ON THE NORM OF ELEMENTARY OPERATORS IN A STANDARD OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{B}(H)$ and $\mathcal{A}$ be a C^*-algebra of all bounded linear operators on a complex Hilbert space H and a complex normed algebra, respectively. For $A, B \in \mathcal{A}$, define a basic elementary operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ by $M_{A,B}(X) = AXB$. An elementary operator is a finite sum $R_{A,B} = \sum_{i=1}^{n} M_{A_i,B_i}$ of the basic ones, where $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ are two n-tuples of elements of $\mathcal{A}$.

If $\mathcal{A}$ is a standard operator algebra of $\mathcal{B}(H)$, it is proved that:

(i) [4] $\left\| M_{A,B} + M_{B,A} \right\| \geq 2(\sqrt{2} - 1) \|A\| \|B\|$, for any $A, B \in \mathcal{A}$

(ii)[1] $\left\| M_{A,B} + M_{B,A} \right\| \geq \|A\| \|B\|$, for $A, B \in \mathcal{A}$, such that $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| = \|A\|$ or $\inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| = \|B\|$,

(iii)[3] $\left\| M_{A,B} + M_{B,A} \right\| = 2 \|A\| \|B\|$, if $\|A + \lambda B\| = \|A\| + \|B\|$, for some unit scalar λ.

In this note, we are interested in the general situation where $\mathcal{A}$ is a standard operator algebra acting on a normed space. We shall prove that $\left\| R_{A,B} \right\| \geq \sup_{f,g \in (\mathcal{A}^*)_1} \left| \sum_{i=1}^{n} f(A_i)g(B_i) \right|$, for any two n-tuples $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ of elements of $\mathcal{A}$ (where $(\mathcal{A}^*)_1$ is the unit sphere of $\mathcal{A}^*$). As a consequence of this result, we show that the results (i), (ii) and (iii) remain true in this general situation.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}(H)$ be a complex normed algebra and a C^*-algebra of all bounded linear operators on a complex Hilbert space H, respectively. For $A, B \in \mathcal{A}$, define a basic elementary operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ by $M_{A,B}(X) = AXB$. An elementary operator is a finite sum $R_{A,B} = \sum_{i=1}^{n} M_{A_i,B_i}$ of the basic ones, where $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ are two n-tuples of elements of $\mathcal{A}$.

Many facts about the relation between the spectrum of $R_{A,B}$ and spectrums of A_i, B_i are known. For the case with the relation between the operator norm of $R_{A,B}$ and norms of A_i, B_i, the problem here is of course a useful lower estimate for the norm of $R_{A,B}$ because some upper estimates such as $\|R_{A,B}\| \leq \sum_{i=1}^{n} \|A_i\| \|B_i\|$ are trivial. In a prime C^*-algebra (A prime C^*-algebra is a C^*-algebra which $M_{A,B} = 0$ implies $A = 0$ or $B = 0$), Mathieu [2] was proved that $\|M_{A,B}\| =$

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$\|A\| \|B\|$ and $\|M_{A,B} + M_{B,A}\| \geq \frac{2}{3} \|A\| \|B\|$. The most abvious prime C^*-algebra are $\mathcal{B}(H)$ and $\mathcal{C}_\infty(H)$ (where $\mathcal{C}_\infty(H)$ is the C^*-algebra of all compact operators on H), respectively. In [4] ,Stacho and Zalar are interested in a standard operator algebra of $\mathcal{B}(H)$(a standard operator algebra of $\mathcal{B}(H)$ is a subalgebra of $\mathcal{B}(H)$ containing all finite rank operators; it is not assumed that is seladjoint or closed with respect to any topology), where they proved that $\|M_{A,B} + M_{B,A}\| \geq 2(\sqrt{2} - 1) \|A\| \|B\|$ and they conjectured the following:

Conjecture 1.1. *Let $\mathcal{A}$ be a standard operator algebra of $\mathcal{B}(H)$. If $A, B \in \mathcal{A}$, then the estimate $\|M_{A,B} + M_{B,A}\| \geq \|A\| \|B\|$ holds.*

Note that this conjecture was verified in the two following cases:
(i) [5] , in the Jordan algebra of symmetric operators of $\mathcal{B}(H)$,
(ii) [1] for $A, B \in \mathcal{B}(H)$ such that $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| = \|A\|$ or $\inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| = \|B\|$.

Here, we are interested in the case where $\mathcal{A}$ is a standard operator algebra acting on a complex normed space. We shall prove that $\|R_{A,B}\| \geq \sup_{f,g \in (\mathcal{A}^*)_1} \left| \sum_{i=1}^{n} f(A_i) g(B_i) \right|$, for any two n-tuples $A = (A_1, ..., A_n)$, $B = (B_1, ..., B_n)$ of elements of $\mathcal{A}$ (where $(\mathcal{A}^*)_1$ is the unit sphere of $\mathcal{A}^*$). As a consequence of this main result in our general situation, we show that the Stacho-Zalar lower bound remains true, and the estimate $\|M_{A,B} + M_{B,A}\| \geq \|A\| \|B\|$ holds if one of the two conditions is satisfied:
(1) $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| = \|A\|$ or $\inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| = \|B\|$,
(2) $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| \leq \frac{\|A\|}{2}$ or $\inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| \leq \frac{\|B\|}{2}$.

So the conjecture of Stacho-Zalar remains unknown only in the case where
(3) $\frac{\|A\|}{2} < \inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| < \|A\|$ and $\frac{\|B\|}{2} < \inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| < \|B\|$.

On the other hand, we are intersted to the following question:

Question. Let $\mathcal{A}$ be a standard operator algebra acting on a normed space. For which $A, B \in \mathcal{A}$ such that $\|R_{A,B}\| = \sum_{i=1}^{n} \|A_i\| \|B_i\|$?

2. Preliminaries

Definition 2.1. *Let Ω be a complex Banach algebra with unity I.*
(1) The set of states on Ω is by definition:

$$P(\Omega) = \{f \in \Omega^* : f(I) = 1 = \|f\|\}$$

(2) The numerical range of an element A in Ω is by definition:

$$W_0(A) = \{f(A) : f \in P(\Omega)\}$$

(3) The numerical radius of an element A in Ω is by definition:

$$w(A) = \sup \{|\lambda| : \lambda \in W_0(A)\}$$

(4) The usual numerical range of an element A in $\mathbb{B}(H)$ is by definition:

$$W(A) = \{\langle Ax, x\rangle : x \in H, \ \|x\| = 1\}$$

(5) The joint numerical range of a n-tuple $A=(A_1,...,A_n)$ *of elemnts of* Ω *is by definition the set:*

$$W_0(A)=\{(f(A_1),...,f(A_n)) : f\in P(\Omega)\}$$

It is known that for any $A\in \mathsf{B}(H)$, then $W_0(A)=W(A)^-$, see [6] (where$W(A)^-$ is the closure of $W(A)$).

Definition 2.2. *Let* E *be a complex normed space and let* $\mathsf{B}(E)$ *denote the complex normed algebra of all bounded linear operators on* E.

(i) $\mathcal{A}$ *is called a standard operator algebra of* $\mathsf{B}(E)$, *if it is a subalgebra of* $\mathsf{B}(E)$ *that contains all finite rank operators.*

(ii) For $x\in E$ *and* $f\in E^*$, *define the operator* $x\otimes f$ *on* E *by* $(x\otimes f)\,y=f(y)x$.

Notation. (i) For any normed space Y, we denote by $(Y)_1$ the unite sphere of Y, i.e. $(Y)_1=\{x\in Y : \|x\|=1\}$.

(ii) For $A,B\in \mathsf{B}(E)$, we put $U_{A,B}=M_{A,B}+M_{B,A}$ and $V_{A,B}=M_{A,B}-M_{B,A}$.

(iii) For $K\subset \mathbb{C}$, we put $|K|=\sup_{\lambda\in K}|\lambda|$.

(iv) For $M,N\subset \mathbb{C}^n$, we put $M\circ N=\left\{\sum_{i=1}^{n}\alpha_i\beta_i : (\alpha_1,...,\alpha_n)\in M,\ (\beta_1,...,\beta_n)\in N\right\}$.

Proposition 2.1. *Assume* $\mathcal{A}$ *is a standard operator algebra on a normed space* E. *Then* $\|M_{A,B}\|=\|A\|\,\|B\|$, *for any* $A,B\in\mathcal{A}$.

Proof. It is clear that $\|M_{A,B}\|\le\|A\|\,\|B\|$.

Now, let $x,y\in(E)_1$ and $f\in(E^*)_1$. Since $x\otimes f\in\mathcal{A}$ and $\|x\otimes f\|=1$, then

$$\begin{aligned}\|M_{A,B}\| &\ge \|A(x\otimes f)B\|\\ &\ge \|A(x\otimes f)By\|\\ &\ge |f(By)|\,\|Ax\|\end{aligned}$$

Hence $\|M_{A,B}\|\ge\|Ax\|\sup_{f\in(E^*)_1}|f(By)|=\|Ax\|\,\|By\|$.

So that $\|M_{A,B}\|\ge\|A\|\,\|B\|$. Therefore $\|M_{A,B}\|=\|A\|\,\|B\|$. ∎

Theorem 2.2. [4] *Assume* $\mathcal{A}$ *is a standard operator algebra of* $\mathsf{B}(H)$. *Then* $\|U_{A,B}\|\ge 2(\sqrt{2}-1)\,\|A\|\,\|B\|$, *for all* $A,B\in\mathcal{A}$.

Theorem 2.3. [1] *Let* $A,B\in\mathsf{B}(H)$ *such that* $\inf_{\lambda\in\mathbb{C}}\|A+\lambda B\|=\|A\|$ *or* $\inf_{\lambda\in\mathbb{C}}\|B+\lambda A\|=\|B\|$. *Then* $\|U_{A,B}\|\ge\|A\|\,\|B\|$.

Theorem 2.4. [3] *Assume* $\mathcal{A}$ *is a standard operator algebra of* $\mathsf{B}(H)$. *Let* $A,B\in\mathcal{A}$ *such that* $w(A^*B)=\|A\|\,\|B\|$. *Then* $\|U_{A,B}\|=2\,\|A\|\,\|B\|$.

Definition 2.3. *Let* Y *be a normed space and* $x,y\in Y$. *We say that* x *is orthogonal to* y $(x\perp y)$, *if* $\inf_{\lambda\in\mathbb{C}}\|\lambda x+y\|=\|y\|$.

Note that if Y is a Hilbert space, then $x\perp y$ iff $\langle x,y\rangle=0$.

Proposition 2.5. *Let* Y *be a normed space and* $x,y\in Y$. *Then the following properties are equivalent:*

(1) $x\perp y$,

(2) $\exists f\in(Y^*)_1 :\ f(x)=0,\ f(y)=\|y\|$.

Proof. If x or y is zero, then the proof is trivial.

Now, assume x and y are not zero.

(1) implies (2).

It is clear that $\|\lambda x+\mu y\|\geq|\mu|\,\|y\|$, for all $\lambda,\ \mu\in\mathbb{C}$. Let F be the subspace of Y generated by x and y.

Define the functional linear g on F by $g(x)=0$ and $g(y)=\|y\|$. Then, $|g(\lambda x+\mu y)|=|\mu|\,\|y\|\leq\|\lambda x+\mu y\|$, for all $\lambda,\ \mu\in\mathbb{C}$. Since $g(\frac{y}{\|y\|})=1$, and $\frac{y}{\|y\|}\in(F)_1$, we have $\|g\|=1$. Therefore, the condition (2) follows immediately by using the prolongement theorem of Hahn-Banach theorem.

(2) implies (1) is trivial. ∎

Remark 2.1. *By using the previous theorem, Theorem 2.3 may be reformulated as follows: If $A\perp B$ or $B\perp A$, then $\|U_{A,B}\|\geq\|A\|\,\|B\|$.*

Proposition 2.6. *Let Y be a normed space and $x_1,...,x_n\in Y$. Then the following properties are equivalent:*

(1) $\left\|\sum_{i=1} x_i\right\|=\sum_{i=1}\|x_i\|$,

(2) $\exists f\in(Y^*)_1:\ f(x_i)=\|x_i\|,\ i=1,...,n.$

Proof. (1) implies (2).

By Hahn-Banach theorem, there exist $f\in(Y^*)_1$ such that $f(\sum_{i=1} x_i)=\left\|\sum_{i=1} x_i\right\|$. Then, $\sum_{i=1} Ref(x_i)=\sum_{i=1}\|x_i\|$. Since $Ref(x_i)\leq\|x_i\|$,$i=1,...,n$, then, we have $Ref(x_i)=\|x_i\|$, $i=1,...,n$. Therefore $f(x_i)=\|x_i\|$, $i=1,...,n$.

(2) implies (1).

It is clear that:

$$\begin{aligned}\sum_{i=1}\|x_i\| &= \sum_{i=1} f(x_i)\\ &= \left|f(\sum_{i=1} x_i)\right|\\ &\leq \left\|\sum_{i=1} x_i\right\|\\ &\leq \sum_{i=1}\|x_i\|\end{aligned}$$ ∎

Theorem 2.7. *Let $\mathcal{B}$ be a C^*-algebra and let $A,B\in\mathcal{B}$. Then $\|A+B\|=\|A\|+\|B\|$ holds iff $\|A\|\,\|B\|\in W_0(A^*B)$.*

Proof. We can assume A and B are not zero.

Assume that $\|A+B\|=\|A\|+\|B\|$. Then we have $\|(A+B)^*(A+B)\|=\|A\|^2+\|B\|^2+2\|A\|\,\|B\|$. On the other hand, there exist $f\in P(\mathcal{B})$ such that $\|(A+B)^*(A+B)\|=f(A^*A)+f(B^*B)+2Ref(A^*B)$, and since $f(A^*A)\leq\|A\|^2$, $f(B^*B)\leq\|B\|^2$ and $Ref(A^*B)\leq\|A\|\,\|B\|$, then we have $Ref(A^*B)=\|A\|\,\|B\|$ and since $|f(A^*B)|\leq\|A\|\,\|B\|$, then we obtain $f(A^*B)=\|A\|\,\|B\|$, so that $\|A\|\,\|B\|\in W_0(A^*B)$.

Now assume that $\|A\|\|B\| \in W_0(A^*B)$. Then there exist $f \in P(\mathcal{B})$ such that $f(A^*B) = \|A\|\|B\|$, and since $|f(A^*B)|^2 \leq f(A^*A)f(B^*B)$, $f(A^*A) \leq \|A\|^2$ and $f(B^*B) \leq \|B\|^2$, then we obtain $f(A^*A) = \|A\|^2$, $f(B^*B) = \|B\|^2$, therefore $f(A^*A) + f(B^*B) + 2Ref(A^*B) = (\|A\| + \|B\|)^2$, thus $(\|A\| + \|B\|)^2 = f((A+B)^*(A+B)) \leq \|(A+B)^*(A+B)\| = \|A+B\|^2 \leq (\|A\| + \|B\|)^2$, we can deduce that $\|A+B\| = \|A\| + \|B\|$. ∎

Corollary 2.8. *Let $\mathcal{B}$ be a C^*-algebra and let $A, B \in \mathcal{B}$. Then the following properties are equivalent:*
(1) $w(A^*B) = \|A\|\|B\|$,
(2) $\exists \lambda \in (\mathbb{C})_1 : \|A + \lambda B\| = \|A\| + \|B\|$.

Proof. (1) implies (2).
Since $W_0(A^*B)$ is compact, then there exist $\mu \in (\mathbb{C})_1$ such that $\|A\|\|B\|\mu \in W_0(A^*B)$. Put $C = \overline{\mu}B$, then $\|A\|\|C\| \in W_0(A^*C)$. Then, by the Theorem2.7, $\|A+C\| = \|A\| + \|C\|$. Therefore $\|A + \lambda B\| = \|A\| + \|B\|$, where $\lambda = \overline{\mu}$.
(2) implies (1).
It is clear, if $C = \lambda B$, then, by the Theorem2.7, $\|A\|\|B\| = \|A\|\|C\| \in W_0(A^*C)$. So we obtain, $\|A\|\|B\| \leq w(A^*C) = w(A^*B)$. Since $w(A^*B) \leq \|A^*B\| \leq \|A\|\|B\|$, the condition (1) follows immediately.

Remark 2.2. *By using the above theorem, Theorem 2.4 may be reformulated as follows:*
If $\|A + \lambda B\| = \|A\| + \|B\|$, for some unit scalar λ, then $\|U_{A,B}\| = 2\|A\|\|B\|$. ∎

3. A lower bound of the norm of $R_{A,B}$

In this section, we consider the case where $\mathcal{A}$ is a standard operator algebra acting on a complex normed space E.

Theorem 3.1. *Let $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ are two n-tuples of elements of $\mathcal{A}$. Then*

$$\|R_{A,B}\| \geq \sup_{f,g \in (\mathcal{A}^*)_1} \left| \sum_{i=1}^{n} f(A_i)g(B_i) \right|$$

Proof. Let $x, y \in (E)_1$, $f, g \in (\mathcal{A}^*)_1$ and $h \in (E^*)_1$. Then, we have:

$$\begin{aligned} \|R_{A,B}\| &\geq \left\| \sum_{i=1}^{n} A_i(x \otimes h)B_i \right\| \\ &\geq \left\| \sum_{i=1}^{n} A_i(x \otimes h)B_i y \right\| \\ &= \left\| \sum_{i=1}^{n} h(B_i y)A_i x \right\| \end{aligned}$$

Thus, $\|R_{A,B}\| \geq \sup_{\|x\|=1} \left\| \sum_{i=1}^{n} h(B_i y)A_i x \right\| = \left\| \sum_{i=1}^{n} h(B_i y)A_i \right\|$.

So that, $\|R_{A,B}\| \geq \left| \sum_{i=1}^{n} h(B_i y)f(A_i) \right| = \left| h(\sum_{i=1}^{n} f(A_i)B_i y) \right|$.

Then, $\|R_{A,B}\| \geq \sup_{h\in(E^*)_1} \left\| h(\sum_{i=1}^{n} f(A_i)B_i y)\right\| = \left\|\sum_{i=1}^{n} f(A_i)B_i y\right\|$, therefore $\|R_{A,B}\| \geq \sup_{\|y\|=1} \left\|\sum_{i=1}^{n} f(A_i)B_i y\right\| = \left\|\sum_{i=1}^{n} f(A_i)B_i\right\|$. Therefore $\|R_{A,B}\| \geq \left|\sum_{i=1}^{n} f(A_i)g(B_i)\right|$. ∎

Corollary 3.2. *Assume E is a Banach space and $\mathcal{A} = \mathcal{B}(E)$. Lat $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ are two n-tuples of operators on E. Then*

$$\|R_{A,B}\| \geq |W_0(A) \circ W_0(B)|$$

Proof. Since $P(\mathcal{A}) \subset (\mathcal{A}^*)_1$, then

$$\begin{aligned}\sup_{f,g\in(\mathcal{A}^*)_1} \left|\sum_{i=1}^{n} f(A_i)g(B_i)\right| &\geq \sup_{f,g\in P(\mathcal{A})} \left|\sum_{i=1}^{n} f(A_i)g(B_i)\right| \\ &= |W_0(A) \circ W_0(B)|\end{aligned}$$

So the resul follows immediately.

Corollary 3.3. *Let $A, B \in \mathcal{A}$. then, we have:*

$$\|U_{A,B}\| \geq \sup_{f,g\in(\mathcal{A}^*)_1} |f(A)g(B) + f(B)g(A)|$$

Proof. This result follows immediately, by Theorem 3.1, since $U_{A,B} = R_{(A,B),\,(B,A)}$. ∎

Corollary 3.4. *Let $A = (A_1, ..., A_n)$ and $B = (B_1, ..., B_n)$ are two n-tuples of elements of $\mathcal{A}$ such that $\left\|\sum_{i=1}^{n} A_i\right\| = \sum_{i=1}^{n} \|A_i\|$ and $\left\|\sum_{i=1}^{n} B_i\right\| = \sum_{i=1}^{n} \|B_i\|$. Then $\|R_{A,B}\| = \sum_{i=1}^{n} \|A_i\| \|B_i\|$.*

Proof. By Proposition 2.6, there exist $f, g \in (\mathcal{A}^*)_1$ such that $f(A_i) = \|A_i\|$ and $g(B_i) = \|B_i\|$, for $i = 1, ..., n$.

By using Theorem 3.1, we obtain $\|R_{A,B}\| \geq \left|\sum_{i=1}^{n} f(A_i)g(B_i)\right| = \sum_{i=1}^{n} \|A_i\| \|B_i\| \geq \|R_{A,B}\|$. ∎

Corollary 3.5. *Let $A, B \in \mathcal{A}$ such that $\|A + B\| = \|A\| + \|B\|$. Then $\|U_{A,B}\| = 2\|A\| \|B\|$.*

Proof. Since $U_{A,B} = M_{A,B} + M_{B,A}$, this corollary is a particular case of the previous Corollary.

Remark 3.1. *In the previous corollary, we can replace the condition $\|A + B\| = \|A\| + \|B\|$, by $\|A + \lambda B\| = \|A\| + \|B\|$, for some unit scalar λ, since $\|U_{A,B}\| = \|U_{A,\lambda B}\| = 2\|A\| \|\lambda B\| = 2\|A\| \|B\|$. Using Corollary 2.8, this give a general form of Theorem 2.4.*

Theorem 3.6. *Let $A, B \in \mathcal{A}$. Then $\|U_{A,B}\| \geq 2(\sqrt{2} - 1)\|A\| \|B\|$, for any $A, B \in \mathcal{A}$.*

Proof. We may assume without lost of the generality that $\|A\| = \|B\| = 1$.

By Corollary 3.3, we have,

$$\|U_{A,B}\| \geq |f(A)g(B) + f(B)g(A)| \quad (1)$$

for any $f, g \in (\mathcal{A}^*)_1$.

Apply (1), for $g = f$, we obtain:

$$\|U_{A,B}\| \geq 2\,|f(A)f(B)| \quad (2)$$

By Hahn-Banach theorem, there exist $f_0, g_0 \in (\mathcal{A}^*)_1$, such that $f_0(B) = 1 = g_0(A)$. Put $f_0(A) = \alpha$ and $g_0(B) = \beta$.

Inequality (1) yields for $f = f_0$ and $g = g_0$, $\|U_{A,B}\| \geq |1 + \alpha\beta| \geq 1 - |\alpha\beta|$.

Apply inequality (2) twice, for $f = f_0$ and for $g = g_0$, we obtain $\|U_{A,B}\| \geq 2\,|\alpha|$ and $\|U_{A,B}\| \geq 2\,|\beta|$.

Therefore $\|U_{A,B}\|^2 + 4\,\|U_{A,B}\| \geq 4\,|\alpha\beta| + 4(1 - |\alpha\beta|) = 4$. We deduce $\|U_{A,B}\| \geq 2(\sqrt{2} - 1)\,\|A\|\,\|B\|$. ∎

Corollary 3.7. *Let $A, B \in \mathcal{A}$ such that $A \perp B$ or $B \perp A$, then:*

(i) $\|U_{A,B}\| \geq \|A\|\,\|B\|$,

(ii) $\|V_{A,B}\| \geq \|A\|\,\|B\|$.

Proof. (i) Assume $A \perp B$. By Proposition 2.5, there exist $f \in (\mathcal{A}^*)_1$, such that $f(A) = 0$ and $f(B) = \|B\|$. Then for all $g \in (\mathcal{A}^*)_1$, we have $\|U_{A,B}\| \geq |f(A)g(B) + f(B)g(A)| = \|B\|\,|g(A)|$. Therefore, $\|U_{A,B}\| \geq \|B\| \sup_{g \in (\mathcal{A}^*)_1} (|g(A)|) = \|A\|\,\|B\|$.

By the same, we obtain the second implication.

By a similar proof, we obtain (ii). ∎

Theorem 3.8. *Let $A, B \in \mathcal{A}$, such that $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| \leq \frac{\|A\|}{2}$ or $\inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| \leq \frac{\|B\|}{2}$. Then $\|U_{A,B}\| \geq \|A\|\,\|B\|$.*

Proof. By a simple computation, we obtain, $V_{A,B} = V_{A+\lambda B,B}$, for all complex λ. Then $\|V_{A,B}\| \leq 2 \inf_{\lambda \in \mathbb{C}} \|A + \lambda B\|\,\|B\|$.

If $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| \leq \frac{\|A\|}{2}$, then $\|V_{A,B}\| \leq \|A\|\,\|B\|$. By Proposition 2.1, we have $\|U_{A,B}\| + \|V_{A,B}\| \geq 2\,\|M_{A,B}\| = 2\,\|A\|\,\|B\|$. It follows that $\|U_{A,B}\| \geq \|A\|\,\|B\|$. By the same, we obtain the inequality with the second condition. ∎

Remark 3.2. *1- The Theorem 3.6 is a general form of Theorem 2.2.*

2- The Corollary 3.7.i is a general form of Theorem 2.3.

3- By Corollary 3.7.i and Theorem 3.8, the conjecture of Stacho-Zalar is satisfied in the two following cases:

(i) $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| = \|A\|$ *or* $\inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| = \|B\|$,

(ii) $\inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| \leq \frac{\|A\|}{2}$ *or* $\inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| \leq \frac{\|B\|}{2}$.

Then, it remains unknown only in the case where $\frac{\|A\|}{2} < \inf_{\lambda \in \mathbb{C}} \|A + \lambda B\| < \|A\|$ *and* $\frac{\|B\|}{2} < \inf_{\lambda \in \mathbb{C}} \|B + \lambda A\| < \|B\|$.

Note that, the conjecture of Stacho-Zalar is given in particualar case of Hilbert space, but our partial results are given in a general situation of normed space.

Theorem 3.9. *Let $A, B \in \mathcal{A}$. Then* $\|U_{A,B}\| \geq \frac{1}{2}\|V_{A,B}\|$.

Proof. We may assume that $\|A\| = \|B\| = 1$.

By Hahn-Banach theorem, there exist $f \in (\mathcal{A}^*)_1$ such that $f(B) = 1$. Put $f(A) = \mu$.

It follows, from Corollary 3.3, that $\|U_{A,B}\| \geq \sup_{g \in (\mathcal{A}^*)_1} |f(A)g(B) + f(B)g(A)| = \|A + \mu B\|$. Since $\|V_{A,B}\| = \|V_{A+\mu B,B}\| \leq 2\|A + \mu B\|$, it follows that $\|U_{A,B}\| \geq \frac{1}{2}\|V_{A,B}\|$. ∎

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