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NEW VERSION OF KKM THEOREM IN PROBABILISTIC METRIC SPACES WITH APPLICATIONS*

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Abstract

In this paper we first introduce the concept of probabilistic interval space. Under this framework, a new version of KKM theorem is obtained. As application, we utilize this result to study some new minimax theorem, section theorem, matching theorem, coincidence theorem and fixed point theorem in probabilistic metric spaces. The results presented in this paper not only contain the main result of von Neumann[7] *as its special case but also extend the corresponding results of* [1, 3, 4, 6, 8] *to the case of probabilistic metric spaces.*

Key words probabilistic metric space, probabilistic interval space, chainability, W-chainability, coincidence point

I. Introduction and Preliminaries

As it is known to all, the KKM theorem plays an important role in the theory of nonlinear functional analysis.

Since 1929 the Polish mathematicians Knaster, Kuratowski and Mazurkiewicz established the famous KKM theorem in finite dimensional space [5], in 1961, Ky Fan extended this theorem to the case of infinite dimension spaces. Recently, KKM theorem has been extended and generalized in various directions by many mathematicians to study a large class of problems arising in variational inequalities, minimax and coincidence point theory, etc.

The purpose of this paper is to introduce the concept of probabilistic interval space. Under this framework, by using the chainability to replace the convexity, a new version of KKM theorem is obtained. As applications, we utilize this result to study the minimax problem, coincidence point problem, section theorem and matching theorem in probabilistic interval space. Our results not only contain the following theorem which is the main result of von Neumann[7] as its special case:

Theorem A Let M and N be two finite dimensional simplexes, $f: M\times N\to R$ be a continuous function such that $x\mapsto f(x, y)$ is quasi-concave and $y\mapsto f(x, y)$ is quasi-

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convex. Then

$$\sup_{x\in M}\inf_{y\in N} f(x, y)=\inf_{y\in N}\sup_{x\in M} f(x, y)$$

but also extend the corresponding results in Komorink[6], Fan[3, 4], Chang and Ma[1] and Park[8] to the cases of probabilistic metric spaces.

Throughout this paper, the concepts, notations, terminologies and properties relating to probabilistic metric spaces are adopted from [2, 9, 12].

In the sequel, we denote by $\mathscr{F}(X)$ the family of all nonempty finite subsets of X.

Definition 1.1 Let Z be a linear ordered space. (1) Z is called to be **complete**, if each nonempty subset in Z has a least upper bound; (2) Z is called to be **dense**, if for any $\alpha, \beta\in Z, \alpha<\beta$, then there exists a $\delta\in Z$ such that $\alpha<\delta<\beta$.

In the sequel, we always assume that Z is a complete dense linear ordered space.

Definition 1.2 Let $(X, \mathscr{F})$ be a probabilistic metric space, and $D\subset X$ be a subset. D is called to be **chainable**, if for any a, $b\in D$ and for any $\varepsilon>0$, $\lambda\in(0, 1]$ there exists a finite subset $\{a=p_0, p_1, \cdots, p_{n-1}, p_n=b\}\subset D$ such that $F_{p_i, p_{i-1}}(\varepsilon)>1-\lambda$ $(i=1, 2, \cdots, n)$. The set $\{p_0, p_1, \cdots, p_n\}$ is called a (ε,λ)-chain joining a and b.

In what follows, we always assume that the empty set ϕ is chainable.

Definition 1.3 Let $(X, \mathscr{F}, \Delta)$ be a Menger probabilistic metric space. $(X, \mathscr{F}, \Delta)$ is called a **probabilistic interval space**, if there exists a mapping $[\cdot,\cdot]: X\times X\to 2^X$ such that for any $x_1, x_2\in X$, $[x_1, x_2]=[x_2, x_1]$ is a chainable subset in X containing x_1 and x_2. The set $[x_1, x_2]$ is called a probabilistic interval in X.

Definition 1.4 Let $(X, \mathscr{F}, \Delta)$ be a probabilistic interval space. A subset $D\subset X$ is called W-**chainable**, if for any $x_1, x_2\in D$, then $[x_1, x_2]\subset D$; $f: X\to Z$ is called probabilistic **quasi convex (resp. concave)**, if the set $\{x\in X: f(x)\leqslant r\}$ $(\{x\in X: f(x)\geqslant r\})$ is W-chainable for all $r\in Z$.

Definition 1.5[6] Let X be a topological space. A mapping $f: X\to Z$ is called **upper (resp. lower) semi-continuous**, if for any $r\in Z$, the set $\{x\in X: f(x)\geqslant r\}$ (resp. $\{x\in X: f(x)\leqslant r\}$) is closed; $f: X\to Z$ is called **upper compact,** if for any $r\in Z$, the set $\{x\in X: f(x)\geqslant r\}$ is compact.

Remark If X is compact and $f: X\to Z$ is upper semi-continuous, then f must be upper compact.

Definition 1.6 A probabilistic interval space $(X, \mathscr{F}, \Delta)$ is called to be **Dadekind complete**, if for any $x_1, x_2\in X$ and for any W-chainable subsets H_1, H_2 in X with $x_1\in H_1, x_2\in H_2$ and $[x_1, x_2]\subset H_1\cup H_2$ then there exists an $x_0\in x$ such that either $x_0\in H_1$, $[x_2, x_0):=[x_2, x_0]\setminus\{x_0\}\subset H_2$ or $x_0\in H_2$ and $[x_1, x_0)\subset H_1$.

A Dadekind complete probabilistic interval space $(X, \mathscr{F}, \Delta)$ is called a **strongly Dadekind complete**, if for any $x_1, x_2\in X$ and for any n points $u_1, \cdots, u_n\in[x_1, x_2)$, then $\bigcap_{i=1}^{n}[u_i, x_2)\neq\phi$.

Proposition 1.1 The intersection of any family of W-chainable subsets in probabilistic interval space is still W-chainable (we stipulate that the empty set ϕ is W-chainable).

Proposition 1.2 Let $(E,\mathscr{F},\Delta)$ be a probabilistic normed linear space and X be a nonempty convex subset of E. Then X is a strongly Dadekind complete probabilistic interval space, where $[x_1, x_2]:=\mathrm{co}\{x_1, x_2\}$ for all $x_1, x_2\in X$.

Proof It is obvious that $(X, \tilde{\mathscr{F}}, \Delta)$ is a probabilistic metric space, where

$\tilde{\mathscr{F}}(a, b) := \mathscr{F}(a-b)$, i. e., $\tilde{F}_{a,b} := F_{a-b}$ for all $a, b \in X$.

Next, we prove that any connected subset M in X is chainable.

Suppose the contrary, there exist $\varepsilon > 0$ and $\lambda \in (0,1]$ and $x, y \in M$ such that there is no any (ε, λ)-chain joining x and y in M. Let A be the subset of all points in M such that each point in A can be joined with X using a (ε, λ)-chain. Denote $B = M \backslash A$, then $X \in A$ and $y \in B$.

If there exists a $p \in A \cap \bar{B}$, then there exists a $q \in B$ such that $\tilde{F}_{p,q}(\varepsilon) > 1 - \lambda$, and so $q \in A$. Hence $q \in A \cap B$, a contradiction (since $A \cap B = \phi$). This shows $A \cap \bar{B} = \phi$.

Similarly, we can prove that $\bar{A} \cap B = \phi$. This implies that A and B are separated. However $M = A \cup B$. This contradicts M being connected. The desired conclusion is proved.

Because $[x_1, x_2] := \mathrm{co}\{x_1, x_2\}$, therefore $[x_1, x_2]$ is a chainable subset in X containing x_1 and x_2. This implies that $(X, \tilde{\mathscr{F}}, \Delta)$ is a probabilistic interval space. By Lemma 1 in [11], we know that X is Dadekind complete, and hence $(X, \tilde{\mathscr{F}}, \Delta)$ is Dadekind complete probabilistic interval space.

Besides, for any $x_1, x_2 \in X$ and for any $z_1, \cdots, z_n \in [x_1, x_2)$ letting $z_i = (1-\lambda_i)x_2 + \lambda_i x_1$, then $0 < \lambda_i \leqslant 1$. Let $\lambda = \min\{\lambda_i : i = 1, \cdots, n\}$, then $0 < \lambda \leqslant 1$. Taking $x_0 = (1-\lambda)x_2 + \lambda x_1$, we have $x_0 \in [x_1, x_2)$. Again taking $a_i = \lambda / \lambda_i$, we have $0 < a_i \leqslant 1$, $i = 1, 2, \ldots, n$ and $x_0 = (1-a_i)x_2 + a_i z_i$, $i = 1, 2, \cdots, n$. This implies that $x_0 \in \bigcup_{i=1}^{n} [z_i, x_2)$.

Therefore $(X, \tilde{\mathscr{F}}, \Delta)$ is a strongly Dadekind complete probabilistic interval space.

This completes the proof.

As pointed out in [10] that if $(X, \mathscr{F}, \Delta)$ is a Menger probabilistic metric space with a continuous t-norm Δ, then X is a Hausdorff topological space in the topology induced by the family of (ε, λ)-neighborhoods $\{U_p(\varepsilon, \lambda) : p \in X, \varepsilon > 0, \lambda > 0\}$, where $U_p(\varepsilon, \lambda) = \{x \in X : F_{x,p}(\varepsilon) > 1 - \lambda\}$.

A subset D in a probabilistic interval space $(X, \mathscr{F}, \Delta)$ is called to be **interval closed (resp. open)** if for any interval $[x_1, x_2] \subset X$, $D \cap [x_1, x_2]$ is a relatively closed (resp. open) set in $[x_1, x_2]$.

A subset D in a topological space X is called **compactly open (resp. closed),** if for any compact subset M in X, $D \cap M$ is a relatively open (resp. closed) set in M.

Let X and Y be two topological spaces, we denote by $\mathscr{C}(X, Y)$ the set of all continuous mappings from X into Y, and we denote

$$\mathscr{C}^*(X, Y) := \{s \in \mathscr{C}(X, Y) : s^{-1} \text{ makes any connected subset in } Y \text{ a connected subset in } X\}.$$

II. New Versions of KKM Theorem in Probabilistic Interval Spaces

In order to prove our main results, we need the following lemmas:

Lemma 2.1 Let $(X, \mathscr{F}, \Delta)$ be a probabilistic interval space, Y be a nonempty set and $G : X \to 2^Y$ be a mapping. Then for any $y \in Y$, $X \backslash G^{-1}(y)$ is W-chainable if and only if for $x_1, x_2 \in X$, we have $G(x) \subset \bigcup_{i=1}^{2} G(x_i)$ for all $x \in [x_1, x_2]$.

Proof If for any $y \in Y$, the set $X \backslash G^{-1}(y)$ is W-chainable, then for any $x_1, x_2 \in X$ and for any $x \in [x_1, x_2]$ when $y \notin G(x_1) \cup G(x_2)$, we have $\{x_1, x_2\} \subset X \backslash G^{-1}(y)$, an so $[x_1, x_2] \subset X \backslash G^{-1}(y)$. Therefore $x \notin G^{-1}(y)$, i. e., $y \notin G(x)$. This implies that $G(x) \subset \bigcup_{i=1}^{2} G(x_i)$ for all $x \in [x_1, x_2]$.

Conversely, if for any x_1, $x_2 \in X$ and for any $x \in [x_1, x_2]$, $G(x) \subset \bigcup_{i=1}^{2} G(x_i)$. Hence for any $y \in Y$, any $a, b \in X \backslash G^{-1}(y)$ and for any $x \in [a, b]$, since $a, b \notin G^{-1}(y)$, we know that $y \notin G(a) \cup G(b)$. Since $G(x) \subset G(a) \cup G(b)$, we have $y \notin G(x)$, i. e., $x \in X \backslash G^{-1}(y)$, and so $[a, b] \subset X \backslash G^{-1}(y)$. This implies that $X \backslash G^{-1}(y)$ is W-chainable. This completes the proof.

Lemma 2.2 Let $(X, \mathscr{F}, \Delta)$ be a Menger probabilistic metric space (M-PM space), let $\{p_n\}$, $\{q_n\} \subset X$ be two sequences such that $p_n \to p$, and $q_n \to q$. If $F_{p_n, q_n}\left(\frac{1}{n}\right) > 1 - \frac{1}{n}$ for all $n \in N$ (the set of all natural numbers), then $p = q$.

Proof For any $\varepsilon > 0$ and $\lambda > 0$, it follows from the continuity of Δ that there exists a $\lambda' (0, 1]$ such that $\Delta(1-\lambda', 1-\lambda') > 1-\lambda$. Hence there exists $n_0 \in N$ large enough such that whenever $n \geqslant n_0$ we have $\frac{1}{n} < \min\left\{\frac{\varepsilon}{2}, \lambda'\right\}$ and $F_{p_n, p}\left(\frac{\varepsilon}{2}\right) > 1 - \lambda'$. Therefore for any $n \geqslant n_0$ we have

$$
\begin{aligned}
F_{p, q_n}(\varepsilon) &\geqslant \left(\Delta F_{p, p_n}\left(\frac{\varepsilon}{2}\right), F_{p_n, q_n}\left(\frac{\varepsilon}{2}\right)\right) \\
&\geqslant \Delta(1-\lambda', 1-\lambda') \\
&> 1-\lambda
\end{aligned}
$$

This implies that $q_n \to p$. Since $q_n \to q$, we have $p = q$

This completes the proof.

Theorem 2.1 Let $(X, \mathscr{F}, \Delta)$ be a probabilistic interval space with a continuous t-norm Δ, $(Y, \widetilde{\mathscr{F}}, \widetilde{\Delta})$ be a strongly Dadekind complete probabilistic interval space with a continuous t-norm $\widetilde{\Delta}$ and G: $Y \to 2^X$ be a mapping with nonempty compact values. If the following conditions are satisfied:

(i) For any $A \in \mathscr{J}(Y)$, $\bigcap_{y \in A} G(y)$ is chainable;

(ii) For any $x \in X$, $Y \backslash G^{-1}(x)$ is W-chainable;

(iii) For any $x \in X$, $G^{-1}(x)$ is interval closed;

then $\bigcap_{y \in Y} G(y) \neq \phi$.

Proof First we prove that $\{G(y): y \in Y\}$ has the finite intersection property. We use induction to prove.

Since G has the nonempty values, for any $y \in Y$, $G(y) \neq \phi$. Suppose that for any n elements of $\{G(y): y \in Y\}$ their intersection is nonempty. Now we prove that for any $n+1$ elements of $\{G(y): y \in Y\}$ their intersection is also nonempty. Suppose the contrary, then there exists $\{y_1, \cdots, y_{n+1}\} \subset Y$ such that $\bigcap_{i=1}^{n+1} G(y_i) = \phi$. Letting $H = \bigcap_{i=3}^{n+1} G(y_i)$, then we have

$$
(H \cap G(y_1)) \cap (H \cap G(y_2)) = \phi \tag{2.1}
$$

By the assumption of induction and condition (i) we know that

$$
H \cap G(y) \text{ is nonempty and chainable for all } y \in Y \tag{2.2}
$$

By condition (ii) and Lemma 2.1, for any $u, v \in Y$ we have

$$
G(y) \subset G(u) \cup G(v) \quad \text{for all } y \in [u, v] \tag{2.3}
$$

Especially we have

$$G(y)\subset G(y_1)\cup G(y_2) \text{ for all } y\in[y_1, y_2] \tag{2.4}$$

Therefore we have

$$H\cap G(y)\subset(H\cap G(y_1))\cup(H\cap G(y_2)) \text{ for all } y\in[y_1, y_2]$$

It follows from the chainability of $H\cap G(y)$ and (2.1) that for any $y\in[y_1, y_2]$, we must have that

$$\text{either } H\cap G(y)\subset H\cap G(y_1) \text{ or } H\cap G(y)\subset H\cap G(y_2) \tag{2.5}$$

In fact, if there exist x_1, $x_2\in H\cap G(y)$ such that $x_1\notin H\cap G(y_2)$, and $x_2\notin H\cap G(y_1)$. Hence $x_i\in H\cap G(y_i)$, $i=1, 2$. Since $H\cap G(y)$ is chainable for all $n\in N$ (the set of all natural numbers), there exists a $\left(\frac{1}{n}, \frac{1}{n}\right)$-chain joining x_1 and x_2 in $H\cap G(y)$. Therefore there exist $a_n\in H\cap G(y_1)$ and $b_n\in H\cap G(y_2)$ such that

$$F_{a_n, b_n}\left(\frac{1}{n}\right)>1-\frac{1}{n}$$

On the other hand, since $H\cap G(y_i)$ is compact ($i=1, 2$), without loss of generality we can assume that $a_n\to a\in H\cap G(y_1)$, $b_n\to b\in H\cap G(y_2)$. By Lemma 2.2 we have $a=b$. This contradicts (2.1). Thus (2.5) is true.

Denote $E_i=\{y\in Y: H\cap G(y)\subset H\cap G(y_i)\}$, $i=1, 2$. It is obvious that $y_i\in E_i$, $i=1, 2$ and $[y_1, y_2]\subset E_1\cup E_2$. Again by (2.3), we know that E_1 and E_2 both are W-chaniable subsets in Y. By the Dadekin completeness of Y, there exist $k\in\{1,2\}$ and $y_0\in E_k$ such that $[y_i, y_0)\subset E_i$, where $i\in\{1,2\}\setminus\{k\}$, without loss of generality we can assume that $k=1$, and so $y_0\in E_1$, $[y_2, y_0)\subset E_2$. Hence we get

$$\left.\begin{aligned} &H\cap G(y_0)\subset H\cap G(y_1)\\ &H\cap G(y)\subset H\cap G(y_2), \quad \text{for all } y\in[y_2, y_0)\end{aligned}\right\} \tag{2.6}$$

Besides, since $[y_2, y_0]$ is chainable, for any $n\in N$ there exists $p_n\in[y_2, y_0)$ such that

$$\bar{F}_{p_n; y_0}\left(\frac{1}{n}\right)>1-\frac{1}{n}$$

This implies that $p_n\to y_0$.

Since Y is strongly Dadekind complete, for any $\{u_1, \cdots, u_m\}\subset[y_2, y_0)$, $\bigcap_{i=1}^{m}[u_i,y_0)\cap[y_2, y_0)\neq\phi$. Therefore there exists a $\bar{y}\in[u_i, y_0)\cap[y_2, y_0)$, $i=1, 2, \ldots, m$. It follows from (2.3) and (2.6) that

$$H\cap G(\bar{y})\subset H\cap(G(u_i)\cup G(y_0))\cap(H\cap G(y_2))\subset H\cap G(u_i),\quad i=1, 2, \ldots, m$$

Since $H\cap G(\bar{y})\neq\phi$, $\bigcap_{i=1}^{m}H\cap G(u_i)\neq\phi$. This implies that $\{H\cap G(y): y\in[y_2, y_0)\}$ has the finite intersection property. Since $H\cap G(y)$ is compact for any $y\in Y$, $\bigcap_{y\in[y_2,y_0)} H\cap G(y)\neq\phi$.

Therefore there exists an $x_0\in X$ such that $x_0\in\bigcap_{y\in[y_2,y_0)} H\cap G(y)\subset\bigcap_{n=1}^{\infty}H\cap G(p_n)$.

This implies for any $n\in N$, $x_0\in G(p_n)$, i. e., for any $n\in N$, $p_n\in G^{-1}(x_0)$. Hence $\{p_n\}\subset$

$G^{-1}(x_0)\cap[y_2, y_0]$. It follows from $p_n\to y_0$ and condition (iii) that $y_0\in G^{-1}(x_0)$, i. e., $x_0\in G(y_0)$.

On the other hand, in view of $x_0\in H\cap G(y_2)$ and condition (2.1), we know that $x_0\notin H\cap G(y_1)$. By (2.6), $x_0\notin H\cap G(y_0)$, and so $x_0\notin G(y_0)$, a contradiction. This shows that the intersection of any $n+1$ elements in $\{G(y): y\in Y\}$ is nonempty.

This implies that $\{G(y): y\in Y\}$ has the finite intersection property. Since $G(y)$ is compact for all $y\in Y$, $\bigcap_{y\in Y} G(y)\neq\phi$.

Corollary 2.1 Let $(X, \mathscr{F}, \Delta)$ be a probabilistic interval space with a continuous t-norm Δ, $(Y, \widetilde{\mathscr{F}}, \widetilde{\Delta})$ be a strongly Dadekind complete probabilistic interval space with a continuous t-norm $\widetilde{\Delta}$ and $G: Y\to 2^X$ be a mapping with a nonempty compact W-chainable values. If for any $x\in X$, $G^{-1}(x)$ is interval closed, and $Y\backslash G^{-1}(x)$ is W-chainable, then $\bigcap_{y\in Y} G(y)\neq\phi$.

Corollary 2.2 Let $(X, \mathscr{F}, \Delta)$ be a probabilistic interval space with a continuous t-norm Δ, $(\widetilde{E}, \widetilde{\mathscr{F}}, \widetilde{\Delta})$ be a probabilistic normed lintear space with a continuous t-norm $\widetilde{\Delta}$, Y be a nonempty convex subset in $\widetilde{E}$, and $G: Y\to 2^X$ be a mapping with nonempty W-chainable values. If for any $x\in X$, $G^{-1}(x)\cap \mathrm{co}\{y_1, y_2\}$ is a relatively closed set in $\mathrm{co}\{y_1, y_2\}$ for all $y_1, y_2\in Y$ and $Y\backslash G^{-1}(x)$ is convex. Then $\bigcap_{y\in Y} G(y)\neq\phi$.

Proof For any $y_1, y_2\in Y$, letting $[y_1, y_2]:=\mathrm{co}\{y_1, y_2\}$, by Proposition 1.2, we know that Y is a strongly Dadeking complete probabilistic interval space. Therefore the conclusion is obtained from Proposition 2.4.

Remark Theorem 2.1, Corollaries 2.1 and 2.2 are all the versions of KKM theorem which are first established in probabilistic metric spaces. These results generalize and unify many recent results related to KKM theorem.

III. Minimax Inequalities in Probabilistic Interval Spaces

Theorem 3.1 Let $(X, \mathscr{F}, \Delta)$ be a probabilistic interval space with a continuous t-norm Δ, $(Y, \widetilde{\mathscr{F}}, \widetilde{\Delta})$ be a strongly Dadekind complete probabilistic interval space with a continuous t-norm $\widetilde{\Delta}$ and $\varphi: X\times Y\to Z$ be a mapping satisfying the following conditions:

(i) For any $A\in\mathscr{J}(Y)$ and for any $r\in Z$, $\bigcap_{y\in A}\{x\in X\}: \varphi(x, y)\geqslant r\}$ is chainable;

(ii) For any $y\in Y$, $\varphi(\cdot, y)$ is upper compact;

(iii) For any $x\in X$, $\varphi(x, \cdot)$ is probabilistic quasi-convex and is upper semi-continuous on any probabilistic interval of Y.

Then

$$z_*:=\sup_{x\in X}\inf_{y\in Y}\varphi(x, y)=\inf_{y\in Y}\sup_{x\in X}\varphi(x, y):=z^*$$

Proof First we prove that $z_*\geqslant z^*$.

Suppose the contrary, then $z_*<z^*$. By the density of Z, there exists an $\alpha\in Z$ such that $z_*<\alpha<z^*$. Now we define a set-valued mapping $G: Y\to 2^X$ as follows:

$$G(y):=\{x\in X: \varphi(x, y)\geqslant\alpha\}, \quad \text{for all } y\in Y$$

By condition (ii), for any $y\in Y$, $G(y)$ is compact. By the choice of α, for any $y\in Y$, $G(y)\neq\phi$. Again by condition (i), we know that the condition (i) in Theorem 2.1 is satisfied. On the other hand for any $x\in X$ we have

$$Y\backslash G^{-1}(x)=\{y\in Y:\varphi(x,\ y)<a\}$$

By the probabilistic quasi-convexity of $\varphi(x,\ \cdot)$, $Y\backslash G^{-1}(x)$ is W-chainable. This implies that the condition (ii) in Theorem 2.1 is satisfied. Besides, for any probabilistic interval $[y_1,\ y_2]$, $G^{-1}(x)\cap[y_1,\ y_2]=\{y\in[y_1,\ y_2]:\ \varphi(x,\ y)\geqslant a\}$. By the assumption that $\varphi(x,\ \cdot)$ is upper semi-continuous on any probabilistic interval in Y, we know that $G^{-1}(x)\cap[y_1,\ y_2]$ is a relatively closed subset in $[y_1,\ y_2]$. This implies that the condition (iii) in Theorem 2.1 is satisfied. By Theorem 2.1, $\bigcap\limits_{y\in Y}G(y)\neq\phi$. Therefore there exists an $\bar{x}\in G(y)$ for all $y\in Y$, i. e., $\varphi(\bar{x},\ y)\geqslant a$ for all $y\in Y$. Hence $\inf\limits_{y\in Y}\varphi(\bar{x},\ y)\geqslant a$. Therefore $\sup\limits_{x\in X}\inf\limits_{y\in Y}\varphi(x,\ y)\geqslant a$, i. e., $z^*\geqslant a$. This contradicts the choice of α. Therefore $z_*\geqslant z^*$ is proved.

On the other hand, it is easy to prove $z^*\geqslant z_*$. Hence $z_*=z^*$.

This completes the proof.

Corollary 3.1. Let $(X,\ \mathscr{F},\ \Delta)$ be a probabilistic interval space with a continuous t-norm Δ, $(Y,\ \tilde{\mathscr{F}},\ \tilde{\Delta})$ be a strongly Dadekind complete probabilistic interval space with a continuous t-norm $\tilde{\Delta}$, and $\varphi:\ X\times Y\to Z$ be a mapping satisfying the following conditions:

(i) for any $y\in Y$, $\varphi(\cdot,\ y)$ is upper compact and probabilistic quasi-concave;

(ii) for any $x\in X$, $\varphi(x,\ \cdot)$ is probabilistic quasi-convex and it is upper semi-continuous on any probabilistic interval of Y.

Then

$$\sup_{x\in X}\inf_{y\in Y}\varphi(x,\ y)=\inf_{y\in Y}\sup_{x\in X}\varphi(x,\ y)$$

Corollary 3.2 Let $(X,\ \mathscr{F},\ \Delta)$ be a probabilistic interval space with a continuous t-norm Δ, $(\tilde{E},\ \tilde{\mathscr{F}},\ \tilde{\Delta})$ be a probabilistic normed linear space with a continuous t-norm $\tilde{\Delta}$, Y be a nonempty convex subset in $\tilde{E}$, and $\varphi:\ X\times Y\to Z$ be a mapping satisfying the following conditions:

(i) for any $y\in Y$, $\varphi(\cdot,\ y)$ is upper compact and probabilistic quasi-concave;

(ii) for any $x\in X$, $\varphi(x,\ \cdot)$ is quasi-convex and is upper semi-continuous on any segment in Y.

Then

$$\sup_{x\in X}\inf_{y\in Y}\varphi(x,\ y)=\inf_{y\in Y}\sup_{x\in X}\varphi(x,\ y)$$

Proof The conclusion can be obtained from Proposition 1.2 and Corollary 3.1 immediately.

Remark. Corollary 3.2 (and so Theorem 3.1 and Corollary 3.1) not only contains the main result in Neumann[7] as a special case but also extends Theorem 3 in V. Komorinik[6] to the case of probabilistic interval spaces.

IV. Section Theorems and Matching Theorems

The following section theorem is equavalent to Theorem 2.1.

Theorem 4.1 (Section Theorem) Let $(X,\ \mathscr{F},\ \Delta)$ be a probabilistic interval space with a continuous t-norm Δ, $(Y,\ \tilde{\mathscr{F}},\ \tilde{\Delta})$ be a strongly Dadekind complete probabilistic interval space with a continuous t-norm $\tilde{\Delta}$ and $A\subset X\times Y$ be a subset. If the following conditions are satisfied:

(i) for any $y\in Y$, the section $A(y):=\{x\in X: (x, y)\in A\}$ is nonempty compact, and for any $S\in\mathscr{F}(Y), \bigcap_{y\in S} A(y)$ is chainable;

(ii) for any $x\in X$, $A(x):=\{y\in Y:(x,y)\in A\}$ is interval closed, and $Y\backslash A(x)$ is W-chainable, then there exists an $\bar{x}\in X$ such that $\{\bar{x}\}\times Y\subset A$.

Proof Theorem 2.1 $\Rightarrow$ Theorem 4.1.

For any $y\in Y$, let $G(y)=A(y)$, then for any $x\in X$ we have $G^{-1}(x)=A(x)$. It is easy to claim that under the conditions in Theorem 4.1, all conditions in Theorem 2.1 are satisfied. By Theorem 2.1, there exists an $\bar{x}\in G(y)$ for all $y\in Y$. This implies that there exists an $\bar{x}\in X$ such that $\{\bar{x}\}\times Y\subset A$.

Theorem 4.1 $\Rightarrow$ Theorem 2.1.

Under the conditions in Theorem 2.1, letting $A=\{(x, y)\in X\times Y: x\in G(y)\}$, then $A(y)=G(y)$, $A(x)=G^{-1}(x)$ for all $x\in X, y\in Y$. Hence it is easy to prove that all conditions in Theorem 4.1 are satisfied. By Theorem 4.1, there exists an $\bar{x}\in X$ such that $\{\bar{x}\}\times Y\subset A$, i. e. $(\bar{x}, y)\in A$ for all $y\in Y$. This implies that $\bar{x}\in\bigcap_{y\in Y} G(y)$. Hence $\bigcap_{y\in Y} G(y)\neq\phi$.

Corollary 4.1 Let $(X, \mathscr{F}, \Delta)$, $(Y, \tilde{\mathscr{F}}, \tilde{\Delta})$ and A be the same as in Theorem 4.1. If the following conditions are satisfied:

(i) for any $y\in Y$, the section $A(y)$ is nonempty compact, W-chainable;

(ii) for any $x\in X$, the section $A(x)$ is interval closed and $Y\backslash A(x)$ is W-chainable,

then there exists an $\bar{x}\in X$ such that $\{\bar{x}\}\times Y\subset A$.

Remark Corollary 4.1 generalizes the section theorem in Ky Fan[3] to the case of probabilistic interval space.

Corollary 4.2 Let $(X, \mathscr{F}, \Delta)$ be a strongly Dadekind complete probabilistic interval space with a continuous t-norm Δ and $A\subset X\times X$ be a subset. If the following conditions are satisfied:

(i) for any $y\in X$, the section $A_1(y):=\{x\in X: (x, y)\in A\}$ is compact W-chainable;

(ii) for any $x\in X$, the section $A_2(x):=\{y\in X: (x, y)\in A\}$ is interval closed and $X\backslash A_2(x)$ is W-chainable, then either there exists an $\bar{x}\in X$ such that $(\bar{x}, \bar{x})\notin A$ or there exists an $\bar{x}\in X$ such that $\{\bar{x}\}\times X\subset A$.

Proof If for any $x\in X$, $(x, x)\in A$, then all conditions in Corollary 4.1 are satisfied. It follows from Corollary 4.1 that there exists an $\bar{x}\in X$ such that $\{\bar{x}\}\times X\subset A$.

Theorem 4.2 Let $(X, \mathscr{F}, \Delta)$ be a strongly Dadekind complete compact probabilistic interval space with a continuous t-norm Δ, $(Y, \tilde{\mathscr{F}}, \tilde{\Delta})$ be a probabilistic interval space and $H: X\to 2^Y$ be a set-valued mapping with compactly open values. If the following conditions are satisfied:

(i) for any $x\in X$, $H^c(x):=Y\backslash H(x)$ is W-chainable;

(ii) for any $y\in Y$, $H^{-1}(y)$ is interval open;

(iii) $H(X)=Y$,

then for any $s\in\mathscr{E}^*(X, Y)$ either there exists an $\bar{x}\in X$ such that $s(X)\cap H^c(\bar{x})=\phi$, or there exist $x_1, x_2\in X$ and an $x_0\in[x_1, x_2]$ such that

$$s^{-1}(H^c(x_0)\cap H(x_1)\cap H(x_2))\neq\phi$$

Proof For any given $s\in\mathscr{E}^*(X, Y)$, if for any $x\in X$, $s(X)\cap H^c(x)\neq\phi$, then letting $G(x)=H^c(x)$, we know that $s^{-1}G:X\to 2^X$ is a set-valued mapping with nonempty closed vaues. By condition (i) and $s\in\mathscr{S}^*(X, Y)$, it is easy to know that $s^{-1}G$ satisfies the condition (i) in Theorem 2.1. Besides, for any $x\in X$, $(s^{-1}G)^{-1}(x)=G^{-1}(s(x))=X\backslash H^{-1}(s(x))$. By condition (ii), we know that $(s^{-1}G)^{-1}(x)$ is interval closed. This implies that $s^{-1}G$ satisfies the condition (iii) in Theorem 2.1.

Next we prove that there exist $x_1, x_2\in X$ and $x_0\in[x_1, x_2]$ such that $s^{-1}(H^c(x_0)\cap H(x_1)\cap H(x_2))\neq\phi$. Suppose the contrary, for any $x_1, x_2\in X$ and for any $x\in[x_1, x_2]$ we have

$$s^{-1}(H^c(x)\cap H(x_1)\cap H(x_2))=\phi$$

Hence we have $s(z)\notin H(x_1)\cap H(x_2)$ for all $z\in s^{-1}(H^c(x))$, and so $s(z)\in H^c(x_1)\cup H^c(x_2)$, i. e., $z\in s^{-1}G(x_1)\cup s^{-1}G(x_2)$. Therefore we have $s^{-1}G(x)\subset s^{-1}G(x_1)\cup s^{-1}G(x_2)$. By Lemma 2.1, we know that $s^{-1}G$ satisfies the condition (ii) in Theorem 2.1. It follows from Theorem 2.1 that $\bigcap\limits_{x\in X} s^{-1}G(x)\neq\phi$, and so $\bigcap\limits_{x\in X} G(x)\neq\phi$, i. e., $Y\backslash H(X)\neq\phi$. This contradicts condition (iii). Therefore the conclusion of Theorem 4.2 is true.

Corollary 4.3 Let $(X, \mathscr{F}, \Delta)$ be a strongly Dadekind complete compact probabilistic interval space with a continuous t-norm Δ and $H:X\to 2^X$ be a set-valued mapping with open values. If the following conditions are satisfied:

(i) for any $x\in X$, $H^c(x)$ is W-chainable;

(ii) for any $y\in X, H^{-1}(y)$ is interval open;

(iii) $H(X)=X$ and for any $x\in X$, $H(x)\neq X$,

then there exist $x_1, x_2\in X$ and $x_0\in[x_1, x_2]$ such that $H^c(x_0)\cap H(x_1)\cap H(H_2)\neq\phi$.

Proof Taking $s=I$ (the identity mapping on X) in Theorem 4.2 and letting $Y=X$, then all conditions in Theorem 4.2 are satisfied. Besides since for any $x\in X$, $H(x)\neq X$, we know that $s(X)\subset H^c(x)\neq\phi$ for all $x\in X$. By Theorem 4.2, there exists $x_1, x_2\in X$ and $x_0\in[x_1, x_2]$ such that

$$H^c(x_0)\cap H(x_1)\cap H(x_2)\neq\phi$$

Remark Theorem 4.2 is a new version of matching theorem in probabilistic interval space. It can be compared with the matching theorems in Park[8], Fan[4] and Chang and Ma[1].

V. Coincidence Point Theorems and Fixed Point Theorems

Theorem 5.1 Let $(X, \mathscr{F}, \Delta)$ be a strongly Dadekind complete compact probabilistic interval space with a continuous t-norm Δ, $(Y, \mathscr{F})$ be a probabilistic metric space, $s\in\mathscr{S}^*(X, Y)$ be a given mapping and $G:X\to 2^Y$ be a mapping with compactly closed values. If the following conditions are satisfied:

(i) for any $A\in\mathscr{J}(X)$, $\bigcap\limits_{x\in A} G(x)$ is chainable;

(ii) for any $y\in Y$, $G^{-1}(y)$ is interval closed;

(iii) $s(X)\cap G(x)\neq\phi$ for all $x\in X$;

(iv) for any $x\in X$, $X\backslash G^{-1}(s(x))$ is W-chainable.

then there exists an $\bar{x}\in X$ such that $s(\bar{x})\in G(\bar{x})$.

Proof Now we consider the set-valued mapping $s^{-1}G:X\to 2^X$. Since G is compactly closed values, by (iii), $s^{-1}G$ has nonempty compact values. Again by condition (i) and $s\in\mathscr{S}^*(X, Y)$, $s^{-1}G$ satisfies the condition (i) in Theorem 2.1. From conditions (ii) and (iv), we know

$s^{-1}G$ satisfies conditions (iii) and (ii) in Theorem 2.1. Therefore we have $\bigcap_{\bar{x}\in X} s^{-1}G(x)\neq\phi$, and so there exists an $\bar{x}\in X$ such that $\bar{x}\in s^{-1}G(x)$ for all $x\in X$, i. e. $s(\bar{x})\in G(\bar{x})$.

This completes the proof.

Theorem 5.2 Let $(X, \mathscr{F}, \Delta)$ be a strongly Dadekind complete compact probabilistic interval space with a continuous t-norm Δ, $(Y, \mathfrak{F}, \bar{\Delta})$ be a probabilistic interval space, $G: X\to 2^Y$ be a set-valued mapping with compactly open values, $s\in\mathscr{E}^*(X, Y)$. If the following conditions are satisfied:

(i) for any $x\in X$, $G^c(x):=Y\backslash G(x)$ is W-chainable;

(ii) for any $x\in X$, $G^{-1}(s(x))$ interval open and W-chainable;

(iii) $G(X)=Y$,

there exists an $\bar{x}\in X$ such that $s(\bar{x})\in G(\bar{x})$.

Proof Considering the proof of Theorem 4.2, we know that for given $s\in\mathscr{E}^*(X, Y)$, if the condition (ii) in Theorem 4.2 is replaced by "for any $x\in X$, $H^{-1}(s(x))$ is interval open", then the conclusion of Theorem 4.2 is still true.

On the other hand, for any $x_1, x_2\in X$ and for any $x\in[x_1, x_2]$, if $z\in s^{-1}G(x_1)\cap s^{-1}G(x_2)$, then $s(z)\in G(x_1)\cap G(x_2)$, and so $\{x_1, x_2\}\subset G^{-1}(s(z))$. Since $G^{-1}(s(z))$ is W-chainable, $[x_1, x_2]\subset G^{-1}(s(z))$, and hence $x\in G^{-1}(s(z))$, i. e., $s(z)\in G(x)$. Hence $z\in s^{-1}G(x)$. This implies that $s^{-1}G(x_1)\cap s^{-1}G(x_2)\subset s^{-1}G(x)$. Besides, since $s^{-1}(G^c(x))=X\backslash s^{-1}G(x)$, we have

$$s^{-1}(G^c(x)\cap G(x_1)\cap G(x_2))\neq\phi$$

By Theorem 4.2, there exists an $\bar{x}\in X$ such that $s(X)\cap G^c(\bar{x})=\phi$, i. e., $s(X)\subset G(\bar{x})$, and so $s(\bar{x})\in G(\bar{x})$. This completes the proof.

Especially, if s is an identity mapping on X, and $Y=X$, we can obtain the following fixed point theorem:

Corollary 5.1 Let $(X, \mathscr{F}, \Delta)$ be a strongly Dadekind complete compact probabilistic interval space with a continuous t-norm Δ, and $G: X\to 2^X$ be a mapping with open values. If the following conditions are satisfied:

(i) for any $x\in X$, $G^c(x):=X\backslash G(x)$ is W-chainable;

(ii) for any $x\in X$, $G^{-1}(x)$ is interval open and W-chainable;

(iii) $G(X)=X$,

then G has a fixed point in X.

Theorem 5.3 Let $(X, \mathscr{F}, \Delta)$ be a strongly Dadekind complete probabilistic interval space and $G: X\to 2^X$. If the following conditions are satisfied:

(i) for any $y\in X$, $G^c(y):=X\backslash G(y)$ is compact W-chainable;

(ii) for any $x\in X$, $G^{-1}(x)$ is nonempty and interval open, and $X\backslash G^{-1}(x)$ is W-chainable,

then G has a fixed poin in X.

Proof Letting $A=\{(x, y)\in X\times X: x\notin G(y)\}$, then

$$A_1(y):=\{x\in X: (x, y)\in A\}=\{x\in X: x\notin G(y)\}=G^c(y)$$

$$A_2(x):=\{y\in X: (x, y)\in A\}=X\backslash G^{-1}(x)$$

It follows from conditions (i) and (ii) that all conditions in Corollary 4.2 are satisfied. On the other hand, since $G^{-1}(x)\neq\phi$ for all $x\in X$, there exists a $y\in G^{-1}(x)$, and $x\in G(y)$, i. e.

$(x, y) \notin A$. Hence for any $x \in X$, we have $\{x\} \times X \not\subset A$. By Corollary 4.2, there exists an $\bar{x} \in X$ such that $(\bar{x}, \bar{x}) \notin A$, i. e. $\bar{x} \in G(\bar{x})$. This implies that G has a fixed point in $\bar{x}$.

References

[1] Zhang Shisheng and Ma Yihai, Generalized KKM theorem on H-space with applications, *J. Math. Anal. Appl.*, **163** (1992), 406—421.

[2] Zhang Shisheng, Basic theory and applications of probabilistic metric spaces (I), (II), *Applied Math. and Mech.*, **9**, 2—3 (1988), 123—133, 213—225.

[3] K. Fan,A generalization of Tychonoff's fixed point theorem, *Math. Ann.*, **142** (1961), 303—310.

[4] K. Fan,. Some properties of convex sets related to fixed point theorem, *Math. Ann.*, **266** (1984), 519—537.

[5] B. Knaster, B. Kuratowski and S. Mazurkiewicz, Ein beweis des fixpunktsatzes für *n*-dimensionale simplexe, *Fund. Math.*, **14** (1929), 132—137.

[6] V. Komorink, Minimax theorems for upper semi-continuous functions, *Acta Math. Acad. Sci. Hunger*, **40** (1982), 159—163.

[7] J. von Neumann, Zur theoric der gesellshaftsphiele, *Math. Ann.*, **100** (1928), 295—320.

[8] S. Park, Generalizations of Ky·Fan's matching theorems and their applications, *J. Math. Anal. Appl.*, **141** (1989), 164—176.

[9] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York, Amsterdam, Oxford (1992).

[10] B. Schweizer and A. Sklar, Probabilistic metric spaces, *Pacific J. Math.*, **10** (1960), 313—334.

[11] L. L. Stacho, Minimax theorems beyond topological vector spaces, *Acta Sci. Math.*, **42** (1980), 157—164.

[12] Zhang Shisheng, Yeol Je Cho, Shin Min Kang, *Probabilistic Metric Spaces and Nonlinear Operator Theory*, Sichuan University Publishing House (1994). (in hinese)