x)$ is non-empty, then $\cap_{x\in A}F(x)$ is connected, where $\mathcal{F}(X)$ denotes the family of all non-empty finite subsets of X;*

(iv) *for each $x_1, x_2 \in X$, the set $\{x \in X : F(x) \subset F(x_1) \cup F(x_2)\}$ is connected.*

Then F has a constant selector, i.e., $\cap_{x\in X}F(x) \neq \emptyset$.

We recall that a topological space X is called an *interval space* (e.g., see Stachó [22, p.158]) if there exists a mapping $[\cdot,\cdot] : X \times X \to \{$the family of all non-empty connected subsets of $X\}$ such that $x_1, x_2 \in [x_1, x_2] = [x_2, x_1]$ for all $x_1, x_2 \in X$.

As the condition (iv) of Theorem A above implies that X is an interval space (the inverval structure mapping $[\cdot,\cdot]$ can be defined by $[x_1, x_2] := \{x \in X : F(x) \subset F(x_1) \cup F(x_2)\}$ for each $x_1, x_2 \in X$), thus Proposition 2 of Stachó in [22] indeed includes Theorem A above as a special case.

If the graph of F in Theorem A above is closed, it then implies the following property (i)$'$:

(i)$'$ $F(x)$ and $F^{-1}(y) := \{x \in X : y \in F(x)\}$ are closed for each $x \in X$ and $y \in Y$.

It is also clear that the converse does not hold in general, i.e., condition (i)$'$ above could not guarantee that the graph of F is closed. On the other hand, motivated by the existence of minimax theorems for separately upper (or lower) semicontinuous functions (e.g., see Remark 4.4 and question 4.5 of Kindler and Trost [12, p.45]; Remark of Joó [6, p.171] and König [15, p.57]), Kindler asked the following question in [9, p.1007]:

QUESTION. *Does the conclusion of Theorem* A *above remain true, if condition* (i) *is replaced by condition* (i)$'$*?*

Being motivated by the above question, we first prove some topological intersection theorems for set-valued mappings without closed graphs. Then we show that one of our intersection theorems answers Kindler's question above in the affirmative when the underlying space X has the so-called α-*connectedness structure* in the sense of Tuy [24, p.145]. Indeed, our result shows that the positive answer for Kindler's question above remains true even when the space Y is not compact provided the space X is a non-empty convex subset of a vector space E without any topology. As applications, topological fixed points and minimax theorems are derived. Finally, some topological variational inequalities are also given. Our results improve and unify many corresponding results in the literature (e.g., see Chang et al [1], Joó [6], Geraghty and Lin [2], Komornik [14], Kindler [9] and so on).

2. Topological intersection theorems

In order to explain our results clearly, we first recall some notations and definitions. Let X be a non-empty set. Then $\mathcal{F}(X)$ and 2^X denote the family of all non-empty finite subsets of X and the family of all subsets of X, respectively. Let X and Y be two sets. A set-valued mapping $F : X \to 2^Y$ is said to have closed (resp., open) inverse values if the set $F^{-1}(y) := \{x \in X : y \in F(x)\}$ is closed (resp., open) in X for each $y \in Y$. If X and Y are topological spaces, the set-valued mapping $F : X \to 2^Y$ is said to be upper (resp., lower) semicontinuous if the set $\{x \in X : F(x) \subset U\}$ is open (resp., closed) for each open (resp., closed) subset U in Y.

We now have the following topological intersection theorem:

THEOREM 1. *Let X and Y be both topological spaces. Suppose that $F : X \to 2^Y$ is a set-valued mapping with non-empty compact values such that*

(1) *for each $x, y \in X$, there exists a continuous mapping $u_{x,y} : [0,1] \to X$ with $u_{x,y}(0) = x$, $u_{x,y}(1) = y$ and $F((u_{x,y}(t)) \subset F(u_{x,y}(t_1)) \cup F(u_{x,y}(t_2))$ for each $t \in [t_1, t_2] \subset [0,1]$ (resp., the set $\{x \in X : F(z) \subset F(x) \cup F(y)\}$ is connected);*

(2) *for each $A \in \mathcal{F}(X)$, if the set $\cap_{x\in A} F(x)$ is non-empty, then $\cap_{x\in A} F(x)$ is connected;*

(3) *for each $y \in Y$, the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is closed (resp., $F^{-1}(y)$ is open (and thus F is lower semicontinuous)) in X.*

Then $\cap_{x\in X} F(x) \neq \emptyset$.

PROOF. We shall first prove that $F(x) \cap F(y) \neq \emptyset$ for each $x, y \in X$. Suppose $F(x) \cap F(y) = \emptyset$. By (1), there exists a continuous function $u_{x,y} : [0,1] \to X$ such that $F(u_{x,y}(t)) \subset F(x) \cup F(y)$ for each $t \in [0,1]$. Let

$M_1 = \{t \in [0,1] : F(u_{x,y}(t)) \subset F(x)\}$ and $M_2 = \{t \in [0,1] : F(u_{x,y}(t)) \subset F(y)\}$. Then M_1 and M_2 are non-empty and disjoint. Note that for each $t \in [0,1]$, $F(u_{x,y}(t)) \subset F(x) \cup F(y)$, $F(u_{x,y}(t))$ is connected, and $F(x)$ and $F(y)$ are both closed and disjoint, so that $F(u_{x,y}(t)) \subset F(x)$ or $F(u_{x,y}(t)) \subset F(y)$. Thus t must be in M_1 or M_2. Therefore $M_1 \cup M_2 = [0,1]$.

Next we shall prove that both M_1 and M_2 are open in $[0,1]$. Let $t_0 \in M_1$ and $x_0 = u_{x,y}(t_0)$. Then $F(x_0) \subset F(x)$. For each $z \in F(y)$, we have $z \notin F(x_0)$, i.e., $x_0 \in X \setminus F^{-1}(z)$. By (3), $X \setminus F^{-1}(z)$ is open in X, and thus there exists a non-empty open neighborhood V_z of x_0 such that $z \notin F(x')$ for each $x' \in V_z$. Let $U_z = u_{x,y}^{-1}(V_z)$. Then U_z is an open neighborhood of t_0 in $[0,1]$. Thus there exist $\lambda'_z, \lambda''_z \in [0,1]$ with $\lambda'_z < t_0 < \lambda''_z$ such that $t_0 \in [\lambda'_z, \lambda''_z] \subset U_z$. Therefore $z \notin F(u_{x,y}(\lambda'_z)) \cup F(u_{x,y}(\lambda''_z))$. Since $F(u_{x,y}(\lambda'_z))$ and $F(u_{x,y}(\lambda''_z))$ are compact, there exist non-empty open neighborhoods V'_z and V''_z of z such that $z' \notin F(u_{x,y}(\lambda'_z))$ and $z'' \notin F(u_{x,y}(\lambda''_z))$ for all $z' \in V'_z$ and $z'' \in V''_z$. Let $V_z = V'_z \cap V''_z$. Then V_z is a non-empty open neighborhood of z such that $z' \notin F(u_{x,y}(\lambda'_z)) \cup F(u_{x,y}(\lambda''_z))$ for each $z' \in V_z$. Taking over $z \in F(y)$, then the family $\{V_z : z \in F(y)\}$ is an open cover of $F(y)$. By the compactness of $F(y)$, there exists a finite number of subsets $\{V_{z_1}, \dots, V_{z_n}\}$ of the family $\{V_z : z \in F(y)\}$ such that $\cup_{i=1}^n V_{z_i} \supset F(y)$. Now let $\lambda' := \max\{\lambda'_{z_i} : i = 1, \dots, n\}$ and $\lambda'' := \min\{\lambda''_{z_i} : i = 1, \dots, n\}$. Then $\lambda' < t_0 < \lambda''$ and (λ', λ'') is a non-empty open neighborhood of t_0 in $[0,1]$. Moreover, for each $t \in (\lambda', \lambda'')$, $z \notin F(u_{x,y}(t))$ for all $z \in F(y)$. Thus $(\lambda', \lambda'') \subset M_1$. It follows that M_1 is open in $[0,1]$. Similarly, the set M_2 is also open in $[0,1]$. Thus the segment $[0,1]$ is the union of two non-empty, disjoint open subsets M_1 and M_2, which is impossible. Therefore we must have $F(x) \cap F(y) \neq \emptyset$ for each $x, y \in X$.

(Resp., if the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for each $y \in Y$, we shall be able to show that $F(x) \cap F(x) \neq \emptyset$ for each $x, y \in X$. If not, there exist $x_1, x_2 \in X$ such that $F(x_1) \cap F(x_2) = \emptyset$. Let $C = \{x \in X : F(x) \subset F(x_1) \cup F(x_2)\}$ and $M_i = \{x \in C : F(x) \subset F(x_i)\}$ for $i = 1, 2$. Then C is connected and $M_1 \cup M_2 = C$ by the condition (1). Note that $F(x_1) \cap F(x_2) = \emptyset$, so that $M_1 \cap M_2 = \emptyset$. By (3), F is lower semicontinuous and $F(x_i)$ is closed, hence the set $M'_i = \{x \in X : F(x) \subset F(x_i)\} = \{x \in X : F(x) \subset X \setminus F(x_{3-i})\}$ is closed in X. Note that $M'_i \cap C = M_i$, so that M_i is closed in C. As C is connected, both M_1 and M_2 are non-empty closed in C, $M_1 \cap M_2 = \emptyset$ and $C = M_1 \cup M_2$, which is a contraction. This contradiction shows that $F(x_1) \cap F(x_2) \neq \emptyset$ for each $x_1, x_2 \in X$).

Finally, we shall prove that $\cap_{x \in A} F(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$ by the induction. Without loss of generality, we may assume that $\cap_{i=1}^n F(x_i) \neq \emptyset$ for each $x_1, \dots, x_n \in X$, where $n \in \mathbf{N}$ and $n \geqq 2$. Define a mapping $F_1 : X \to 2^Y$ by $F_1(x) = \cap_{i=1}^{n-1} F(x_i) \cap F(x)$ for each $x \in X$. Then $F_1(x) \neq \emptyset$ for each

$x \in X$. Moreover it is easy to verify that F_1 satisfies all hypotheses (1)–(3). Thus $F_1(x') \cap F_1(x'') \neq \emptyset$ for each $x', x'' \in X$ by the earlier proof above. Let $x' = x_n$ and $x'' = x_{n+1}$, then $\cap_{i=1}^{n+1} F(x_i) \neq \emptyset$. Therefore the compact family $\{F(x) : x \in X\}$ has the finite intersection property, so that $\cap_{x \in X} F(x) \neq \emptyset$. □

REMARK 1. We would like to point out that the condition (1) of Theorem 1 implies that X is an interval space with a practically interval mapping $[\cdot,\cdot]$ defined by $[x, y] := \{u_{x,y}(t) : t \in [0,1]\}$ for each $x, y \in X$. However we do not know whether Theorem 1 is still true if X is a general interval space such that the condition (1) of Theorem 1 is replaced by the following:

(1)′ $F(z) \subset F(x) \cup F(y)$ for all $z \in [x, y]$ for each $x, y \in X$.

We also remark that Theorem 1 not only improves corresponding results of Theorem 1 of Chang et al [1], but it also shows that the condition '*X is a W-space*' is superfluous which was given by Chang et al [1] (we recall that a topological space is called to be a *W-space* (e.g., see Chang et al. [1, p.231]) if there exists a family $\{C_A : A \in \mathcal{F}(X)\}$ of non-empty connected subsets of X, indexed by $A \in \mathcal{F}(X)$ such that $A \subset C_A$).

Let X be a non-empty convex subset of a vector space E. In what follows, we shall denote by $[x_1, x_2]$ the line segment $\{tx_1 + (1-t)x_2 : t \in [0,1]\}$, equipped with the Euclidean topology (of course, each topological vector space E with this structure $[x, y]$ is an interval space). A function $f : X \to \mathbf{R} \cup \{-\infty, +\infty\}$ is said to *be segment upper (resp., lower) semicontinuous* if the function $t \to f(tx_1 + (1-t)x_2)$ is upper (resp., lower) semicontinuous on $[0,1]$ for each given $x_1, x_2 \in X$.

When X is a non-empty subset of a vector space, the following theorem can be shown to follow from Theorem 1. However we provide a direct proof here.

THEOREM 2. *Let X be a convex of a vector space E and Y be a topological space. Suppose that $F : X \to 2^Y$ is a set-valued mapping with non-empty compact values such that*

(1) *for each $x, y \in X$, $F(z) \subset F(x) \cup F(y)$ for each $z \in [x, y]$;*

(2) *for each $A \in \mathcal{F}(X)$, if the set $\cap_{x \in A} F(x)$ is non-empty, then $\cap_{x \in A} F(x)$ is connected;*

(3) *for each $y \in Y$, the set $F^{-1}(y)$ is closed in the line segment $[x_1, x_2]$ for each $x_1, x_2 \in X$.*

Then $\cap_{x \in X} F(x) \neq \emptyset$.

PROOF. *Step 1.* We shall first prove that $F(x_1) \cap F(x_2) \neq \emptyset$ for each $x_1, x_2 \in X$. If it were false, then there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $F(x_1) \cap F(x_2) = \emptyset$. Now let $M_i := \{x \in [x_1, x_2] : F(x) \subset F(x_i)\}$ for $i = 1,2$. Note that $F(x)$ is non-empty connected for each $x \in X$, we must have $[x_1, x_2] = M_1 \cup M_2$ by the condition (1). For simplicity, we shall denote the set

$[x_1, x_2] \setminus \{x_2\}$ by $[x_1, x_2) = (x_2, x_1]$. Now we claim that both M_1 and M_2 are open in $[x_1, x_2]$. If not, without loss of generality, we assume M_1 is not open in $[x_1, x_2]$. Then there exist a point $x_0 \in M_1$ and a convergent net $\{x_\alpha\}_{\alpha \in I}$ in M_2 such that $x_\alpha \to x_0$. In order to prove $\cap_{x \in M_2} F(x) \neq \emptyset$, it suffices to prove that the compact family $\{F(z) : z \in M_2\}$ has the finite intersection property. For each $z_1, z_2 \in M_2$, there exist $\lambda_1, \lambda_2 \in [0,1]$ such that $z_i = (1 - \lambda_i)x_1 + \lambda_i x_2$ for $i = 1, 2$. Without loss of generality, we may assume that $\lambda_1 \leqq \lambda_2$, so that $[x_1, z_1] \subset [x_1, z_2]$. Note that $z_1 \in [x_1, z_2]$, so that $F(z_1) \subset F(x_1) \cup F(z_2)$ by the condition (1). Since $F(z_1) \subset F(x_2)$ and $F(x_1) \cap F(x_2) = \emptyset$, so that $F(z_1) \subset F(z_2)$. Therefore $F(z_1) \cap F(z_2) \neq \emptyset$ for each $z_1, z_2 \in M_2$. Next we shall prove that $\cap_{z \in A} F(z) \neq \emptyset$ for each $A \in \mathcal{F}\big((x_0, x_2]\big)$. Assume that $\cap_{i=1}^n F(x_i) \neq \emptyset$ for each $x_1, x_2, \ldots, x_n \in X$ and $n \geqq 2$. Now define a mapping $F_1 : X \to 2^Y$ by $F_1(x) = \cap_{i=1}^{n-1} F(x_i) \cap F(x)$ for each $x \in X$. Then it is clear that F_1 satisfies all hypotheses (1)–(3). By repeating the same arguments above, we have that $F_1(x') \cap F_1(x'') \neq \emptyset$ for each $x', x'' \in (x_0, x_2] = M_2$. Let $x' = x_n$ and $x'' = x_{n+1}$. Then $\cap_{i=1}^{n+1} F(x_i) \neq \emptyset$. Thus the compact family $\{F(x) : x \in (x_0, x_2]\}$ has the finite intersection property. Therefore $\cap_{z \in (x_0, x_2]} F(z) \neq \emptyset$. Taking $y_0 \in \cap_{z \in (x_0, x_2]} F(z)$, then $y_0 \in F(x)$ for each $x \in (x_0, x_2]$. Therefore $(x_0, x_2] \subset F^{-1}(y_0)$. Since $\{x_\alpha\}_{\alpha \in I}$ is a net in M_2 such that $x_\alpha \to x_0 \in M_1$, so $x_\alpha \in F^{-1}(y_0)$. Note that $F^{-1}(y_0)$ is closed in $[x_1, x_2]$ by (3), we have that $x_0 \in F^{-1}(y_0)$, i.e., $y_0 \in F(x_0)$. Therefore $y_0 \in F(x_0) \cap F(x_2)$, i.e., $F(x_0) \cap F(x_2) \neq \emptyset$ which contradicts that $F(x_0) \cap F(x_2) \subset F(x_1) \cap F(x_2) = \emptyset$. This contradiction shows that $F(x_1) \cap F(x_2) \neq \emptyset$ for each $x_1, x_2 \in X$.

Step 2. By induction, we shall prove that $\cap_{x \in A} F(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$. Without loss of generality, we may assume that $\cap_{i=1}^n F(x_i) \neq \emptyset$ for each $x_1, \ldots, x_n \in X$, where $n \in \mathbf{N}$ and $n \geqq 2$. Define a mapping $F_2 : X \to 2^Y$ by $F_2(x) = \cap_{i=1}^{n-1} F(x_i) \cap F(x)$ for each $x \in X$. Then $F_2(x) \neq \emptyset$ for each $x \in X$. Moreover it is clear that the mapping F_2 satisfies all hypotheses (1)–(3). Therefore $F_2(x') \cap F_1(x'') \neq \emptyset$ for each $x', x'' \in X$ by the proof of Step 1. Let $x' = x_n$ and $x'' = x_{n+1}$. Then $\cap_{i=1}^{n+1} F(x_i) \neq \emptyset$. Therefore the compact family $\{F(x) : x \in X\}$ has the finite intersection property, so that $\cap_{x \in X} F(x) \neq \emptyset$. □

REMARK 2. Theorem 1 answers Kindler's problem above in the affirmative when the underlying topological space X has some so-called α-connected structure which was first introduced by Tuy [25, pp.145–146]. When X is not a topological space, Theorem 2 also shows that Kindler's question above still holds positively when the underlying space X is a non-empty convex subset of a vector space and the topological space Y may not be compact. But we still do not know if the above result remains true in the case X is a topological space instead of having α-connected structure in the sense of Tuy [25] or being a convex subset of a vector space.

Let Y be a regular space and $F : X \to Y$ be upper semicontinuous with closed values. Then F has a closed graph, which in turn implies that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is closed in X for each $y \in Y$. As consequences of Theorem 1, we have the following:

COROLLARY 3. *Let X and Y be both topological spaces. Suppose that $F : X \to 2^Y$ is a set-valued mapping with non-empty compact values such that*

(1) *for each $x, y \in X$, there exists a continuous mapping $u_{x,y} : [0,1] \to X$ with $u_{x,y}(0) = x$, $u_{x,y}(1) = y$ and $F((u_{x,y}(t)) \subset F(u_{x,y}(t_1)) \cup F(u_{x,y}(t_2))$ for each $t \in [t_1, t_2] \subset [0,1]$;*

(2) *for each $A \in \mathcal{F}(X)$, if the set $\cap_{x\in A} F(x)$ is non-empty, then $\cap_{x\in A} F(x)$ is connected;*

(3) *the graph of F is closed in $X \times Y$.*

Then $\cap_{x\in X} F(x) \neq \emptyset$.

As the condition (3) of Theorem 1 could not, in general, guarantee that the graph of F is closed, thus Theorem 1 is a topological intersection theorem for set-valued mappings which may not have closed graphs.

3. Topological fixed points and minimax inequalities

As applications of Theorems 1 and 2, we have the following topological fixed point theorems:

THEOREM 4. *Let X be a topological space. Suppose that $F : X \to 2^X$ is a set-valued mapping with non-empty compact values such that*

(1) *for each $x, y \in X$, there exists a continuous mapping $u : [0,1] \to X$ with $u(0) = x$, $u(1) = y$ and $F((u(t)) \subset F(u(t_1)) \cup F(u(t_2))$ for each $t \in [t_1, t_2] \subset [0,1]$ (resp., the set $\{x \in X : F(x) \subset F(x) \cup F(y)\}$ is connected);*

(2) *for each $A \in \mathcal{F}(X)$, if the set $\cap_{x\in A} F(x)$ is non-empty, then $\cap_{x\in A} F(x)$ is connected;*

(3) *for each $y \in X$, the set $F^{-1}(y)$ is closed (resp., $F^{-1}(y)$ is open) in X.*

Then there exists $x_0 \in X$ such that $x_0 \in F(x_0)$.

PROOF. By Theorem 1, $\cap_{x\in X} F(x) \neq \emptyset$. Take any fixed $x_0 \in \cap_{x\in X} F(x)$. Then $x_0 \in F(x_0)$. □

Similarly, as an application of Theorem 2, we have the following:

THEOREM 5. *Let X be a convex subset of a vector space E (which may not be a topological vector space). Suppose that $F : X \to 2^X$ is a set-valued mapping with non-empty compact values such that*

(1) *for each $x, y \in X$, $F(z) \subset F(x) \cup F(y)$ for each $z \in [x, y]$, where $[x, y]$ is a line segment;*

(2) *for each $A \in \mathcal{F}(X)$, if the set $\cap_{x\in A} F(x)$ is non-empty, then $\cap_{x\in A} F(x)$*

is connected;

(3) *for each $y \in X$, the set $F^{-1}(y)$ is closed in the line segment $[x_1, x_2]$ for each $x_1, x_2 \in X$.*

Then there exists $x_0 \in X$ such that $x_0 \in F(x_0)$.

Let $P := \cap_{x\in X} F(x)$ in Corollaries 4 and 5. Then P is, in fact a non-empty fixed point set of the mapping F. We also note that Theorems 4 and 5 improve and unify corresponding fixed point theorems of Joó [5–6] and Proposition 2 of Stachó [22] in several aspects.

As another application of the topological intersection Theorem 2 above, we have the following topological minimax theorem:

THEOREM 6. *Let X be a compact topological space and Y be a non-empty convex subset of a vector space E. Suppose $f : X \times Y \to \mathbf{R} \cup \{-\infty, +\infty\}$ is such that*

(1) *the set $\cap_{y\in A}\{x \in X : f(x,y) \geqq a\}$ is connected or empty for each $A \in \mathcal{F}(Y)$, where $a := \inf_{y\in Y} \max_{x\in X} f(x,y)$;*

(2) *for each fixed $y \in Y$, the mapping $x \mapsto f(x,y)$ is upper semicontinuous;*

(3) *for each fixed $x \in X$, the set $\{y \in Y : f(x,y) < a\}$ is convex in Y; and*

(4) *for each fixed $x \in Y$, the mapping $y \mapsto f(x,y)$ is segment upper semicontinuous.*

Then

$$\max_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{y\in Y} \max_{x\in X} f(x,y).$$

PROOF. We define a set-valued mapping $F : Y \to 2^X$ by

$$F(y) = \{x \in X : f(x,y) \geqq a\}$$

for each $y \in Y$. Since X is compact, $F(x)$ is non-empty and closed for each $y \in Y$ by (2). Moreover we have:

(i) the set $\cap_{y\in A} F(y)$ is connected or empty for each $A \in \mathcal{F}(Y)$ by (1);

(ii) for each $y_1, y_2 \in Y$, $F(z) \subset F(y_1) \cup F(y_2)$ for each $z \in [y_1, y_2]$ by (3);

(iii) for each $x \in X$, the set $F^{-1}(x) = \{y \in Y : x \in F(y)\} = \{y \in Y : f(x,y) \geqq a\}$ which is closed in the segment $[y_1, y_2]$ for each given $y_1, y_2 \in Y$ by (4).

Therefore F satisfies all hypotheses of Theorem 2. By Theorem 2, there exists $x_0 \in X$ such that $x_0 \in \cap_{y\in Y} F(y)$. Thus $f(x_0, y) \geqq \inf_{y\in Y} \max_{x\in X} f(x,y)$ for all $y \in Y$, so that $\max_{x\in X} \inf_{y\in Y} f(x,y) \geqq \inf_{y\in Y} f(x_0,y) \geqq \inf_{y\in Y} \max_{x\in X} f(x,y)$. Note that $\max_{x\in X} \inf_{y\in Y} f(x,y) \leqq \inf_{y\in Y} \max_{x\in X} f(x,y)$ holds in general, we must have

$$\max_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{y\in Y} \max_{x\in X} f(x,y).$$

Theorem 6 improves Theorem 3 of Komornik [14] in the sense that X need not be an interval space. Theorem 6 also includes corresponding results of Geraghty and Lin [2] and Example 5.1 of Kindler [9] as special cases. For other extensive study of topological minimax theorems by employing connectedness, we refer to Kindler [11], König [15], Simons [19] and his survey paper [20].

Before we conclude this paper, we would like to give the following topological variational inequalities.

THEOREM 7. *Let X be a non-empty convex subset of a vector space and Y be a compact topological space. Let (E,C) be a topological Riesz space, where C is a closed cone with $C^o \neq \emptyset$, where C^o denotes the interior of C. Suppose $r \in E$ and $f : X \times Y \to E$ satisfies the following conditions:*

(1) the set $\{y \in Y : f(x,y) \notin r + C^o\}$ is non-empty and closed in Y for each $x \in X$;

(2) for each $y \in Y$, the set $\{x \in X : f(x,y) \notin r + C^o\}$ is closed in each line segment and is convex in X; and

(3) the set $\bigcap_{x \in A}\{y \in Y : f(x,y) \notin r + C^o\}$ is connected or empty for each $A \in \mathcal{F}(X)$.

Then there exists $y_0 \in Y$ such that

$$f(x,y_0) \notin r + C^o \text{ for all } x \in X.$$

PROOF. We define a mapping $F : X \to 2^Y$ by

$$F(x) = \{y \in Y : f(x,y) \notin r + C^o\}$$

for each $x \in X$. Then it is easy to verify that the mapping F satisfies all hypotheses of Theorem 2. By Theorem 2, $\bigcap_{x \in X} F(x) \neq \emptyset$. Taking $y_0 \in \bigcap_{x \in X} F(x)$, then we have $f(x,y_0) \notin r + C^o$ for all $x \in X$. □

As an immediate consequence of Theorem 7, we have the following:

COROLLARY 8. *Let X be a non-empty convex subset of a vector space and Y be a compact topological space. Suppose $f : X \times Y \to \mathbf{R}$ is a real-valued function such that*

(1) for each $x \in X$, there exists $y \in Y$ such that $f(x,y) \leqq 0$;

(2) for each fixed $y \in Y$, $x \mapsto f(x,y)$ is segment lower semicontinuous in X;

(3) for each fixed $x \in X$, $y \mapsto f(x,y)$ is lower semicontinuous in Y;

(4) for each $A \in \mathcal{F}(X)$, the set $\{y \in Y : f(x,y) \leqq 0\}$ is connected or empty; and

(5) for each $y \in Y$, the set $\{x \in X : f(x,y) \leqq 0\}$ is convex.

Then there exists $y_0 \in Y$ such that

$$f(x,y_0) \leqq 0 \quad \text{for all} \quad x \in X.$$

We would like to mention how the concept of 'connectedness' and related 'topological intersection theorems' play roles in the study of some problems in functional analysis could be found in a recent paper of Ricceri [17].

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References

[1] S. S. Chang, Y. J. Cho, X. Wu and Y. Zhang, The topological versions of KKM theorems and Fan's matching theorem with applications, *Topological Methods in Nonlinear Analysis,* **1** (1993), 231–245.
[2] M. A. Geraghty and B. L. Lin, Topological minimax theorems, *Proc. Amer. Math. Soc.,* **91** (1984), 377–380.
[3] C. W. Ha, Minimax and fixed point theorem, *Math. Ann.,* **248** (1980), 73–77.
[4] C. Horvath, Quelques théorémes en théorie des mimi-max, *C. R. Acad. Sci. Paris,* **310** (1990), 269–272.
[5] I. Joó, A simple proof of von Neumann's minimax theorem, *Acta Sci. Math. (Szeged),* **42** (1980), 91–94.
[6] I. Joó, On a fixed point theorem, *Publ. Math. Debrecen,* **36** (1989), 127–128.
[7] I. Joó and L. L. Stachó, A note on Ky Fan's minimax theorem, *Acta Math. Acad. Sci. Hungar.,* **39** (1982), 401–407.
[8] I. Joó and A. P. Sövegjártó, A fixed point theorem, *Ann. Univ. Sci. Budapest. Sectio Math.,* **24** (1981), 9–11.
[9] J. Kindler, Topological intersection theorems, *Proc. Amer. Math. Soc.,* **117** (1993), 1003–1011.
[10] J. Kindler, Intersection theorems and minimax theorems based on connectedness, *J. Math. Anal. Appl.,* **178** (1993), 529–546.
[11] J. Kindler, Intersecting sets in midset spaces I and II, *Arch. Math.,* **62** (1994), 49–57 and 168–176.
[12] J. Kindler and R. Trost, Minimax theorems for interval spaces, *Acta Math. Hungar.,* **54** (1989), 39–49.
[13] H. Komiya, On minimax theorems without linear structure, *Hiyoshi Review of Natural Science,* **8** (1990), 74–78.
[14] V. Komornik, Minimax theorems for upper semicontinuous functions, *Acta Math. Acad. Sci. Hungar.,* **40** (1982), 159–163.
[15] H. König, A general minimax theorem based on connectedness, *Arch. Math.,* **59** (1992), 55–64.
[16] B. L. Lin and X. C. Quan, A symmetric minimax theorem without linear structure, *Arch. Math.,* **52** (1989), 367–370.
[17] B. Ricceri, Recent uses of connectedness in functional analysis, in *Nonlinear Analysis and Convex Analysis* (ed. by W. Takahashi), RIMS 939, Research Institute for Mathematical Science, Kyoto University, Japan, (1996), pp. 11–22.
[18] S. Simons, Minimax theorems with staircases, *Arch. Math.,* **57** (1991), 169–179.
[19] S. Simons, A flexible minimax theorem, *Acta Math. Hungar.,* **63** (1994), 119–132.
[20] S. Simons, *Minimax theorems and their proofs,* in: Minimax and Applications, eds.: D. Z. Du and P. Pardalos, Kulwer Academic Publisher (in press, 1996).
[21] M. Sion, On general minimax theorems, *Pacific. J. Math.,* **8** (1958), 171–176.

[22] L. L. Stachó, Minimax theorems beyond topological vector spaces, *Acta Sci. Math. (Szeged)*, **42** (1980), 157–164.
[23] F. Terkelsen, Some minimax theorems, *Math. Scand.*, **31** (1971), 177–194.
[24] H. Tuy, On a general minimax theorem, *Soviet. Math. Dokl.*, **15** (1974), 1689–1693.
[25] H. Tuy, On the general minimax theorem, *Colloq. Math.*, **33** (1975), 145–158.
[26] Wen-Tsün Wu, A remark on the fundamental theorem in the theory of games, *Science Record (new series)*, **3** (1959), 229–233.

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