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ON THE EQUIVALENCE OF MINIMAX THEOREMS

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Abstract. In this note we show that Ky Fan's minimax theorem and its several generalizations such as König's minimax theorem [6], M. Neumann's minimax theorem [8] and Fuchssteiner–König's minimax theorem [3] are equivalent. We also give a direct proof for Fuchssteiner–König's minimax theorem on the basis of Eidelheit's well-known seperation theorem.

1. Introduction

In 1953, Ky Fan [2] proved a minimax theorem for a function with convexlike-concavelike properties generalizing the Kneser minimax and then the von Neumann minimax theorem. Since then, there is a living interest for the axiomatic character of minimax theorems. In 1968, König [6] extended the Ky Fan minimax theorem to the case where the function has mid-convexlike-concavelike properties. The König minimax theorem was further extended by M. Neumann [8] in 1977, and the result of M. Neumann was subsequently extended by Fuchssteiner and König [3] in 1980 by weakening the convexlike-concavelike conditions. Borwein and Zhuang [1] and Kassay [5] gave a simple proof of the Fan and König minimax theorem, respectively. In 1994, Stachó [12] derived the König minimax theorem (1968) from the Ky Fan minimax theorem by using a function lifting. In this note we give a simple proof for the Fuchssteiner–König minimax theorem from the Ky Fan minimax theorem. This implies that all the minimax theorems mentioned above are equivalent. Finally, we also give an elementary proof of the Fuchssteiner–König minimax theorem using a standard separation theorem. For a survey of minimax theorems, we refer to Simons [11].

2. Minimax theorems

DEFINITION 1 [9]. Let $f : X \times Y \to R$, where X and Y are arbitrary nonempty sets. The function f is said to be nearly subconvexlike on Y, if

$$(2.1)\qquad \begin{cases} \exists \alpha \in (0,1), \quad \forall \varepsilon > 0, \quad \forall y_1, y_2 \in Y, \quad \exists y_3 \in Y, \quad \forall x \in X, \\ \text{such that} \quad f(x, y_3) \leqq \alpha f(x, y_1) + (1-\alpha) f(x, y_2) + \varepsilon; \end{cases}$$

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f is said to be nearly subconcavelike on X, if

$$(2.2)\qquad \begin{cases} \exists \beta \in (0,1), \quad \forall \varepsilon > 0, \quad \forall x_1, x_2 \in X, \quad \exists x_3 \in X, \quad \forall y \in Y, \\ \text{such that} \quad f(x_3, y) \geqq \beta f(x_1, y) + (1-\beta) f(x_2, y) - \varepsilon. \end{cases}$$

The function f is said to be

(i) subconvexlike on Y [resp. subconcavelike on X], if (2.1) [resp. (2.2)] holds for all $\alpha \in (0,1)$ [resp. all $\beta \in (0,1)$];

(ii) nearly convexlike on Y [resp. nearly concavelike on X], if (2.1) [resp. (2.2)] holds for $\varepsilon = 0$;

(iii) convexlike on Y [resp. concavelike on X], if (2.1) [resp. (2.2)] holds for $\varepsilon = 0$ and all $\alpha \in (0,1)$ [resp. all $\beta \in (0,1)$];

(iv) mid-convexlike on Y [resp. mid-concavelike on X], if (2.1) [resp. (2.2)] holds for $\varepsilon = 0$ and $\alpha = \frac{1}{2}$ [resp. $\beta = \frac{1}{2}$].

REMARK 1. Obviously, we have the following implications: convexlike $\Longrightarrow$ mid-convexlike $\Longrightarrow$ nearly convexlike $\Longrightarrow$ nearly subconvexlike; convexlike $\Longrightarrow$ subconvexlike $\Longrightarrow$ nearly subconvexlike. For the concavelike properties, we have similar implications. From Remark 4.2 of [9], one can conclude that, if Y (X) is a compact topological space and f is lower (upper) semicontinuous on Y (X), then all the convexlike (concavelike) properties for f defined in Definition 2.1 are equivalent. Note that, in Theorem 4.1 and Remark 4.2 of [9], the Hausdorff assumption on Y is not necessary.

In 1953, Ky Fan established the following result generalizing the Kneser minimax theorem:

THEOREM 1 [2]. *Let X be a nonempty set and Y a nonempty compact topological space. Let $f : X \times Y \to R$ be lower semicontinuous, convexlike on Y and concavelike on X. Then*

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In [1], Borwein and Zhuang gave a very short proof of Theorem 1 by using the Eidelheit separation theorem. In 1968, König proved the following result generalizing Theorem 1:

THEOREM 2 [6]. *Let X be a nonempty set and Y a nonempty compact topological space. Let $f : X \times Y \to R$ be lower semicontinuous, mid-convexlike on Y and mid-concavelike on X. Then*

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In [5], Kassay gave an elementary proof of Theorem 2 by using so-called methods of level sets and cones. Theorem 2 was further extended by M. Neumann in 1977 and by Fuchssteiner–König in 1980.

THEOREM 3 [3], [7]. *Let X be a nonempty set and Y a nonempty compact topological space. Let $f : X \times Y \to R$ be lower semicontinuous, nearly subconvexlike on Y and nearly subconcavelike on X. Then*

$$\min_Y \sup_X f = \sup_X \min_Y f.$$

In [8], Neumann proved Theorem 3 by assuming that f is nearly concavelike on X instead of that f is nearly subconcavelike on X.

In [4], Jeyakumar proved a generalization of Theorem 1 for a function f with subconvexlike-subconcavelike property by a theorem of the alternative. His result is a special case of Theorem 3.

In [12], Stachó gave an immediate deduction of Theorem 2 from Theorem 1. In the following, we shall give a simple proof of Theorem 3 from Theorem 1. For this purpose, we need the following lemma.

A function $f : X \times Y \to R$ is said to nearly τ-subconcavelike on X if (2.2) holds for $\beta = \tau$.

LEMMA 1 [9]. *If f is nearly subconcavelike on X, then*

$$\Omega = \left\{ \tau \in (0,1),\ f \text{ is nearly } \tau\text{-subconcavelike on } X \right\}$$

is dense in $[0,1]$.

REMARK 2. By S^m we denote the $m-1$-dimensional simplex, i.e.

$$S^m = \left\{ (\lambda_1, \dots, \lambda_m) \in R^m : \lambda_i \geqq 0, \sum_{i=1}^m \lambda_i = 1 \right\}.$$

Let M be a subset of S^m with the following property: for every $t = (t_1, \dots, t_m) \in M$, $x_1, \dots, x_m \in X$ and $\varepsilon > 0$ there exists $x_t \in X$ such that

$$\sum_{i=1}^m t_i f(x_i, y) \leqq f(x_t, y) + \varepsilon, \quad \text{for all} \quad y \in Y.$$

From Lemma 1, it is easy to show that M is dense in S^m if f is nearly subconcavelike on X.

PROOF OF THEOREM 3. Let α be an arbitrary real number strictly less than $\min_Y \sup_X f$. Let $C(x) = \{ y \in Y : f(x,y) > \alpha \}$. Then $Y = \cup_{x \in X} C(x)$. Since $C(x)$ is open for each $x \in X$ and Y is compact, one can find $x_1, \dots, x_m \in X$ such that

$$\alpha < \min_Y \max_{1 \leqq i \leqq m} f(x_i, y).$$

Define a function $\phi : S^m \times Y \to R$ by

$$\phi(\lambda, y) = \sum_{i=1}^{m} \lambda_i f(x_i, y).$$

Clearly, we have

$$\min_Y \max_{1 \leqq i \leqq m} f(x_i, y) = \min_Y \sup_{S^m} \phi(\lambda, y).$$

It is clear that ϕ is affine (finite, convex and concave) in its first variable and then concavelike on S^m, and lower semicontinuous in its second variable. From Remark 1, one can see that ϕ is convexlike on Y. By Theorem 1, we have

$$\min_Y \sup_{S^m} \phi(\lambda, y) = \sup_{S^m} \min_Y \phi(\lambda, y).$$

Thus

$$\min_Y \max_{1 \leqq i \leqq m} f(x_i, y) = \sup_{S^m} \min_Y \phi(\lambda, y).$$

Since, for each $y \in Y$, $\lambda \to \phi(\lambda, y)$ is a continuous affine function, by Theorem 10.2 of [10], the function $\lambda \to \min_Y \phi(\lambda, y)$ is continuous on S^m. Thus there exists some $\lambda' \in S^m$ such that

$$\min_Y \max_{1 \leqq i \leqq m} f(x_i, y) = \min_Y \phi(\lambda', y).$$

Since $\lambda \to \min_Y \phi(\lambda, y)$ is continuous on S^m and the set M (in Remark 2) is dense in S^m, for any $\varepsilon > 0$ there exists an $\mu \in M$ such that

$$\min_Y \max_{1 \leqq i \leqq m} f(x_i, y) < \min_Y \phi(\mu, y) + \varepsilon = \min_Y \sum_{i=1}^{m} \mu_i f(x_i, y) + \varepsilon.$$

The definition of the set M implies that there exists $\bar{x} \in X$ such that

$$\min_Y \max_{1 \leqq i \leqq m} f(x_i, y) < \min_Y f(\bar{x}, y) + 2\varepsilon \leqq \sup_X \min_Y f(x, y) + 2\varepsilon.$$

Hence

$$\alpha \leqq \sup_X \min_Y f(x, y).$$

From the choice of α, we have

$$\min_Y \sup_X f(x, y) \leqq \sup_X \min_Y f(x, y),$$

which completes the proof. □

Therefore, we conclude that Theorems 1, 2, 3 and Corollary 3.1 of [4] are equivalent.

REMARK 3. We give an elementary proof of Theorem 3 using the Eidelheit separation theorem. The proof is adapted from that of Theorem A in [1] and Theorem in [5].

PROOF. Let α be an arbitrary real number strictly less than $\min_Y \sup_X f$. Let $C(x) = \{ y \in Y : f(x,y) > \alpha \}$. Then $Y = \cup_{x\in X} C(x)$. Since $C(x)$ is open for each $x \in X$ and Y is compact, one can find $x_1, \dots, x_m \in X$ such that

$$\alpha < \min_Y \max_{1 \leqq i \leqq m} f(x_i, y).$$

Define a mapping $\psi : Y \to R^m$ as follows

$$\psi(y) = (f(x_1, y) - \alpha_1, \dots, f(x_m, y) - \alpha_1),$$

where $\alpha_1 = \min_Y \max_{1\leqq i \leqq m} f(x_i, y)$. Since f is nearly subconvexlike on Y, we have $\exists \beta \in (0,1)$, $\forall \varepsilon > 0$, $\forall y_1, y_2 \in Y$, $\exists y_3 \in Y$, such that $\varepsilon s + \beta\psi(y_1) + (1-\beta)\psi(y_2) - \psi(y_3) \in R^m_+$, where $s = (1, \dots, 1) \in \operatorname{int} R^m_+$. This means that ψ is a nearly R^m_+-subconvexlike mapping (see [9]). By Theorem 3.1 of [9], we see that $\psi(Y) + \operatorname{int} R^m_+$ is convex and so

$$\overline{\psi(Y) + R^m_+} = \overline{\psi(Y) + \operatorname{int} R^m_+}$$

is convex.

From the definition of ψ, it is clear that

$$(\psi(Y) + R^m_+) \cap (-\operatorname{int} R^m_+) = \emptyset.$$

Since $\operatorname{int} R^m_+$ is open, we have that

$$\overline{\psi(Y) + R^m_+} \cap (-\operatorname{int} R^m_+) = \emptyset.$$

By the Eidelheit separation theorem, there exists $t = (t_1, \dots, t_m) \in R^m_+ \setminus \{0\}$ such that

$$\sum_{i=1}^m t_i(f(x_i, y) - \alpha_1) \geqq 0 \quad \text{for all} \quad y \in Y.$$

Let $\lambda'_i = t_i / \sum_{j=1}^m t_j$. Then $\lambda' = (\lambda'_1, \dots, \lambda'_m) \in S^m$ and

$$\min_Y \max_{1 \leqq i \leqq m} f(x_i, y) \leqq \min_Y \sum_{i=1}^m \lambda'_i f(x_i, y) = \min_Y \phi(\lambda', y).$$

The inverse inequality is obviously true and thus

$$\min_Y \max_{1\leqq i\leqq m} f(x_i, y) = \min_Y \phi(\lambda', y).$$

By using exactly the same arguments as in the previous proof we can finish the proof. □

In view of the proofs of Theorem 3 above, we obtain a stronger version of Theorem 3 by replacing the nearly subconcavelikeness of f on X with the assumption that the set M (in Remark 2) is dense in S^m.

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