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A GENERAL SADDLE POINT THEOREM AND ITS APPLICATIONS

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Let X and Y be nonempty sets, f and g be real-valued functions on the Cartesian product $X\times Y$ of these sets. A point (x, y) in $X\times Y$ is said to be a *saddle point* of the functions f, g if

(SP) $$g(u, y) \leqq f(x, v) \quad \text{for every} \quad (u, v) \quad \text{in} \quad X\times Y$$

holds true. For a single function f the well-known notion of saddle point follows here by letting $g\equiv f$ in (SP). It should also be noted that the existence of a saddle point implies the following minimax inequality

(MMI) $$\inf_{y\in Y}\sup_{x\in X} g(x, y) \leqq \sup_{x\in X}\inf_{y\in Y} f(x, y).$$

In the case when $f\leqq g$, especially when g equals f, the latter property is known as the statement of the two variable generalized version of the celebrated von Neumann's minimax theorem, namely

(MME) $$\inf_{y\in Y}\sup_{x\in X} g(x, y) = \sup_{x\in X}\inf_{y\in Y} f(x, y).$$

Our aim is to prove a general but rather elementary theorem first on the existence of saddle points (Theorem 1), secondly, as a consequence, on the existence of minimax inequality and equality respectively — giving necessary and sufficient conditions for them. Our condition is general enough and not only of convexity type. The results so obtained are a common generalization of our previous ones and many other known theorems of concave-convex type. Our approach is essentially the same as our earlier one. We use the finite dimensional separation argument for disjoint convex sets in a similar but essentially simpler way as in [1, Theorem 2.5.1] and Riesz's well-known theorem concerning a common point of compact sets with finite intersection property. The compactness here follows by Alexander's subbase theorem [6].

Concerning minimax type inequalities see S. Simmons [10], J. Kindler [5] and Z. Sebestyen [8, 9]. Minimax theorems are e.g. in Belakrishnan [1], Z. Sebestyen [7, 8, 9], I. Joó [3] and I. Joó—L. L. Stachó [4].

Let now f, g be two real-valued functions defined on the Cartesian product $X\times Y$ of two nonempty sets X, Y. As a notation, for a nonemty set $K\subset X\times Y$, for a point (u, v) in $X\times Y$ and for a positive real number c let

$$K_{u,v}^{c} = \{(x, y)\in K\colon\ 0 \leqq f(x, v)-g(u, y)+c\}.$$

This is why for a point (x, y) in $X\times Y$ to be a saddle point is nothing else but each $K_{u,v}^{c}$ being nonempty for the one point set $K=\{(x, y)\}$.

THEOREM 1. *Let f, g be real-valued functions on $X\times Y$. There exists a saddle point for f, g if and only if there exists a nonempty set $K\subset Y\times Y$ such that :*

(1) $$\min_{(u,v)\in G} \sum_{(x,y)\in F} \lambda(x,y)[f(x,v)-g(u,y)] \leqq \sup_{(x,y)\in K} \min_{(u,v)\in G} [f(x,v)-g(u,y)]$$

for all finite sets $F\subset K$, $G\subset X\times Y$ and a probability measure λ on F;

(2) $$0\leqq \inf_{(u,v)\in X\times Y} \sup_{(x,y)\in K} [f(x,v)-g(u,v)] \leqq \sup_{(x,y)\in K} \sum_{(u,v)\in G} \mu(u,v)[f(x,v)-g(u,y)]$$

for every finite set $G\subset X\times Y$ and a probability measure μ on G;

(3) *if $D\subset(0,+\infty)\times X\times Y$ has the property that for any (x,y) in K there exists (c,u,v) in D with $f(x,v)-g(u,y)+c<0$, then a finite subset of D exists with the same property.*

PROOF. Assume first that a point (x,y) in $X\times Y$ is a saddle point for the functions f, g on $X\times Y$. The one point subset $K=\{(x,y)\}$ of $X\times Y$ clearly satisfies conditions (1), (2) and (3)

To prove the sufficiently let K be as in the assumption. Let further $U^c_{u,v}=K\setminus K^c_{u,v}$ be the complements in K of the subsets $K^c_{u,v}$ introduced before.

Topologize K by taking $\{U^c_{u,v}: (c,u,v)\in(0,+\infty)\times X\times Y\}$ as a family of open subbase for this topology. Condition (3) says that if K is covered by a subfamily $\{U^c_{u,v}: (c,u,v)\in D\}$ then K is also covered by a finite subcollection of the family indexed by D. By Alexander's well-known subbase lemma K is thus compact in the topology so introduced. But the subsets $K^c_{u,v}$ of K are thus closed hence compact with respect to this topology on K. Now a point (x,y) in $X\times Y$ satisfies (SP) if and only if

$$0\leqq f(x,v)-g(u,y)+c \quad \text{holds for all} \quad (c,u,v)\in(0,+\infty)\times X\times Y,$$

in other words (x,y) belongs to each of $K^c_{u,v}$. To prove that a saddle point exists is therefore nothing else but to prove that the sets $K^c_{u,v}$ have a common point. But the compactness of $K^c_{u,v}$'s allows us, refering to Riesz, to prove the finite intersection property of the family $K^c_{u,v}$. Let $0<c_i$, $(u_i,v_i)\in X\times Y$ for $i=1,2,\dots,n$ have a finite family of subsets $K^{c_i}_{u_i,v_i}$ in K indexed by $i=1,2,\dots,n$. Since with $c=\{\min c_i: 1\leqq i\leqq n\}$

$$K^c_{u_i,v_i}\subset K^{c_i}_{u_i,v_i} \quad \text{for} \quad i=1,2,\dots,n,$$

$\bigcap_{i=1}^n K^c_{u_i,v_i}\neq\emptyset$ will imply the desired nonvoid intersection property for the chosen finite family $\{K^{c_i}_{u_i,v_i}: i=1,2,\dots n\}$. Assume the contrary: $\bigcap_{i=1}^n K^c_{u_i,v_i}=\emptyset$. Then we conclude that for any (x,y) in K there exists a natural number i, $1\leqq i\leqq n$ such that $(x,y)\notin K^c_{u_i,v_i}$, i.e. $f(x,v_i)-g(u_i,y)+c<0$.

This implies the following property:

(4) $$\min_{1\leqq i\leqq n} [f(x,V_i)-g(u_i,y)] < -c \quad \text{for any} \quad (x,y) \quad \text{in} \quad K.$$

Let now Φ_c be the R^n-valued function on K defined as follows:

$$\Phi_c(x,y):=\big(f(x,v_1)-g(u_1,y)+c, \dots, f(x,v_n)-g(u_1,y)+c\big).$$

We have thus that $\Phi_c(X, Y)$, the range of Φ_c, does not meet

$$\mathbf{R}^n_+ := \{t = (t_1, \dots, t_n) \in \mathbf{R}^n\colon\ 0 \leqq t_i \text{ for } i = 1, 2, \dots, n\},$$

the positive cone in $\mathbf{R}^n$. But we state more that this, namely that the convex hull of $\Phi_c(X, Y)$ also does not meet but the interior of $\mathbf{R}^n_+$. Otherwise there would be a finite set $F \subset X \times Y$, probability measure λ on F such that

$$0 < \sum_{(x,y)\in F} \lambda(x, y)[f(x, v_i) - g(u_i, y) + c] \quad \text{for} \quad i = 1, 2, \dots, n.$$

But then, in view of (1) and (4), we have

$$-c < \min_{1\leqq i\leqq n} \sum_{(x,y)\in F} \lambda(x, y)[f(x, v_i) - g(u_i, y) \leqq \sup_{(x,y)\in K} \min_{1\leqq i\leqq n} [f(x, v_i) - g(u_i, y)] \leqq -c,$$

a contradiction. The separation argument thus applies: there exists a nonzero vector $\mu = (\mu_1, \dots, \mu_n) \in \mathbf{R}^n$ separating the mentioned two convex sets in $\mathbf{R}^n$. This can be expressed by the following property

$$\sum_{i=1}^n \mu_i[f(x, v_i) - g(u_i, y) + c] \leqq \sum_{i=1}^n \mu_i t_i \quad \text{for} \quad (x, y) \quad \text{in} \quad K, 0 \leqq t_i, \quad i = 1, 2, \dots, n.$$

As an easy consequence we have $0 \leqq \mu_i$ for $i = 1, 2, \dots, n$. We can thus assume $\sum_{i=1}^n \mu_i = 1$, i.e. that μ is a probability measure on the finite set $G = \{(u_i, v_i)\colon i = 1, 2, \dots, n\}$. But (2) thus gives us the following contradiction

$$0 \leqq \inf_{(u,v)\in X\times Y} \sup_{(x,y)\in K} [f(x, v) - g(u, y)] \leqq \sup_{(x,y)\in K} \sum_{i=1}^n \mu_i[(f(x, v_i) - g(u_i, y))] \leqq -c.$$

The proof of the theorem is now complete.

COROLLARY 1. *Let X, Y be convex subsets of real linear spaces, and let f, g be real-valued functions on $X \times Y$ such that* (5) *$f(-g)$ is concave in its first (second), and convex in its second (first) variable.*

Then there exists a saddle point in $X \times Y$ for f, g if and only if there exists a nonempty subset K in $X \times Y$ with (3) *and such that*

$$0 \leqq \sup_{(x,y)\in K} [f(x, v) - g(u, y)] \quad \textit{for every} \quad (u, v) \in X \times Y. \tag{6}$$

PROOF. For concave-convex functions f, $-g$, as (5) assumes, we have for every finite sets $F, G \subset X \times Y$ and probability measures λ, μ on them, respectively, such that

$$\sum_{(x,y)\in F} \lambda(x, y)[f(x, v) - g(u, y)] \leqq f\Big(\sum_{(x,y)\in F} \lambda(x, y)x, v\Big) - g\Big(u, \sum_{(x,y)\in K} \lambda(x, y)y\Big),$$

$$f\Big(x, \sum_{(u,v)\in G} \mu(u, v)v\Big) - g\Big(\sum_{(u,v)\in G} \mu(u, v)u, y\Big) \leqq \sum_{(u,v)\in G} \mu(u, v)[f(x, v) - g(u, v)].$$

Properties (1), (2) are easy consequences of these and (6). Therefore Theorem 1 applies.

REMARK 1. A known result is a consequence of Corollary 1 in the case when X, Y are compact convex subsets of real topological linear spaces and $f=g$ is continuous (or at least $f(x, v)-g(u, y)$ is upper semicontinuous in (x, y) for every (u, v)) concave-convex real-valued function on $X\times Y$.

THEOREM 2. *Let g, f be real-valued functions on the Cartesian product $X\times Y$ of the nonempty sets X, Y. The minimax inequality* (MMI) *holds true for f, g if and only if for each positive real number ε there exists a nonempty subset K_ε of $X\times Y$ such that conditions* (1), (2), (3) *of Theoerm* 1 *hold true with K_ε, $f+\varepsilon$ instead of K and f, respectively.*

PROOF. The minimax inequality (MMI) clearly holds if and only if for any $\varepsilon>0$ the following inequality is satisfied

$$\inf_{y\in Y}\sup_{x\in X} g(x, y) < \sup_{x\in X}\inf_{y\in Y}\big(f(x, y)+\varepsilon\big).$$

Equivalently, when there exists y_ε in Y such that

$$\sup_{x\in X} g(x, y_\varepsilon) < \sup_{x\in X}\inf_{y\in Y}\big(f(x, y)+\varepsilon\big),$$

then there exists x_ε in X such that

$$\sup_{x\in X} g(x, y_\varepsilon) < \inf_{y\in Y}\big(f(x_\varepsilon, y)+\varepsilon\big).$$

But this is (SP) for $f+\varepsilon$, g with $(x_\varepsilon, y_\varepsilon)$ in $X\times Y$ indeed. Theorem 1 is therefore applies.

As a further consequence we have [3, Theorem] in an improved form instead of its minimax formulation in [3]:

COROLLARY 1a. *Let f be a real-valued function on $X\times Y$ such that $\inf_{y\in Y}\sup_{u\in X} f(u, y)\in \mathbf{R}$. There exists $x_0\in X$ such that*

$$\inf_{y\in Y}\sup_{x\in X} f(u, y) \leqq f(x_0, v) \quad \textit{for every} \quad v\in Y \tag{7}$$

holds if and only if there exists a nonempty set $X_0\subset X$ such that the following properties hold:

$$\min_i \sum_j \lambda_j f(x_j, v_i) \leqq \sup_{x\in X_0}\min_i f(x, v_i) \tag{8}$$

for any finite sets $(x_j, v_i)\in X_0\times Y$ and $\lambda_j\geqq 0$, $\sum_j \lambda_j=1$;

$$\inf_{v\in Y}\sup_{x\in X_0} f(x, v) \leqq \sup_{x\in X_0}\sum_i \mu_i f(x, v_i) \tag{9}$$

for any finite sets $v_i\in Y$ and $\mu_i\geqq 0$, $\sum_i \mu_i=1$;

(10) *if $C\subset(0, +\infty)\times X$ has the property that for any $x\in X_0$ there exists (c, v)* in *C with*

$$f(x, v)+c < \inf_{y\in Y}\sup_{u\in X_0} f(u, y)$$

then a finite subset of C exists with the same property.

PROOF. The validity of the conditions (8), (9), (10) for the one point set $X_0=\{x_0\}$ where x_0 satisfies (7), is evident. Thus the necessity part of the theorem is clear. To prove the sufficiency let g be the constant

$$g := \inf_{u\in Y} \sup_{u\in X_0} f(u, y)$$

as a function on $X\times Y$ to apply Theorem 1 with $X_0\times\{y_0\}$ as K in the hope that (x_0, y_0) is saddle point for f, g with any $y_0\in Y$, as (7) requires. With this K, f and g conditions (1)—(3) reduce clearly to conditions (8)—(10), respectively. Theorem 1 hence applies, thus the proof is complete.

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