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On the Numerical Range and Norm of Elementary Operators

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Let $\mathcal{B}(E)$ be the complex Banach algebra of all bounded linear operators on a complex Banach space E. For n-tuples $A=(A_1,\ldots,A_n)$ and $B=(B_1,\ldots,B_n)$ of operators on E, let $R_{A,B}$ denote the operator on $\mathcal{B}(E)$ defined by $R_{A,B}(X)=\sum_{i=1}^{n}A_iXB_i$.

For $A,B\in\mathcal{B}(E)$, we put $U_{A,B}=R_{(A,B),(B,A)}$.

In this note, we prove that

$$co\left\{\sum_{i=1}^{n}\alpha_i\beta_i\colon(\alpha_1,\ldots,\alpha_n)\in V(A),\ (\beta_1,\ldots,\beta_n)\in V(B)\right\}^{-}\subset W_0(R_{A,B}|J)$$

where $V(\cdot)$ is the joint spatial numerical range, $W_0(\cdot)$ is the algebraic numerical range and J is a norm ideal of $\mathcal{B}(E)$. We shall show that this inclusion becomes an equality when $R_{A,B}$ is taken to be a derivation. Also, we deduce that $w(U_{A,B}|J)\geq 2(\sqrt{2}-1)w(A)w(B)$, for $A,B\in\mathcal{B}(E)$ and J is a norm ideal of $\mathcal{B}(E)$, where $w(\cdot)$ is the numerical radius.

On the other hand, in the particular case when E is a Hilbert space, we shall prove that the lower estimate bound $\|U_{A,B}|J\|\geq 2(\sqrt{2}-1)\|A\|\|B\|$ holds, if one of the following two conditions is satisfied:

(i) J is a standard operator algebra of $\mathcal{B}(E)$ and $A,B\in J$.
(ii) J is a norm ideal of $\mathcal{B}(E)$ and $A,B\in\mathcal{B}(E)$.

Keywords: Numerical range; Numerical radius; Norm ideal; Standard operator algebra

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1. INTRODUCTION

All operators considered here are linear bounded operators on a complex Banach space E. The collection of operators on E is denoted by $\mathcal{B}(E)$.

Notation 1

(i) If $M\subset\mathbb{C}$, we denote by M^{-}, $co\,M$ and $\overline{M}$, respectively the closure of M, the convex hull of M, and the set $\{\overline{\lambda}:\lambda\in M\}$.

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(ii) For $(x,f) \in E \times E^*$, we denote by $x \otimes f$ the operator on E given by $(x \otimes f)(y) = f(y)x$.
(iii) If E is a Hilbert space and if $x, y \in E$, we denote by $x \otimes y$ the operator on E given by $(x \otimes y)(z) = \langle z, y\rangle x$.
(iv) If $K, L \subset \mathbb{C}^n$, we put $K \circ L = \left\{\sum_{i=1}^{n} \alpha_i \beta_i \colon (\alpha_1, \ldots, \alpha_n) \in K, (\beta_1, \ldots, \beta_n) \in L\right\}$.

Definition 1 Let Ω be a complex unital Banach algebra with identity I and let $A \in \Omega$.

(1) We define:

(i) the spectrum of A by:

$$\sigma(A) = \{\lambda \in \mathbb{C} \colon A - \lambda I \text{ is not invertible in } \Omega\}$$

(ii) the spectral radius of A by:

$$r(A) = \sup\left\{|\lambda| : \lambda \in \sigma(A)\right\}$$

(iii) the set of states on Ω by:

$$\mathcal{P}(\Omega) = \left\{ f \in \Omega^* \colon f(I) = \| f \| = 1\right\}$$

(iv) the algebraic numerical range of A by:

$$W_0(A) = \left\{ f(A) \colon f \in \mathcal{P}(\Omega)\right\}$$

(v) the numerical radius of A by:

$$w(A) = \sup\left\{|\lambda| : \lambda \in W_0(A)\right\}$$

(2) A is called convexiod if $W_0(A) = co\,\sigma(A)$.

It is known that $W_0(A)$ is convex and compact (this result follows at once from the corresponding properties of the set of states) and contains $\sigma(A)$ (see [16]). If $\Omega = \mathcal{B}(E)$ and E is a Hilbert space, then $w(A) = \|A\|$ iff $r(A) = \|A\|$(see [6]).

Definition 2 For $A \in \mathcal{B}(E)$, define the spatial numerical range of A by:

$$V(A) = \left\{ f(Ax) \colon (x,f) \in \Pi\right\}$$

where $\Pi = \left\{(x,f) \in E \times E^* \colon \|x\| = \|f\| = f(x) = 1\right\}$.

This notion of spatial numerical range is introduced by Lumer in [7], where it is proved that $W_0(A) = co\,V(A)^-$, for every $A \in \mathcal{B}(E)$. In the particular case, when E is a Hilbert space, it is known that $W_0(A) = W(A)^-$, where $W(A) = \{\langle Ax, x\rangle \colon x \in E, \|x\| = 1\}$ is the numerical range of A.

Definition 3 For n-tuples $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ of operators on E, we define:

(i) the joint spatial numerical range of A (see [4]) by:

$$V(A) = \left\{(f(A_1 x), \ldots, f(A_n x)) \colon (x,f) \in \Pi\right\}$$

(ii) the joint numerical range of A by:

$$W(A) = \{(\langle A_1x, x\rangle, \ldots, \langle A_nx, x\rangle): x \in E, \|x\| = 1\}$$

(iii) the elementary operator $R_{A,B}$: $\mathcal{B}(E) \longrightarrow \mathcal{B}(E)$ by:

$$\forall X \in \mathcal{B}(E): R_{A,B}(X) = \sum_{i=1}^{n} A_iXB_i$$

Definition 4 For $A, B \in \mathcal{B}(E)$, define the particular elementary operators:

(i) the left multiplication operator L_A: $\mathcal{B}(E) \longrightarrow \mathcal{B}(E)$ by:

$$\forall X \in \mathcal{B}(E): L_A(X) = AX$$

(ii) the right multiplication operator R_B: $\mathcal{B}(E) \longrightarrow \mathcal{B}(E)$ by:

$$\forall X \in \mathcal{B}(E): R_B(X) = XB$$

(iii) the generalized derivation (induced by A, B) by $\delta_{A,B} = L_A - R_B$.
(iv) the elementary multiplication operator (induced by A, B) by $M_{A,B} = L_AR_B$
(v) the operator $U_{A,B}$, by $U_{A,B} = M_{A,B} + M_{B,A}$.

In the sequel, $T_{A,B}$ will stand for any one of the above linear operators.

Let J be a standard operator algebra or a norm ideal of $\mathcal{B}(E)$. Note that a standard operator algebra of $\mathcal{B}(E)$ is a subalgebra of $\mathcal{B}(E)$ associated with the usual operator norm and containing all finite rank operators, and a norm ideal of $\mathcal{B}(E)$ is a two-sided ideal of $\mathcal{B}(E)$ associated with a symmetric norm ideal (which satisfies axioms like those in Hilbert space case (see [5,10,13])) . We denote by $\|.\|_J$ the norm on J.

If J is a norm ideal, then $T_{A,B}(J\ ) \subset J$, so we can define the operator $T_{J,A,B}$ on J by $T_{J,A,B}(X) = T_{A,B}(X)$.

If J is a standard operator algebra and $A, B \in J$, define $U_{J,A,B} : J \to J$ by $U_{J,A,B}(X) = U_{A,B}(X)$.

Many facts about the relation betwen the spectrum of $R_{A,B}$ and the joint spectrum (spectrum in the sense of Taylor (see [17])) of two commuting n-tuples A and B of operators on E are known (see [3]). Recently in [11,12], we are interested in the relation betwen the numerical range of $R_{J,A,B}$ and the joint numerical ranges of any n-tuples A and B of operators on E, in the particular case where E is a Hilbert space and J is $\mathcal{B}(E)$ or a Schatten p-ideal of $\mathcal{B}(E)$. For any $A, B \in \mathcal{B}(E)$, we have proved that:

(i) $co(W(A) \circ W(B))^- \subset W_0(R_{J,A,B})$,
(ii) $W_0(\delta_{J,A,B}) = W_0(A) - W_0(B)$.

Section 2 of this note was motivated by the question: To what extent do the properties (i) and (ii) hold in the general situation of Banach space? It will be shown that for any norm ideal J the properties (i) and (ii) remain true, but the condition (i) may be modified by taking $V(\cdot)$ instead of $W(\cdot)$. As a consequence of the main result of this section (Theorem 2), we shall prove that $w(U_{J,A,B}) \geq 2(\sqrt{2} - 1)w(A)w(B)$, for any $A, B \in \mathcal{B}(E)$.

While the proof of the main result of this section is simple, it leads to some rather surprising consequences such as $W_0(L_{J,A}) = W_0(R_{J,A}) = W_0(A)$ and $W_0(\delta_{J,A,B}) = W_0(A) - W_0(B)$ independent of which symmetric norm ideal one chooses.

In Section 3, we establish a lower estimate bound for the norm of $U_{J,A,B}$. Note that, Stachó and Zalar are interested to know whether there exists a uniform lower for the norm of the operator $U_{J,A,B}$, in the case where E is a Hilbert space, J is a standard operator algebra and $A, B \in J$. Especially, in [14] they proved that $(*)$ $\|U_{J,A,B}\| \geq 2(\sqrt{2}-1)\|A\|\|B\|$, and in [15], they obtained the best estimate $(**)$ $\|U_{A,B}\| \geq \|A\|\|B\|$, for symmetric operators A and B. Also, Barraa and Boumazgour [1], proved that $(**)$ holds if $\inf_{\lambda\in\mathbb{C}}\|A-\lambda B\| = \|A\|$ or $\inf_{\lambda\in\mathbb{C}}\|B-\lambda A\| = \|B\|$. Here, we shall give an easy proof of $(*)$ if one of the two conditions is satisfied:

(i) J is a standard operator algebra and $A, B \in J$,
(ii) J is a norm ideal and $A, B \in \mathcal{B}(E)$.

So, the Stachó–Zalar lower estimate becomes a particular case of our work. In the end of this section, we exhibit some classes of operators A, B such that $\|U_{J,A,B}\| \geq \|A\|\|B\|$, in particular we shall give a general form of the result of Barraa–Boumazgour.

In Section 4, we are interested in the characterization of the operators A, B such that $\|U_{J,A,B}\| = 2\|A\|\|B\|$ in the particular case of Hilbert space. In particular, we shall prove that if J is the Hilbert–Schmidt class, then $\|U_{J,A,B}\| = 2\|A\|\|B\|$ iff $w(A^*B) = \|A\|\|B\|$.

2. THE NUMERICAL RANGE AND NUMERICAL RADIUS OF ELEMENTARY OPERATORS

In this section, we assume that J is a norm ideal.

THEOREM 1 *Assume E is a Hilbert space and let A and B be two n-tuples of operators on E. Then $co(W(A) \circ W(B))^- \subset W_0(R_{J,A,B})$.*

Proof For $J = \mathcal{B}(E)$ (resp. $J = \mathcal{C}_p(E)$, the Schatten p-ideal), then the result is obtained in [11, Theorem 1] (resp. [12, Theorem 4.1]).

For any norm ideal J, the proof is analogous to that of [12, Theorem 4.1]. ■

THEOREM 2 *Let A and B be two n-tuples of operators on E. Then $co(V(A) \circ V(B))^- \subset W_0(R_{J,A,B})$.*

Proof Let $(x,f), (y,g) \in \Pi$. Define the linear functional h on $\mathcal{B}(J)$ by:

$$h(F) = f(F(x \otimes g)y), \quad F \in \mathcal{B}(J)$$

We have $h(I) = f(x)g(y) = 1$, and since $\|x \otimes g\|_J = \|x \otimes g\| = \|x\|\|g\| = 1$, then:

$$\begin{cases} |h(F)| \leq \|F(x\otimes g)y\| \\ \qquad\quad \leq \|F(x\otimes g)\| \\ \qquad\quad \leq \|F(x\otimes g)\|_J \\ \qquad\quad \leq \|F\|\,\|x\otimes g\|_J \\ \qquad\quad \leq \|F\|. \end{cases}$$

So $h(I) = \|h\| = 1$; thus h is a state on $\mathcal{B}(J)$. It is obvious that $h(R_{J,A,B}) = \sum_{i=1}^{n} f(A_i x)g(B_i y)$, therefore $V(A) \circ V(B) \subset W_0(R_{J,A,B})$. Since $W_0(R_{J,A,B})$ is closed and convex, the result follows easily. ■

COROLLARY 1 *Let $A \in \mathcal{B}(E)$. Then $W_0(L_{J,A}) = W_0(R_{J,A}) = W_0(A)$.*

Proof The inclusion $co\, V(A)^- \subset W_0(L_{J,A})$ follows immediately from Theorem 2. Then $W_0(A) = co\, V(A)^- \subset W_0(L_{J,A})$.

Now, let f be a state on $\mathcal{B}(J)$. Define the linear functional g on $\mathcal{B}(E)$ by $g(X) = f(L_{J,X})$. By a simple computation, we find that g is a state on $\mathcal{B}(E)$, so that $g(A) = f(L_{J,A}) \in W_0(A)$. Thus $W_0(L_{J,A}) \subset W_0(A)$, therefore $W_0(L_{J,A}) = W_0(A)$. By the same argument, we find also $W_0(R_{J,A}) = W_0(A)$. ∎

COROLLARY 2 *Let $A, B \in \mathcal{B}(E)$. Then $W_0(\delta_{J,A,B}) = W_0(A) - W_0(B)$.*

Proof By Theorem 2, we have $co(V(A) - V(B))^- \subset W_0(\delta_{J,A,B})$. Then

$$\begin{aligned} W_0(A) - W_0(B) &= co\, V(A)^- - co\, V(B)^- \\ &= co(V(A) - V(B))^- \\ &\subset W_0(\delta_{J,A,B}) \end{aligned}$$

On the other hand using Corollary 1, we have:

$$\begin{aligned} W_0(\delta_{J,A,B}) &= W_0(L_{J,A} - R_{J,B}) \\ &\subset W_0(L_{J,A}) - W_0(R_{J,B}) \\ &= W_0(A) - W_0(B) \end{aligned}$$ ∎

Remark 1 As a consequence of the above Corollary and by the same argument as in [12, Theorem 3.1], we show that $\delta_{J,A,B}$ is convexoid iff A and B are convexoid.

COROLLARY 3 *Let $A, B \in \mathcal{B}(E)$. Then $W_0(A)W_0(B) \subset W_0(M_{J,A,B})$, and thus $w(M_{J,A,B}) \geq w(A)w(B)$.*

Proof By Theorem 2, we obtain $co(V(A)V(B))^- \subset W(M_{J,A,B})$. Then we have:

$$\begin{aligned} W_0(A)W_0(B) &= co\, V(A)^- co\, V(B)^- \\ &= (co\, V(A) co\, V(B))^- \\ &\subset co(V(A)V(B))^- \\ &\subset W_0(M_{J,A,B}) \end{aligned}$$

The inequality follows immediately from this inclusion. ∎

THEOREM 3 *Let $A, B \in \mathcal{B}(E)$. Then $w(U_{J,A,B}) \geq 2(\sqrt{2} - 1)w(A)w(B)$.*

Proof We may assume, without loss of the generality, that $w(A) = w(B) = 1$.

For any $(x,f), (y,g)$ in Π, we have

$$f(Ax)g(By) + f(Bx)g(Ay) \in V(A,B) \circ V(B,A)$$

Since $V(A,B) \circ V(B,A) \subset W_0(U_{J,A,B})$, then

$$w(U_{J,A,B}) \geq \left| f(Ax)g(By) + f(Bx)g(Ay) \right| \tag{1}$$

Applying inequality (1) for $(y,g) = (x,f)$, we obtain:

$$w(U_{J,A,B}) \geq 2\left| f(Ax) \right| \left| f(Bx) \right| \tag{2}$$

Let (x_n, f_n) and (y_n, g_n) be two sequences in Π such that:

$$\lim\left|f_n(Ax_n)\right| = w(A) = 1 = w(B) = \lim\left|g_n(By_n)\right|$$

For $(x, f) = (x_n, f_n)$ and $(y, g) = (y_n, g_n)$, inequality (1) yields:

$$w(U_{J,A,B}) \geq \left|f_n(Ax_n)g_n(By_n) + f_n(Bx_n)g_n(Ay_n)\right| \tag{3}$$

Thus,

$$w(U_{J,A,B}) \geq \left|f_n(Ax_n)g_n(By_n)\right| - \left|f_n(Bx_n)g_n(Ay_n)\right| \tag{4}$$

Applying inequality (2) twice for $(x, f) = (x_n, f_n)$ and for $(x, f) = (y_n, g_n)$, we obtain:

$$\begin{cases} w(U_{J,A,B}) \geq 2\left|f_n(Ax_n)\right|\left|f_n(Bx_n)\right|. & (5) \\ w(U_{J,A,B}) \geq 2\left|g_n(Ay_n)\right|\left|g_n(By_n)\right| & (6) \end{cases}$$

Since the two complex sequences $(f_n(Bx_n))$ and $(g_n(Ay_n))$ are bounded, we can extract a convergent subsequence from each one. We can put $\alpha = \lim|f_n(Bx_n)|$ and $\beta = \lim|g_n(Ay_n)|$.

Letting $n \to +\infty$, in (4), (5) and (6), we obtain,

$$w(U_{J,A,B}) \geq \max\left\{1 - \left|\alpha\beta\right|, 2|\alpha|, 2\left|\beta\right|\right\}$$

Therefore,

$$\begin{cases} w(U_{J,A,B})^2 + 4w(U_{J,A,B}) \geq 4\left|\alpha\beta\right| + 4(1 - \left|\alpha\beta\right|) \\ \qquad\qquad\qquad\qquad\qquad\quad \geq 4. \end{cases}$$

Thus we have $w(U_{J,A,B}) \geq 2(\sqrt{2} - 1)$. ■

3. A LOWER BOUND FOR THE NORM OF $U_{J,A,B}$

In this section, we assume that E is a Hilbert space. Let $A, B \in \mathcal{B}(E)$. We assume that if J is a standard operator algebra, then $A, B \in J$.

Definition 5 We define the numerical range of A^*B relative to B by:

$$W_B(A^*B) = \{\lambda \in \mathbb{C}\colon \lambda = \lim\langle A^*Bx_n, x_n\rangle,\ \lim\|Bx_n\| = \|B\|,\ \|x_n\| = 1\}$$

This concept of this numerical range is introduced by Magajna in [9]. The most interesting properties of $W_B(A^*B)$ are given as below (see [9]):

1. $W_B(A^*B)$ is not empty and compact subset of $\mathbb{C}$,
2. the relation $\inf_{\lambda\in\mathbb{C}}\|B - \lambda A\| = \|B\|$ holds iff $0 \in W_B(A^*B)$.

LEMMA 1 *We have the following properties:*

(i) $\|U_{J,A,B}\| \geq \sup\{|\langle Ax, y\rangle\langle Bu, v\rangle + \langle Bx, y\rangle\langle Au, v\rangle|: \|x\| = \|y\| = \|u\| = \|v\| = 1\}$
(ii) $\|U_{J,A,B}\| \geq 2w(A^*B)$.

Proof
(i) Since $\|x \otimes v\|_J = \|x \otimes v\| = \|x\|\|v\| = 1$, and since $\|X\|_J \geq \|X\|$, for any $X \in J$, then we have;

$$\begin{aligned}\|U_{J,A,B}\| &\geq \|A(x \otimes v)B + B(x \otimes v)A\|_J \\ &\geq \|Ax \otimes B^*v + Bx \otimes A^*v\| \\ &\geq \|\langle Bu, v\rangle Ax + \langle Au, v\rangle Bx\| \\ &\geq |\langle Ax, y\rangle\langle Bu, v\rangle + \langle Bx, y\rangle\langle Au, v\rangle|\end{aligned}$$

(ii) Let x be a unit vector in E such that $Ax \neq 0$. Using (i), we obtain, $\|U_{J,A,B}\| \geq |(1/\|Ax\|)\langle A^*Bx, x\rangle\|Ax\| + \langle A^*Bx, x\rangle\|Ax\||$, then we can deduce immediately that $\|U_{J,A,B}\| \geq 2|\langle A^*Bx, x\rangle|$, for any unit vector x in E. So $\|U_{J,A,B}\| \geq 2w(A^*B)$. ∎

THEOREM 4 *We have the following property:*

$$\|U_{J,A,B}\| \geq 2(\sqrt{2} - 1)\|A\|\|B\|.$$

Proof We may assume, without loss of the generality, that $\|A\| = \|B\| = 1$. Let $\lambda \in W_B(A^*B)$ and $\mu \in W_A(B^*A)$. Then, there exist two sequences (x_n) and (y_n) of unit vectors in E such that $\lim\|Bx_n\| = \lim\|Ay_n\| = 1$, and $\lim\langle A^*Bx_n, x_n\rangle = \lambda$, $\lim\langle B^*Ay_n, y_n\rangle = \mu$. By Lemma 1.(i), we have:

$$\|U_{J,A,B}\| \geq \left|\frac{1}{\|Ay_n\|\|Bx_n\|}\langle A^*By_n, y_n\rangle\langle B^*Ax_n, x_n\rangle + \|Ay_n\|\|Bx_n\|\right|$$

Letting $n \to +\infty$, we get $\|U_{J,A,B}\| \geq |1 + \overline{\lambda}\overline{\mu}| = |1 + \lambda\mu|$.

On the other hand, by Lemma 1.(ii), we have $\|U_{J,A,B}\| \geq \max\{2|\lambda|, 2|\mu|\}$; therefore $\|U_{J,A,B}\| \geq \max\{|1 + \lambda\mu|, 2|\lambda|, 2|\mu|\}$, and by the same argument as in the proof of Theorem 3, we obtain the inequality. ∎

Remark 2 The above Theorem is proved by Stachó and Zalar in [14] in the particular case where J is a standard operator algebra, but here, we have obtained it, in a more general situation by a direct proof.

THEOREM 5 *If A and B are not zero, we have:*

$$\|U_{J,A,B}\| \geq \sup\left\{\left|\|A\|\|B\| + \frac{\lambda\mu}{\|A\|\|B\|}\right|, \lambda \in W_B(A^*B),\ \mu \in W_A(B^*A)\right\}$$

Proof Let $\lambda \in W_B(A^*B)$ and $\mu \in W_A(B^*A)$. By the same argument as in the proof of the Theorem 4, we obtain $\|U_{J,A,B}\| \geq |\|A\|\|B\| + (\lambda\mu/\|A\|\|B\|)|$. ∎

COROLLARY 4 *The inequality* $\|U_{J,A,B}\| \geq \|A\|\|B\|$ *holds, if any one of the following conditions is satisfied:*

(i) $\exists\lambda \in W_B(A^*B), \exists\mu \in W_A(B^*A)$: $\mathrm{Re}(\lambda\mu) \geq 0$,
(ii) $A^*B \geq 0$ or $AB^* \geq 0$,
(iii) $\exists\,\theta \in [0, 2\pi[\, : W(A^*B) \subset \{z \in \mathbb{C} : \theta \leq \arg z \leq \theta + \pi/2\}$.

Proof

(i) Let $\lambda \in W_B(A^*B)$ and $\mu \in W_A(B^*A)$ such that $Re(\lambda\mu) \geq 0$. Then, by Theorem 5, we have $\|U_{J,A,B}\| \geq \|A\|\|B\| + (Re(\lambda\mu)/\|A\|\|B\|)$. Therefore, $\|U_{J,A,B}\| \geq \|A\|\|B\|$.
(ii) If $A^*B \geq 0$, it is clear that $Re(\lambda\mu) \geq 0$, for every $\lambda \in W_B(A^*B)$ and every $\mu \in W_A(B^*A)$, so we deduce the Corollary, by (i). On the other hand, if $AB^* \geq 0$, and since $\|U_{J,A,B}\| = \|U_{J^*,A^*,B^*}\|$ (where $J^* = \{X^*: X \in J\}$), we obtain the Corollary, only by using the first step.
(iii) We put $B_1 = e^{-i\theta}B$, then $W_0(A^*B_1) \subset \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi/2\}$, since $W_{B_1}(A^*B_1) \subset W_0(A^*B_1)$ and $W_A(B_1^*A) \subset \overline{W_0(A^*B_1)}$, so we have $Re(\lambda\mu) \geq 0$, for all $\lambda \in W_{B_1}(A^*B_1)$ and for all $\mu \in W_A(B_1^*A)$. Then we can obtain (iii) immediately using (i) and the fact that $\|U_{J,A,B}\| = \|U_{J,A,B_1}\|$. ■

Remark 3 It is proved in [1] that $\|U_{A,B}\| \geq \|A\|\|B\|$, if $0 \in W_A(B^*A) \cup W_B(A^*B)$, so that the Corollary 4.i, is a generalisation of this result in our general situation.

COROLLARY 5 *The inequality* $\|U_{J,A,B}\| \geq \|A\|\|B\| + (1/\|A\|\|B\|)$ *holds, if* $A = S$ *and* $B = (S^*)^{-1}$, *for some invertible operator* S *on* H.

Proof There exist two sequences (x_n) and (y_n) of unit vectors in E such that $\lim\|Ax_n\| = \|A\| = \|S\|$ and $\lim\|By_n\| = \|B\| = \|S^{-1}\|$; and since $\lim\langle A^*Bx_n, x_n\rangle = \|x_n\|^2 = 1 = \|y_n\|^2 = \lim\langle B^*Ay_n, y_n\rangle$, then $1 \in W_A(B^*A) \cap W_B(A^*B)$, so we have, by Theorem 5, $\|U_{J,A,B}\| \geq \|A\|\|B\| + (1/\|A\|\|B\|)$.

THEOREM 6 *We have* $\|U_{J,A,B}\| \geq \|A\|\|B\|$, *if* $\|B\|^2(A^*A) \leq \|A\|^2(B^*B)$ *or* $\|A\|^2(B^*B) \leq \|B\|^2(A^*A)$.

Proof We can assume $\|A\| = \|B\| = 1$. Then, by Lemma1.(i), we have: $\|U_{J,A,B}\| \geq (1/\|Ax\|\|Bx\|)|\langle A^*Bx, x\rangle|^2 + \|Ax\|\|Bx\|$, for any unit vector x in E such that $Ax \neq 0$ and $Bx \neq 0$. So we obtain $\|U_{J,A,B}\| \geq \|Ax\|\|Bx\|$, for any unit vector x in E. Then, if $A^*A \leq B^*B$, we have $\|U_{J,A,B}\| \geq \|Ax\|^2$; thus $\|U_{J,A,B}\| \geq 1$. By the same argument, the inequality holds with the second condition. ■

4. WHEN IS $\|U_{J,A,B}\| = 2\|A\|\|B\|$?

In this section, we also assume that E is a Hilbert space.

LEMMA 2 *If* $w(A^*B) = \|A\|\|B\|$, *for some* $A, B \in \mathcal{B}(E)$, *then* $\|U_{J,A,B}\| = 2\|A\|\|B\|$.

Proof It follows immediately from Lemma 1.(ii). ■

LEMMA 3 *Let* J *be a standard operator algebra and* $A, B \in J$. *If* $\|U_{J,A,B}\| = 2\|A\|\|B\|$, *then* $\|A^*B\| = \|A\|\|B\|$.

Proof This Lemma is proved by Barraa and Boumazgour in [1] in the particular case $J = \mathcal{B}(E)$. Note that the same proof works in any standard operator algebra.

THEOREM 7 *If* $J = \mathcal{C}_2(E)$ (*the Hilbert–Schmidt class*) *and* $A, B \in \mathcal{B}(E)$, *then* $\|U_{J,A,B}\| = 2\|A\|\|B\|$ *iff* $w(A^*B) = \|A\|\|B\|$.

Proof Assume that $\|U_{J,A,B}\| = 2\|A\|\|B\|$. Since $\|M_{J,A,B}\| = \|M_{J,B,A}\| = \|A\|\|B\|$, then we have $\|U_{J,A,B}\| = \|M_{J,A,B}\| + \|M_{J,B,A}\|$, where $M_{J,A,B}, M_{J,B,A}, U_{J,A,B} \in \mathcal{B}(J)$, and J is a Hilbert space. Thus, by [2], we obtain $\|M_{J,A,B}\|\|M_{J,B,A}\| = \|A\|^2\|B\|^2 \in W_0((M_{J,A,B})^*(M_{J,B,A}))$, and since $(M_{J,A,B})^* = M_{J,A^*,B^*}$, then we have $\|A\|^2\|B\|^2 \in W_0(M_{J,A^*B,AB^*})$. Therefore

$$\begin{cases} \|A\|^2\|B\|^2 \le w(M_{J,\ A^*B,\ AB^*}) \\ \qquad\qquad\quad \le \left\|M_{J,\ A^*B,\ AB^*}\right\| \\ \qquad\qquad\quad = \|A^*B\|\|AB^*\| \\ \qquad\qquad\quad \le \|A\|^2\|B\|^2 \end{cases}$$

So we have $w(M_{J,\ A^*B,\ AB^*}) = \|M_{J,A^*B,AB^*}\| = \|A\|^2\|B\|^2$, which implies $r(M_{J,A^*B,AB^*}) = \|M_{J,A^*B,AB^*}\| = \|A\|^2\|B\|^2$. Since $r(M_{J,A^*B,AB^*}) \le r(A^*B)r(AB^*) \le \|A\|^2\|B\|^2$, and $r(A^*B) = r(BA^*) = r((BA^*)^*) = r(AB^*)$, therefore $r(A^*B) = \|A\|\|B\|$. So we have $r(A^*B) = \|A^*B\| = \|A\|\|B\|$, and thus $w(A^*B) = \|A^*B\| = \|A\|\|B\|$.

The converse implication follows immediately by Lemma 2. ■

THEOREM 8 *Let* J *be a standard operator algebra and let* $A, B \in J$ *be such that* A^*B *is normaloid. Then* $\|U_{J,A,B}\| = 2\|A\|\|B\|$ *iff* $w(A^*B) = \|A\|\|B\|$.

Proof Assume that $\|U_{J,A,B}\| = 2\|A\|\|B\|$. Then, by Lemma 3, we have $\|A^*B\| = \|A\|\|B\|$, and since $w(A^*B) = \|A^*B\|$, we obtain $w(A^*B) = \|A\|\|B\|$. By Lemma 2, we obtain the converse implication. ■

Remark 4 In general, Theorem 8 is not true without the condition that A^*B is normaloid. For example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then $w(A^*B) = (1/2) < 1 = \|A\|\|B\|$ but $\|U_{A,B}\| = 2 = 2\|A\|\|B\|$.

THEOREM 9 *Let* J *be a standard operator algebra and* $A, B \in J$. *Then we have* $\|U_{J,A,B}\| = 2\|A\|\|B\|$ *iff* $\|A^*B\| = \|A\|\|B\|$, *if one of the following conditions is satisfied:*

(i) B normal and $AB = BA$,
(ii) B normal and $A \ge 0$,
(iii) $\|(A^*B)^2\| = \|A^*B\|^2$.

Proof Assume that $\|A\| = \|B\| = 1$. By Lemma 3, we have only to prove that , if $\|A^*B\| = \|A\|\|B\|$ then $\|U_{J,A,B}\| = 2\|A\|\|B\|$. It is clear that $\|U_{J,A,B}\| \ge \|A(B^*B) + (BB^*)A\|$; then by McIntosh's inequality [8], we have $\|U_{J,A,B}\| \ge 2\|B^*AB^*\| = 2\|BA^*B\|$. Then, by this inequality, we may deduce the following implications:

Assume $\|A^*B\| = \|A\|\|B\|$.

(i) Since B is normal, then $\|BA^*B\| = \|B^*A^*B\|$, and by Putnam–Fuglede theorem, we have $AB^* = B^*A$, so we obtain $\|U_{J,A,B}\| \ge 2\|AB^*A^*B\| = 2\|B^*AA^*B\| = 2\|A^*B\|^2 = 2$.

(ii) Since $\|U_{J,A,B}\| \geq 2\|BAB\| = 2\|B^*AB\| = 2\|A^{\frac{1}{2}}B\|^2$, then $\|U_{J,A,B}\| \geq 2\|AB\|^2 = 2$.

(iii) Since $\|U_{J,A,B}\| \geq 2\|BA^*B\|$, then $\|U_{J,A,B}\| \geq 2\|A^*BA^*B\| = 2\|A^*B\|^2 = 2$. ■

Remark 5

(i) Theorem 9.(i) is a general form of the known result $\|U_{A,I}\| = 2\|A\|$, for all $A \in \mathcal{B}(H)$.

(ii) If B is a unitary operator, it is abvious that $\|U_{A,B}\| = 2\|A\|\|B\|$ and $\|A^*B\| = \|A\|\|B\|$, for every operator A.

We may ask the following questions:

Question 1 Does Theorem 9.(i) (resp. Theorem 9.(ii)) remain true with only the condition for B to be normal?

Question 2 Does Theorem 9 remain true if we drop all conditions on A and B?

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