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Inequality Systems and Minimax Theorems*

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This paper is a comprehensive study of the results about the consistency of inequality systems. The minimax theorems are the main tools used in proceeding to this survey. Most of the known results are quoted, and some of them are weakened or generalized to infinite dimensional spaces. Applications to programming are given.

The aim of this paper is to summarize the scattered results concerning the consistency of inequality systems. This problem is closely related to the minimax theorems, and we shall, incidentally, be obliged to recall, in the second section, the main and stronger minimax theorems. It is well known that the minimax theorems more or less involve some convexity in their assumptions. To overstep these assumptions it is useful to immerse the inequality index set into a convex set by taking a measure set on it; this will be done in Sections IV and V. The two following sections are devoted to the infinite systems of inequalities with applications to programming. In the last section we straightforwardly derive the Mazur–Orlicz theorem and some moment theorems from the consistency theorems on inequalities.

The relationships between inequality systems, fixed-point theorems and variational inequalities are not investigated here and will be studied in another paper.

I. Inequality Systems and Minimax

Let C and D be two nonempty subsets of two Hausdorff spaces X and Y, respectively, and f a real-valued function from $C \times D$ into the real line $\mathbb{R}$. We are concerned with the consistency of the following inequality systems:

$$\exists y \in D \quad \forall x \in C \quad f(x, y) \leqslant 0 \tag{1}$$

$$\exists y \in D \quad \forall x \in C \quad f(x, y) < 0. \tag{2}$$

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The subset C is called the index set and we shall often write $f(x,\cdot)$ as $f_x(\cdot)$ or $f(\cdot,y)$ as $f_y(\cdot)$. If there exists $y\in D$ satisfying (1) (resp. (2)) we shall say that (1) (resp. (2)) is consistent on D.

We shall also use a weaker form of (1), which we denote (1′):

$$\forall\varepsilon>0 \quad \exists y\in D \quad \forall x\in C \quad f(x,y)\leqslant\varepsilon. \tag{1′}$$

If (1) (resp. (2)) is consistent on D then necessarily the following condition (A1) (resp. (A2)) holds.

$$\forall x\in C \quad \exists y\in D \quad f(x,y)\leqslant 0 \tag{A1}$$

$$\forall x\in C \quad \exists y\in D \quad f(x,y)<0. \tag{A2}$$

Let us give two obvious results showing the links between the inequality systems and the minimax theorems.

LEMMA 1.1. *Assume that D is compact and that for every $x\in C$ $f(x,\cdot)$ is lower semicontinuous (l.s.c.) on D. Then* (1) *is equivalent to*

$$\inf_{y\in D}\sup_{x\in C} f(x,y)=\min_{y\in D}\sup_{x\in C} f(x,y)\leqslant 0,$$

and (A1) *is equivalent to*

$$\sup_{x\in C}\inf_{y\in D} f(x,y)=\sup_{x\in C}\min_{y\in D} f(x,y)\leqslant 0.$$

LEMMA 1.2. *Assume that C is compact and that for every $y\in D$ $f(\cdot,y)$ is upper semicontinuous (u.s.c.) on C. Then* (2) *is equivalent to*

$$\inf_{y\in D}\sup_{x\in C} f(x,y)=\inf_{y\in D}\max_{x\in C} f(x,y)<0,$$

and (A2) *is equivalent to*

$$\sup_{x\in C}\inf_{y\in D} f(x,y)=\max_{x\in C}\inf_{y\in D} f(x,y)<0.$$

II. MINIMAX THEOREMS

To apply the minimax theorems to the question of the consistency of the system (1) or (2) we shall suppose that f is not symmetrical in x and y. More precisely we assume that very little is known about the function $f(\cdot,y)$ (in particular that C is not necessarily convex), but conversely that $f_x(\cdot)$ has some classical properties (such as convexity). So we do not recall the symmetrical minimax theorems but only those requiring few assumptions on C.

THEOREM 2.1 (Ky Fan [33]). *Let D be a compact set, and for every $x\in C$ let $f(x,\cdot)$ be l.s.c. on D. If f is convex-like on D and concave-like on C then*

$$\min_{y\in D}\sup_{x\in C} f(x,y)=\sup_{x\in C}\min_{y\in D} f(x,y). \tag{3}$$

Before recalling the definition of convex-like, let us introduce two conditions which lead to a minimax theorem and contain the Ky Fan conditions.

(K1) $\exists\mu>0$ $\exists\nu>0$ with $\mu+\nu=1$ such that $\forall(y_1,y_2)\in D^2$ $\forall\varepsilon>0$ $\exists y_3\in D$ such that $\forall x\in C$ $f(x,y_3)\leqslant\mu f(x,y_1)+\nu f(x,y_2)+\varepsilon$.

(K2) $\exists\alpha>0$ $\exists\beta>0$ with $\alpha+\beta=1$ such that $\forall(x_1,x_2)\in C^2$ $\forall\varepsilon>0$ $\exists x_3\in C$ such that $\forall y\in D$ $\alpha f(x_1,y)+\beta f(x_2,y)\leqslant f(x_3,y)+\varepsilon$.

THEOREM 2.2 (Fuchssteiner and König [18]). *Let D be compact, and for every $x\in C$ let $f(x,\cdot)$ be l.s.c. on D. If f satisfies* (K1) *and* (K2) *then* (3) *holds.*

Theorem 2.2 is a slight generalization of the theorem of König [29]. König has given many related results and applications in [30, 31]. The same theorem was also obtained by Simons [60], who has also given some close results in [61].

We can now recall what we mean by convex-like.

DEFINITION 2.3 (Ky Fan [33]). We say that f is convex-like on D (resp. concave-like on C) if (K1) (resp. (K2)) holds with $\varepsilon=0$ for every $(\mu,\nu)\in(]0,1[)^2$ (resp. for every $(\alpha,\beta)\in(]0,1[)^2$).

Remark 2.4. It is clear that f is convex-like if, for instance, D is convex and $f(x,\cdot)$ is convex on D.

Some other minimax theorems with a quasi-convexity assumption are also worth noting.

THEOREM 2.5. *Let D be convex, compact, and for every $x\in C$ let $f(x,\cdot)$ be l.s.c. and quasi-convex on D. Then* (3) *holds in the following two cases:*

(i) (K2) *is satisfied with $\varepsilon=0$* [65, *Theorem* 3];

(ii) *D is a subset of a topological vector space* (TVS), *C is a convex subset of a vector space, and for every $y\in D$ $f(\cdot,y)$ is quasi-concave and u.s.c. on every segment of C* [9, *Proposition* 1].

Remark 2.6. In his proof Terkelsen takes $\alpha=\beta=2^{-1}$, but the proof applies with any α and β such that $\alpha+\beta=1$, $\alpha>0$, $\beta>0$.

Terkelsen has also shown that there exist functions satisfying the assumptions of one of the Theorems 2.1, 2.5(i), or 2.5(ii), whereas they do not satisfy the others. It is essentially due to the fact that quasi-convex does not imply and is not implied by convex-like.

Theorem 2.5(ii) generalizes the famous Sion minimax theorem [63]. Recently Stachó [64] has shown that it is not possible to weaken the l.s.c. assumption for $f(x, \cdot)$ on D in Theorem 2.5(ii).

Now, from Lemma 1.1, we get:

COROLLARY 2.7. *With the assumptions of one of the previous theorems,* (1) *is equivalent to* (A1).

It should be possible to give the theorems symmetrical to the preceding ones. Let us do it for Theorem 2.2.

THEOREM 2.8. *Let C be compact and for every $y \in D$ $f(\cdot, y)$ be u.s.c. on C. If f satisfies* (K1) *and* (K2) *then*

$$\inf_{y\in D} \max_{x\in C} f(x, y) = \max_{x\in C} \inf_{y\in D} f(x, y). \qquad (4)$$

COROLLARY 2.9. *With the assumptions of Theorem* 2.8, (2) *is equivalent to* (A2) *and* (A1) *implies that* (1′) *holds.*

In the preceding theorems either D or C is compact. There is no simple assumption relaxing the compactness hypothesis. One can find symmetrical results in this direction in [50]. We now give some nonsymmetrical results which rely on the study of the convex program "minimize $F(x, 0)$," where $F(x, u) = \sup_{y\in D}(\langle u, y\rangle - f(x, y))$ if $x \in C$, $+\infty$ otherwise; see [53, Chap. XI].

Up to the end of this section we assume that D is a subset of the locally convex TVS Y which is paired with U by the bilinear form $\langle \cdot, \cdot\rangle$. The space U is endowed with its weak* topology. We introduce the set $A_0 = \{(u, t)/\exists x \in C \text{ satisfying } F(x, u) \leqslant t\}$.

THEOREM 2.10 (Lévine–Pomerol, [45]). *Assume that D is closed and convex, that for every $x \in C$ $f(x, \cdot)$ is convex and l.s.c. on D, and that $\bar{l} = \inf_{y\in D} \sup_{x\in C} f(x, y) < +\infty$. Moreover assume that either f is concave-like on C or assumption* (ii) *of Theorem* 2.5 *holds*

Then $\max_{x\in C} \inf_{y\in D} f(x, y) = \inf_{y\in D} \sup_{x\in C} f(x, y)$ if there exist a 0-neighborhood M in U and a neighborhood N of $\bar{l}$ such that $A_0 \cap M \times N$ is closed.

Proof. It results from [45, Prop. 3 and 4]. Q.E.D.

COROLLARY 2.11. *Suppose that the assumptions of Theorem* 2.10 *are satisfied. Then* (A2) *is equivalent to* (2) *and* (A1) *implies* (1′) *provided that the following condition* (C) *holds*:

(C) *$\exists \varepsilon > 0$, a 0-neighborhood M in U and a compact subset K in C such that*

(i) $\forall (u, t) \in A_0 \cap (M \times]-\infty, \varepsilon])$ $\exists x \in K$ *satisfying* $F(x, u) \leqslant t$,

and

(ii) $\forall y \in D$ $f(\cdot, y)$ *is u.s.c. on* K.

Proof. Combining Propositions 3, 4, and 9 in [45] we obtain the conclusion of Theorem 2.10. It immediately implies that (A2) and (2) are equivalent (Lemma 1.2). Moreover (A1) entails that $\inf_{y\in D} \sup_{x\in C} f(x, y) \leqslant 0$, which is another formulation for (1′). Q.E.D.

In the following theorem we assume that X is also a locally convex TVS, V being its topological dual endowed with the weak* topology.

THEOREM 2.12. *Assume that D and C are closed convex subset of Y and X, respectively, and that $\forall x \in C$ $f(x, \cdot)$ is convex and l.s.c. on D while $\forall y \in D$ $f(\cdot, y)$ is concave and u.s.c. on C. We suppose that $-\infty < \sup_{x\in C} \inf_{y\in D} f(x, y) = \underline{l}$. If there exist a 0-neighborhood M in V and a neighborhood N of $\underline{l}$ such that $M \times N \cap \{(v, t)/\exists y \in D \sup_{x\in C}(\langle v, x\rangle + f(x, y)) \leqslant t\}$ is weak*-closed then*

$$\min_{y\in D} \sup_{x\in C} f(x, y) = \sup_{x\in C} \inf_{y\in D} f(x, y).$$

Proof. It is an immediate translation of Proposition 3.3 of Lévine–Pomerol [46]. Q.E.D.

Remark 2.13. In the situation of Theorem 2.12, the generalized Slater condition is: $\varphi(u) = \inf_{x\in X} F(x, u)$ is continuous at 0. When either X is a Banach space or V is normed for a topology compatible with the pairing and U is a Banach space this condition reduces to $0 \in \operatorname{int}\{u / \exists x \in C$ such that $\sup_{y\in D}(\langle u, y\rangle - f(x, y)) < +\infty\}$. Then it implies that the conclusion of Theorem 2.12 is satisfied; see [46].

When Y and U are, as previously, two paired TVS, an application of Theorem 2.10 gives a result which will be useful for the study of the linear inequality systems.

THEOREM 2.14. *Assume that D is a convex subset of Y and C is a closed convex subset of U. Then*

$$\min_{u\in C} \sup_{y\in D} \langle u, y\rangle = \sup_{y\in D} \inf_{u\in C} \langle u, y\rangle$$

provided that:

(i) $\sup_{y\in D} \inf_{u\in C} \langle u, y\rangle < +\infty$;

(ii) *the interior of D (denoted* int *D) is, for a compatible topology, nonempty*;

(iii) $\exists y_0 \in \operatorname{int} D$ *such that* $\inf_{u\in C} \langle u, y_0\rangle > -\infty$.

Proof. Assuming that D is closed, in order to apply Theorem 2.10 to $-\langle u, y\rangle$ it suffices to show that $A_0 = \{(v, t) / \exists u \in C$ such that $\sup_{y\in D}\langle u+v, y\rangle \leqslant t\}$ is closed. Following, mutatis mutandis, the proof of Proposition 1 in [54], we see that if a generalized sequence (v_α, t_α) converges to $(\bar{v}, \bar{t})$ with $(v_\alpha, t_\alpha) \in A_0$, then there exist $u_\alpha \in C$ and $t_0 \geqslant 1$ such that $v_\alpha + u_\alpha \in -t_0(D - y_0)^0$, where $(D - y_0)^0$ is the polar set of $(D - y_0)$ and is weak*-compact. The closedness of A_0 follows as in [54]. To relax the closedness assumption on D one can prove, as in [54, Prop. 2] that $\sup_{y\in \bar{D}} \inf_{u\in C} \langle u, y\rangle = \sup_{y\in D} \inf_{u\in C}\langle u, y\rangle$. Q.E.D.

Remark 2.15. Christiansen [11] has independently obtained Theorem 2.14 by using directly the Hahn–Banach theorem. However, the assumption (iii) is lacking in his version, which makes it fail, as can be seen by the following example. Nevertheless with (iii) his proof works. One can find another approach in [5].

EXAMPLE 2.16. Let D be the ice cream cone $\{(y_1, y_2, y_3) / y_1 \geqslant 0, y_2 \geqslant 0, y_3^2 \leqslant 2y_1 y_2\}$ and $C = \{(u_1, u_2, u_3) / u_1 \leqslant 0, u_2 \leqslant -1, u_3 = 0\}$. The assumptions of Theorem 2.14 are satisfied except (iii). One can check that $\sup_{y\in D} \inf_{u\in C} \langle u, y\rangle = 0$ and $\inf_{u\in C} \sup_{y\in D} \langle u, y\rangle = +\infty$.

We would like to finish this section by saying that the theorems recalled above are by no means the only minimax theorems available in the literature. Besides the references already cited the reader interested in this topic should also consult Hoàng Tuy [20, 21], who has given some very strong theorems based on connectness properties, and Lassonde [42], Penot [52], and Chung-Wei Ha [13], who have obtained a minimax theorem related to the KKM maps. But these last results cannot be easily handled in the framework of inequality systems because f is requested to be l.s.c. of the two variables.

The problem of the relationships between the KKM maps, Ky Fan's inequality [38], and some related fixed-point and minimax theorems is not confronted here; see, e.g., [32, 19, 39, 62, 2]. One can also find a recent survey on minimax theorems and a generalization of the convex-like functions in [22].

III. DUALITY RESULTS

Along the lines of Chung-Wei Ha's work [12], it is possible to obtain some duality results which rely on the properties of the previously introduced set A_0.

We assume in this section that D *is a convex* subset of the locally convex TVS Y which is paired with U. The space U is endowed with its weak* topology.

For every $x \in C$ we suppose that $f(x, \cdot)$ *is defined convex, proper and l.s.c.* on Y. The functional f is allowed to take the value $+\infty$, which means that if f is only defined convex, proper, and l.s.c. on a closed convex subset of Y, we can give it the value $+\infty$ outside. We denote by $\psi_D(\cdot)$ the indicator function of a set D.

We have $\sup_{y\in D}(\langle u, y\rangle - f(x, y)) = (f_x + \psi_D)^*(u)$, where * denotes the conjugate. Using the inf-convolution $f_1 \nabla f_2(x) = \inf_{x_1+x_2=x}(f_1(x_1) + f_2(x_2))$, it can be shown that $(f_x + \psi_D)^*(u) = f_x^* \nabla \psi_D^*(u)$ if either D is closed and $\operatorname{int}(D) \neq \varnothing$ [56, Theorem 20] or D is compact. Moreover when $\operatorname{int}(D) \neq \varnothing$, the infimum in $f_x^* \nabla \psi_D^*$ is a minimum.

THEOREM 3.1. *Assume that D is compact, and that either* (K2) *or* (ii) *in Theorem* 2.5 *is satisfied. Then* (A1) *is equivalent to* (1) *and to*

$$\forall (x, u) \in C \times U, \qquad f_x^*(u) \geqslant \min_{y\in D} \langle u, y\rangle.$$

Proof. By Corollary 2.7 we know that (A1) and (1) are equivalent to $\sup_{x\in C} \min_{y\in D} f(x, y) \leqslant 0$. The latter is equivalent to $\forall x \in C$ $-(f_x + \psi_D)^*(0) \leqslant 0$ or $\forall x \in C$ $0 \leqslant f_x^* \nabla \psi_D^*(0) = \inf_{u\in U}(f_x^*(u) + \sup_{y\in D}\langle -u, y\rangle)$. Q.E.D.

Remark 3.2. It is obvious that (A1) implies the inequality of the above theorem. If f_x takes the value $+\infty$ outside of D, then the inequality of the theorem implies for $u = 0$ that (A1) is satisfied.

A more interesting result is obtained when D is not compact. Let us consider $\mathfrak{W}$ a basis of closed convex 0-neighborhoods in Y. We pick a neighborhood W_0 in $\mathfrak{W}$ and we denote by $\overline{D + W_0}$ the closure of $D + W_0$, which is convex.

THEOREM 3.3. *If for every* $W \in \mathfrak{W}$, $W \subset W_0$, (1) *is consistent on* $D + W$ *then*

$$\forall (x, u) \in C \times U \qquad f_x^*(u) \geqslant \inf_{y\in D} \langle u, y\rangle. \tag{5}$$

Moreover, assume that $\forall x \in C\ \exists y \in Y$ *such that* $f(x, y) < 0$, *and either*

(i) C *is compact,* $\forall y \in D + W_0\ f(\cdot, y)$ *is u.s.c. on* C *and* (K2) *holds on* $D + W_0$, *or*

(ii) *the assumptions of Theorem* 2.10 *are satisfied (including the closedness of* $A_0 \cap M \times N$).

Then (5) *implies that* (2) *is consistent on* $D + W$ *for every* $W \in \mathfrak{W}$.

Proof. Let us begin by the direct assertion. If (1) consistent on $D + W$, then $\inf_{y \in \overline{D+W}} \sup_{x \in C} f(x, y) \leqslant 0$. Thus $\sup_{x \in C} \inf_{y \in \overline{D+W}} f(x, y) \leqslant 0$, which is equivalent, as shown in the previous proof, to $\forall x \in C$ $0 \leqslant (f_x + \psi_{\overline{D+W}})^*(0)$. The interior of $\overline{D+W}$ is nonempty; it follows that $(f_x + \psi_{\overline{D+W}})^*(0) = \min_{u \in U}(f_x^*(u) + \sup_{y \in \overline{D+W}} \langle -u, y\rangle)$. Therefore (1) implies that $\forall W \subset W_0\ \ \forall (x, u) \in C \times U\ \ f_x^*(u) \geqslant \inf_{y \in \overline{D+W}} \langle u, y\rangle$. Our first assertion follows from $\inf_{y \in D} \langle u, y\rangle = \sup_{W \in \mathfrak{W}} \inf_{y \in \overline{D+W}} \langle u, y\rangle$. To show this last point, let us suppose that $\inf_{y \in D} \langle u, y\rangle = \alpha$ and $\sup_{W \in \mathfrak{W}} \inf_{y \in \overline{D+W}} \langle u, y\rangle = \alpha - \varepsilon$ with $\varepsilon > 0$. If we consider $W_1 = \{y / |\langle u, y\rangle| \leqslant \varepsilon 2^{-1}\}$ we get $\inf_{y \in D+W_1} \langle u, y\rangle \geqslant \alpha - 2^{-1}\varepsilon$; therefore $\inf_{y \in \overline{D+W_1}} \langle u, y\rangle \geqslant \alpha - 2^{-1}\varepsilon$, which is absurd.

We now attack the converse assertion. Let $W \in \mathfrak{W}$ be a neighborhood such that $W \subset W_0$ and consider $W' \subset W_0$ such that $\overline{D + W'} \subset D + W$. Then (5) entails that $f_x^*(u) > \inf_{y \in \overline{D+W'}} \langle u, y\rangle$. It is true if $\inf_{y \in D} \langle u, y\rangle = -\infty$ because $f_x^*(u) > -\infty$ (it is a proper, convex functional). When $u \neq 0$ and $\inf_{y \in D} \langle u, y\rangle$ is finite, we can find $y \in W'$ such that $\langle u, y\rangle < 0$, which implies our assertion. When $u = 0$ we have $f_x^*(0) > 0$, since $\forall x \in C\ \exists y \in Y$ such that $f(x, y) < 0$.

Hence we get $(f_x^* + \psi_{\overline{D+W'}})^*(0) = \min_u (f_x^*(u) + \sup_{y \in \overline{D+W'}} \langle -u, y\rangle)$ since $\overline{D + W'}$ has a nonempty interior. It follows that $(f_x + \psi_{\overline{D+W'}})^*(0) > 0$ for every x belonging to C. From Theorem 2.8 or 2.10 we have $\inf_{y \in \overline{D+W'}} \sup_{x \in C} f(x, y) = \max_{x \in C} \inf_{y \in \overline{D+W'}} f(x, y)$, whence $\inf_{y \in \overline{D+W'}} \sup_{x \in C} f(x, y) = -(f_{\bar{x}} + \psi_{\overline{D+W'}})^*(0) < 0$ for a given $\bar{x} \in C$. We conclude that (2) is consistent on $D + W$. Q.E.D.

COROLLARY 3.4. *Assume that either* (i) *or* (ii) *in Theorem* 3.3 *is satisfied. Then* (A2) *implies that* (2) *is consistent on* $D + W$ *for every* $W \in \mathfrak{W}$. *On the other hand* (5) *implies that* $\forall W \in \mathfrak{W}\ \forall \varepsilon > 0\ \exists y \in D + W$ $\forall x \in C\ f(x, y) \leqslant \varepsilon$.

Proof. Condition (A2) implies both $f_x^*(0) > 0$ and $f_x^*(u) > \inf_{y \in D} \langle u, y\rangle$, which proves the first assertion.

We have shown that (5) entails that $\inf_{y \in \overline{D+W'}} \sup_{x \in C} f(x, y) \leqslant 0$, which is exactly our second assertion. Q.E.D.

Remark 3.5. The first assertion of the above corollary is nothing else than Corollary 2.9 when assumption (i) is fulfilled, since (A1) implies (5).

COROLLARY 3.6. *Assume that either* (i) *or* (ii) *in Theorem* 3.3 *is satisfied, and that* $D = Y$ *(the whole space). Then* (A2) *and* (2) *are equivalent, and* $\forall x \in C\ \inf_{y \in Y} f(x, y) \leqslant 0$ *implies that* (1′) *is fulfilled.*

Proof. That (A2) and (2) are equivalent is immediate from Corollary 3.4. We observe that (5) reduces to $f_x^*(0) \geqslant 0$ or $\forall x \in C\ \inf_{y \in Y} f(x, y) \leqslant 0$. Thus the second assertion straightforwardly follows from Corollary 3.4, too. Q.E.D.

The two previous theorems have their source in a paper by Chung-Wei Ha [12], who extends some results of Ky Fan on linear inequality systems [37].

However, our results are not immediately comparable to those of Chung-Wei Ha, mainly because he works on a "convex-conical-closure" of the inequality (5), instead of giving conditions ensuring that (5) is really sufficient. Namely Chung-Wei Ha uses the closed convex conical hull of the set $\{(u, f_x^*(u)) / x \in C,\ u \in \operatorname{dom} f_x^*\}$. If we denote this cone by K then (5) is replaced by $\forall (u, t) \in K\ \inf_{y \in D} \langle u, y\rangle \leqslant t$.

Furthermore let us observe that in his result similar to Theorem 3.3 [12, Theorem 1] the assumption $f_x^*(0) < 0$ is lacking, which makes the theorem false. Actually Chung-Wei Ha cannot be sure in his proof [12, p. 29] that $I_B(u) < I_A(u)$ (it is obviously false, at least when $u = 0$). Let us give an example where Theorem 1 of Chung-Wei Ha [12] fails to be satisfied.

EXAMPLE 3.7. For every integer $n \in \mathbb{N}$, let us consider $f_n(y) = 0$ if $y \leqslant 0$ and $y_n \leqslant -1$, $+\infty$ otherwise, where $y \in l^2$ (the square summable sequence space ordered by its usual positive cone). There does not exist a y such that $\forall n \in \mathbb{N}\ f_n(y) \leqslant 0$. We check that $f_x^*(u) = -u_n$ if $u \geqslant 0$, $+\infty$ otherwise. The set K is the closure of $\{(u, -u_n) / u \geqslant 0, n \in \mathbb{N}\}$ and it is easy to verify that $(0, -1) \notin K$. Thus Theorem 1(a) of Chung-Wei Ha [12] is not satisfied.

IV. CONVEXIFICATION I

As we have already said, generally very little is known about the index set C. In most cases C is discrete and not convex. It is for this reason that it is often useful to immerse C in a convex set, as we are going to do by using different measure spaces.

In this section we assume that:

(a) C is a locally compact set,

(b) $\forall y \in D\ f(\cdot, y)$ is u.s.c. on C,

(c) $\forall y \in D\ \exists \alpha(y) < +\infty$ such that $\forall x \in C\ |f(x, y)| \leqslant \alpha(y)$.

We consider the space $\mathfrak{M}$ of the Radon measures on C [8, Chap. III, Sect. 1, n° 3] endowed with its weak* topology or "vague" topology. The subset $\mathfrak{M}_+^1$ of the positive bounded measures satisfying $\|\mu\| \leqslant 1$ is a weak*-compact subset of $\mathfrak{M}$ [8, Chap. III, Sect. 1, n° 9, Corollary 2, Proposition 15]. In what follows we can replace $\mathfrak{M}_+^1$ by $\mathfrak{N} = \{\mu \in \mathfrak{M} / \mu \geqslant 0 \text{ and } \|\mu\| = 1\}$ [8, Chap. III, Sect. 1, n° 9, Corollary 3, Proposition 15] whenever C is compact.

Let us consider the bifunction K from $\mathfrak{M}_+^1 \times D$ into $\mathbb{R}$ defined by $K(\mu, y) = \langle \mu, f_y \rangle = \int_C f_y(x)\, d\mu(x)$. It is immediate that $\int_C |f_y(x)|\, d\mu(x) \leqslant \alpha(y) \int_C d\mu(x) < +\infty$ since $\mu \in \mathfrak{M}_+^1$, and f_y is therefore integrable [8, Chap. IV, Sect. 4, Corollary 1, Proposition 5].

LEMMA 4.1. (i) *$K(\cdot, y)$ is u.s.c. and linear on $\mathfrak{M}_+^1$.*

(ii) *If f is convex-like on D (resp. satisfies* (K1)) *then $K(\mu, \cdot)$ is convex-like (resp. satisfies* (K1)) *on D.*

Proof. The assertion (i) is a consequence of the definition and of Bourbaki [8, Chap. IV, Sect. 4, Corollary 3, Proposition 5]. Let us prove (ii). For any $\theta \in [0, 1]$, $K(\mu, y_3) = \int_C f(x, y_3)\, d\mu(x) \leqslant \int_C (\theta f(x, y_2) + (1 - \theta) f(x, y_1))\, d\mu(x)$ since $f(x, y_3) \leqslant \theta f(x, y_2) + (1 - \theta) f(x, y_1)$ for every $x \in C$. Q.E.D.

THEOREM 4.2. *Assume that C is compact and that $f(x, \cdot)$ satisfies* (K1). *Then the following assertions are equivalent*:

(i) $\forall \mu \in \mathfrak{N}\ \exists y \in D$ *satisfying* $\int_C f(x, y)\, d\mu(x) < 0$;

(i′) $\exists \alpha < 0$ *such that* $\forall \mu \in \mathfrak{N}\ \exists y \in D$ *satisfying* $\int_C f(x, y)\, d\mu(x) \leqslant \alpha$;

(ii) *the inequality system* (2) *is consistent on D.*

Proof. The first assertion is condition (A2) for the functional $K(\mu, y)$. Thus, by Lemmas 4.1 and 1.2, it is equivalent to (i′) or $\max_{\mu \in \mathfrak{N}} \inf_{y \in D} K(\mu, y) < 0$. This latter is equivalent to $\inf_{y \in D} \max_{\mu \in \mathfrak{N}} K(\mu, y) < 0$ (Theorem 2.8).

Now $K(\mu, y) \leqslant \max_{x \in C} f(x, y)\, \mu(C)$; therefore for every $\mu \in \mathfrak{N}$ $K(\mu, y) \leqslant \max_{x \in C} f(x, y)$. Moreover the inequality holds by choosing μ as the Dirac measure at the point $\bar{x}$ such that $f(\bar{x}, y) = \max_{x \in C} f(x, y)$. So we always have $\max_{\mu \in \mathfrak{N}} K(\mu, y) = \max_{x \in C} f(x, y)$. Then (i) is equivalent to $\inf_{y \in D} \max_{x \in C} f(x, y) < 0$. Q.E.D.

Remark 4.3. The previous theorem relaxes the assumption that (K2) must be satisfied in Corollary 2.9. Taking for C the finite set $\{1, 2, \ldots, m\}$

Theorem 4.2 subsumes the main theorem of Ky Fan, Glicksberg, and Hoffman [40]; see also [35, p. 208].

THEOREM 4.4. *Assume that $f(x, \cdot)$ satisfies* (K1); *then the following assertions are equivalent*:

(i) *the inequality system* (1′) *is consistent on D*;

(ii) $\forall \mu \in \mathfrak{M}_+^1\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\int_C f(x, y)\, d\mu(x) \leqslant \varepsilon$.

Proof. The assertion (i) is equivalent to $\inf_{y \in D} \sup_{x \in C} f(x, y) \leqslant 0$ while the second is equivalent to $\max_{\mu \in \mathfrak{M}_+^1} \inf_{y \in D} K(\mu, y) \leqslant 0$.

As in the previous theorem $\max_{\mu \in \mathfrak{M}_+^1} \inf_{y \in D} K(\mu, y) = \inf_{y \in D} \max_{\mu \in \mathfrak{M}_+^1} K(\mu, y)$.

Assume that (ii) holds; then $\forall \varepsilon \geqslant 0$ there exists $\bar{y} \in D$ such that $\max_{\mu \in \mathfrak{M}_+^1} K(\mu, \bar{y}) \leqslant \varepsilon$. If there exists $\bar{x} \in C$ with $f(\bar{x}, \bar{y}) > \varepsilon$, let us consider the Dirac measure at $\bar{x}$, $\delta_{\bar{x}}$; we get $K(\delta_{\bar{x}}, y) > \varepsilon$, which is absurd. Thus (i) holds.

Conversely we have $K(\mu, y) \leqslant \sup_{x \in C} f(x, y)\, \mu(C) \leqslant \sup_{x \in C} f(x, y)$ if $\sup_{x \in C} f(x, y) \geqslant 0$, and $K(\mu, y) \leqslant 0$ otherwise. It implies that $\sup_{\mu \in \mathfrak{M}_+^1} K(\mu, y) \leqslant \max(0, \sup_{x \in C} f(x, y))$. Assume that (i) holds; then for every $\varepsilon > 0$ there exists $\bar{y} \in D$ such that $\sup_{x \in C} f(x, \bar{y}) \leqslant \varepsilon$. It follows that $\inf_{y \in D} \sup_{\mu \in \mathfrak{M}_+^1} K(\mu, y) \leqslant \sup_{\mu \in \mathfrak{M}_+^1} K(\mu, \bar{y}) \leqslant \varepsilon$, implying that $\inf_{y \in D} \max_{\mu \in \mathfrak{M}_+^1} K(\mu, y) \leqslant 0$. Q.E.D.

COROLLARY 4.5. *Assume that D is compact, that $\forall x \in C\ f(x, \cdot)$ is l.s.c. on D and that $f(x, \cdot)$ satisfies* (K1). *Then the following assertions are equivalent*:

(i) *the inequality system* (1) *is consistent on D*;

(ii) *for every finite family x_i $(i \in I)$ and every nonnegative number α_i $(i \in I)$ with $\sum_{i \in I} \alpha_i = 1$ there exists $y \in D$ such that $\sum_{i \in I} \alpha_i f(x_i, y) \leqslant 0$.*

Proof. With our assumptions, (i) is equivalent to $\min_{y \in D} \sup_{x \in C} f(x, y) \leqslant 0$ (Lemma 1.1), which in turn is equivalent to $\max_{\mu \in \mathfrak{M}_+^1} \inf_{y \in D} K(\mu, y) \leqslant 0$ (Theorem 4.4).

On the other hand the sets $A_x = \{y \in D \mid f(x, y) \leqslant 0\}$ are closed and compact. Then y satisfies (1) if and only if every finite intersection $\bigcap_{i \in I} A_{x_i}$ is nonempty. Thus, setting $C = \bigcup_{i \in I} \{x_i\}$, we can apply the previous theorem and replace $\mathfrak{M}_+^1$ by $\mathfrak{N}$. We observe to finish that

$$\max_{\substack{\alpha_i \geqslant 0 \\ \sum_{i \in I} \alpha_i = 1}} \inf_{y \in D} \sum_{i \in I} \alpha_i f(x_i, y) = \max_{\substack{\alpha_i \geqslant 0 \\ \sum_{i \in I} \alpha_i = 1}} \min_{y \in D} \sum_{i \in I} \alpha_i f(x_i, y). \quad \text{Q.E.D.}$$

Remark 4.6. Corollary 4.5 is essentially due to Ky Fan [35], who assumes that D and $f(x,\cdot)$ are convex entailing (K1). Many consequences of Corollary 4.5 are derived by Ky Fan [35, 36].

The convexification method, as previously developed, goes back to Von Neumann. The Radon measures and the compactness property of $\mathfrak{N}$ or $\mathfrak{M}^1_+$ have of course already been used by many authors; e.g., Ky Fan [32], Peck and Dulmage [51], and Lemaire [44].

On the other hand it should be possible to convexify the set D in the same manner to get some results such as those of Ville [66] (see Ky Fan [32]), or Peck and Dulmage [51] (by taking the finite support probability measures on Y). But the measurability question related to Fubini's theorem arises when $f(\cdot,\cdot)$ is not continuous. This problem should deserve special study; the interested reader is referred to Kindler [26, 27] and Mertens [48].

In order to apply Theorems 3.3 and 3.4 to $K(\mu, y)$, we need the following lemma.

LEMMA 4.7. *Assume that Y is a normed space, that either the family of the functionals $f(x,\cdot)$ $(x\in C)$ is equicontinuous on D, or the functional $\alpha(\cdot)$ (defined in* (c)*) is u.s.c. on D. If for every $x\in C$ $f(x,\cdot)$ is l.s.c. on D then $K(\mu,\cdot)$ is l.s.c. on D for every $\mu\in\mathfrak{M}^1_+$.*

Proof. Let (y_n, t_n) be a sequence converging to $(\bar y, \bar t)$ such that $K(\mu, y_n)\leqslant t_n$. Let us consider $g_k(x)=\inf_{\|y-\bar y\|\leqslant k^{-1}} f(x,y)$. It is clear that $g_k(x)$ is u.s.c. on C. Now $\inf_{\|y-\bar y\|\leqslant k^{-1}} -\alpha(y)\leqslant g_k(x)\leqslant \inf_{\|y-\bar y\|\leqslant k^{-1}}\alpha(y)$; then for a sufficiently large k one has $-1-\limsup_{y\to\bar y}\alpha(y)\leqslant g_k(x)\leqslant\alpha(\bar y)$ or $-1-\alpha(\bar y)\leqslant g_k(x)\leqslant\alpha(\bar y)$. Thus $g_k(x)$ is integrable for every bounded measure. The theorem of the dominated convergence [8, Chap. IV, Sect. 4, Proposition 4] says that $\sup_k g_k(x)$ is integrable and we have

$$\int_C (\sup_k g_k(x))\,d\mu(x)=\lim_{k\to\infty}\int_C g_k(x)\,d\mu(x)=\int_C \liminf_{y\to\bar y} f(x,y)\,d\mu(x)$$

$$=\int_C f(x,\bar y)\,d\mu(x)\leqslant\int_C f(x,y_k)\,d\mu(x)\leqslant t_k$$

$$(\text{with } \|\bar y-y_k\|\leqslant k^{-1}),\ \text{whence } K(\mu,\bar y)\leqslant\bar t.$$

With the alternate assumption one has $\forall x\in C$ $|f(x,y_n)|\leqslant|f(x,\bar y)|+1$, which is integrable for every bounded measure. Since $\lim_{n\to\infty} f(x,y_n)=f(x,\bar y)$, by Lebesgue's theorem [8, Chap. IV, Sect. 4, Theorem 2] it follows that $K(\mu,\bar y)=\lim_{n\to\infty}K(\mu,y_n)$ and $K(\mu,\bar y)\leqslant\bar t$. Q.E.D.

THEOREM 4.8. *Assume that D is a convex subset of a normed TVS Y paired with U. For every $x\in C$ we suppose that $f(x,\cdot)$ is convex proper and l.s.c. on a neighborhood $D+W_0$ of D ($W_0\in\mathfrak{W}$; see Section* III*). Assume that* (a), (b), (c), *and the hypotheses of Lemma* 4.7 *hold on $D+W_0$.*

If for every 0-neighborhood $W\in\mathfrak{W}$ (1) *is consistent on $D+W$, then*

$$\forall(\mu,u)\in\mathfrak{M}^1_+\times U\quad \sup_{y\in Y}\left(\langle u,y\rangle-\int_C f(x,y)\,d\mu(x)\right)\geqslant\inf_{y\in D}(\langle u,y\rangle).\quad (6)$$

Also, (6) *implies that* (1′) *is consistent on every $D+W$ ($W\in\mathfrak{W}$).*

Moreover the strict inequality in (6) *implies that* (2) *is consistent on $D+W$ for every $W\in\mathfrak{W}$.*

Proof. The functional $K(\mu,y)$ satisfies the assumptions of Theorem 3.3(i). The inequality (6) is the translation of (5). The first assertion is nothing else than the first assertion of Theorem 3.3.

By Corollary 3.4, (6) implies that $\forall W\in\mathfrak{W}$ $\forall\varepsilon>0$ $\exists y\in D+W$ such that $\forall\mu\in\mathfrak{M}^1_+$ $K(\mu,y)\leqslant\varepsilon$, which implies that (1′) holds (see the proof of Theorem 4.4).

The last assertion similarly follows from the first part of Corollary 3.4. Q.E.D.

V. CONVEXIFICATION II

We have seen in Theorems 2.10 and 2.12 that it is possible to relax the compactness assumption. To do that, we need a dual system of two locally convex TVS.

The measure theory offers different possibilities. First, as in the previous section, the space of the continuous functions on X with a compact support $\mathfrak{C}_K(X)$ endowed with the topology $\mathfrak{T}_1$ "inductive limit of the sup convergence on every compact." Then the topological dual of $\mathfrak{C}_K(X)$ is the space $\mathfrak{M}$ of the Radon measures (see [8, Chap. III, Sect. 1, n° 3]).

We can also consider the Banach space $\mathfrak{C}_0(X)$ of the continuous functions on X which vanish at infinity, endowed with the topology $\mathfrak{T}_2$ of the sup convergence. Then the dual of $\mathfrak{C}_0(X)$ is the space of the bounded Radon measures $\mathfrak{M}^b$.

Finally we have the space $\mathfrak{C}(X)$ of the continuous functions on X, endowed with the topology $\mathfrak{T}_3$ of the sup convergence on every compact. Its topological dual is the space $\mathfrak{M}^K$ of the Radon measures with a compact support (see [8, Chap. IV, Sect. 4, n° 8]).

In any case the spaces $\mathfrak{M}$, $\mathfrak{M}^b$, $\mathfrak{M}^K$ are topologized with their weak* topologies, the coarser being the weak* topology of $\mathfrak{M}$ (named the "vague" topology).

Throughout this section we assume that conditions (a), (b), and (c) of the previous section hold. In the dual system $\langle\mathfrak{C}(X),\mathfrak{M}^K\rangle$, assumption (c) may be replaced by the following weaker ones:

(c′) $\forall y \in D\ \exists \alpha(y)$ such that $\forall x \in C\ 0 \leqslant f(x, y) + \alpha(y)$, or

(c″) $\forall y \in D\ f(\cdot, y)$ is l.s.c. on C (whence continuous with (b)).

We denote by $\mathfrak{M}_+$, $\mathfrak{M}_+^b$, and $\mathfrak{M}_+^K$ the cones of the positive measures in $\mathfrak{M}$, $\mathfrak{M}^b$, and $\mathfrak{M}^K$, respectively.

We call E, one of the spaces $\mathfrak{C}_K(X)$ (resp. $\mathfrak{C}_0(X)$ or $\mathfrak{C}(X)$) endowed with the topology $\mathfrak{T}_1$, (resp. $\mathfrak{T}_2$ or $\mathfrak{T}_3$), and we consider a weak*-closed subset $\mathfrak{S}$ in the dual of E such that $0 \in \mathfrak{S}$, *$\mathfrak{S}$ is contained in the positive cone* and *contains the Dirac measures*. We introduce the set $A_0 = \{(g, t) \in E \times \mathbb{R} / \exists y \in D$ such that $\sup_{\mu \in \mathfrak{S}} \int_C (g(x) + f(x, y))\, d\mu(x) \leqslant t\}$.

THEOREM 5.1. *Assume that for every $x \in C$ $f(x, \cdot)$ is convex-like on D. If there exist a 0-neighborhood M in E and $\alpha > 0$ such that $A_0 \cap M \times]-\infty, \alpha]$ is closed, then*

$$\min_{y \in D} \sup_{\mu \in \mathfrak{S}} K(\mu, y) = \sup_{\mu \in \mathfrak{S}} \inf_{y \in D} K(\mu, y).$$

Proof. We apply Theorem 2.10 to $-K(\mu, y)$. For every $y \in D$ $-K(\cdot, y)$ is l.s.c. (Lemma 4.1), convex (actually linear) on $\mathfrak{S}$, and $-\sup_{\mu \in \mathfrak{S}} \inf_{y \in D} K(\mu, y) \leqslant 0$ (because $0 \in \mathfrak{S}$). Moreover $-K(\mu, \cdot)$ is concave-like on D (Lemma 4.1). The conclusion follows from Theorem 2.10. Q.E.D.

COROLLARY 5.2. *With the assumptions of Theorem 5.1, suppose that $\mathfrak{S}$ is the cone of the positive measures. Then Theorem 5.1 holds if there exists a 0-neighborhood M in E such that $A_0' \cap M$ is closed, where $A_0' = \{g \in E / \exists y \in D\ \forall x \in C\ g(x) + f(x, y) \leqslant 0\}$.*

Proof. The set $\mathfrak{S}$ being the cone of the positive measures, it is easy to check that

$$A_0 = \{g \in E / \exists\, y \in D\ \forall x \in C\ g(x) + f(x, y) \leqslant 0\} \times [0, +\infty[. \qquad \text{Q.E.D.}$$

COROLLARY 5.3. *With the assumptions of Theorem 5.1 the following assertions are equivalent.*

(i) *the inequality system* (1) *is consistent on D;*

(ii) $\forall \mu \in \mathfrak{S}\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\int_C f(x, y)\, d\mu(x) \leqslant \varepsilon$.

Proof. The assertion (ii) is equivalent to $\sup_{\mu \in \mathfrak{S}} \inf_{y \in D} K(\mu, y) \leqslant 0$. By Theorem 5.1 it is equivalent to $\min_{y \in D} \sup_{\mu \in \mathfrak{S}} K(\mu, y) \leqslant 0$. The latter is clearly implied by (i) since μ is positive. On the other hand it implies (i) because the Dirac measures are contained in $\mathfrak{S}$. Q.E.D.

PROPOSITION 5.4. *Assume that D is closed and that for every $x \in C$ $f(x, \cdot)$ is l.s.c. on D. Then the closedness assumption in Corollary 5.2 is fulfilled in $\mathfrak{C}_0(X)$ (resp. $\mathfrak{C}(X)$) if $\{y / \forall x \in X\, f(x, y) \leqslant \varepsilon\}$ ($\varepsilon > 0$) (resp. $\{y /$ there exists a compact $K \subset X$ such that $\forall x \in K\ f(x, y) \leqslant \varepsilon\}$) is compact.*

Proof. Let us consider the 0-neighborhood in E, $M = \{g / \forall x \in C$ $g(x) \geqslant -\varepsilon\}$ (resp. $M = \{g / \exists K\ \forall x \in K\ g(x) \geqslant -\varepsilon\}$). Let g_α be a converging net in $A_0' \cap M$, $g_\alpha \to g$. For every α there exists $y_\alpha \in D$ such that $\forall x \in C$ $f(x, y_\alpha) \leqslant -g_\alpha(x)$ and $-g_\alpha(x) \leqslant \varepsilon$. With our compactness assumption the net y_α remains in a compact. A subnet $y_{\alpha'}$ converges to $\bar{y} \in D$ and $f(x, \bar{y}) \leqslant -g(x)$. Q.E.D.

Remark 5.5. When $\mathfrak{S}$ is a cone the Slater condition (see Remark 2.13) reduces to:

There exists a 0-neighborhood M in E such that $\forall g \in M\ \exists y \in D$ such that $\forall x \in C\ f(x, y) + g(x) \leqslant 0$. In order words it means that

$$0 \in \operatorname{int} \left\{ g / \exists y \in D \text{ such that } \sup_{\mu \in \mathfrak{S}} \int_C (g(x) + f(x, y))\, d\mu(x) < +\infty \right\}.$$

For instance, with $E = \mathfrak{C}_0(X)$ it becomes "$\exists \varepsilon > 0\ \exists y \in D$ such that $\forall x \in C\ f(x, y) \leqslant -\varepsilon$."

Up to the end of this section we assume Y is a normed TVS paired with U as in Section III, whereas 0 *does not necessarily belong to* $\mathfrak{S}$.

We introduce the set $B_0 = \{(u, t) / \exists \mu \in \mathfrak{S}\ \sup_{y \in D} (\langle u, y \rangle - K(\mu, y)) \leqslant t\}$.

THEOREM 5.6. *Assume that D is closed, convex, and that $\forall x \in C\ f(x, \cdot)$ is convex and* l.s.c. *on D. We suppose that the assumptions of Lemma* 4.7 *are fulfilled and that $-\underline{l} = \inf_{y \in D} \sup_{\mu \in \mathfrak{S}} K(\mu, y) < +\infty$. If there exist a 0-neighborhood M in U and a neighborhood N of $\underline{l}$ such that $B_0 \cap M \times N$ is weak*-closed then*

$$\max_{\mu \in \mathfrak{S}} \inf_{y \in D} K(\mu, y) = \inf_{y \in D} \sup_{\mu \in \mathfrak{S}} K(\mu, y).$$

Proof. We apply Theorem 2.12 to $-K(\mu, y)$. From Lemmas 4.1 and 4.7, $-K(\mu, \cdot)$ is concave and u.s.c. on D. The other assumptions of Theorem 2.12 are also satisfied. Q.E.D.

COROLLARY 5.7. *With the assumptions of Theorem 5.6 the following assertions are equivalent.*

(i) *The inequality system* (1′) *is consistent on D;*

(ii) $\forall \mu \in \mathfrak{S}\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\int_C f(x, y)\, d\mu(x) \leqslant \varepsilon$.

Proof. It is obvious that (i) implies (ii). The converse also holds as in the proof of Theorem 4.4. Q.E.D.

Remark 5.8. The preceding corollary is a generalization of the Theorem 4.4 to a noncompact measure set.

VI. Application to the Infinite Systems of Convex Inequalities

The positive integer set is denoted by $\mathbb{N}$.

THEOREM 6.1. *Assume that D is closed and convex and that $\forall n \in \mathbb{N}$ $f_n(\cdot)$ is convex, proper and l.s.c. on D.*

If there exist $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $D \cap \{y / \forall n \leqslant n_0\ f_n(y) \leqslant \varepsilon\}$ is compact. Then the following assertions are equivalent:

(i) $\exists y \in D$ *such that* $\forall n \in \mathbb{N}\ f_n(y) \leqslant 0$;

(ii) $\forall m\, \forall \lambda_i \geqslant 0$, $i = 1, 2,..., m$, $\forall \varepsilon > 0$ $\exists y \in D$ *such that* $\sum_{i=1}^{m} \lambda_i f_i(y) \leqslant \varepsilon$.

Proof. Let us set $f(n, y) = f_n(y)$ if $y \in D$ and $+\infty$ otherwise. The set $\mathbb{N}$ is endowed with the discrete topology. Then conditions (a), (b), and (c″) are fulfilled. Thus the result is a consequence of Corollary 5.3 and of Proposition 5.4. Q.E.D.

COROLLARY 6.2. *Assume that the hypotheses of Theorem 6.1 hold and that $\forall y \in D\ \exists \alpha(y)$ such that $\forall n \in \mathbb{N}\ |f_n(y)| \leqslant \alpha(y)$.* If $\exists \varepsilon > 0$ such that *$D \cap \{y / \forall n \in \mathbb{N}\ f_n(y) \leqslant \varepsilon\}$ is compact, then the assertion* (i) *of Theorem* 6.1 *is equivalent to*

(ii′) $\forall \lambda_i \geqslant 0$ $\sum_{i=1}^{\infty} \lambda_i < +\infty$ $\forall \varepsilon > 0$ $\exists y \in D$ *such that* $\sum_{i=1}^{\infty} \lambda_i f_i(y) \leqslant \varepsilon$.

Proof. It results from Proposition 5.4, condition (c″) being replaced by (c), and the measures with a compact support by the bounded ones. Q.E.D.

LEMMA 6.3. *With the hypotheses of Theorem 6.1 assume that D is contained in a finite dimensional euclidean space $\mathbb{R}^n$; then the following assertions are equivalent:*

(i) $\exists n_0 \in \mathbb{N}\ \exists \varepsilon > 0$ *such that* $D \cap \{y / \forall n \leqslant n_0\ f_n(y) \leqslant \varepsilon\}$ *is compact*;

(ii) $\exists \varepsilon > 0$ *such that* $D \cap \{y / \forall n \in \mathbb{N}\ f_n(y) \leqslant \varepsilon\}$ *is compact.*

Proof. It suffices to show that (ii) implies (i). Let us assume that (ii) holds and that (i) does not. Then $\forall n \in \mathbb{N}$ $D \cap \{y / \forall i \leqslant n\ f_i(y) \leqslant \varepsilon\}$ is unbounded. Thus a nonzero vector y_n belongs to its recession cone (one can suppose that $\|y_n\| = 1$). A subsequence of y_n converges to $\bar{y}$. For any $i_0 \in \mathbb{N}$ $\forall \lambda > 0$ $f_{i_0}(y + \lambda y_n) \leqslant \varepsilon$ for n sufficiently large. Consequently $f_{i_0}(y + \lambda \bar{y}) \leqslant \varepsilon$, which contradicts (ii). Q.E.D.

Remark 6.4. (a) The assertions (i) and (ii) in Theorem 6.1 are obviously equivalent to:

(iii) $\forall m$ $\forall \lambda_i \geqslant 0$, $i = 1, 2,..., m$, $\exists y \in D$ such that $\sum_{i=1}^{m} \lambda_i f_i(y) \leqslant 0$. The same remark holds in Corollary 6.2 and whenever we are in an analogous situation.

(b) Combining Theorem 6.1 and Lemma 6.3 we obtain Theorem 21.3 of Rockafellar [55].

Let us now consider the following program (P), where D and the functions f_n, $n = 0, 1, 2,...$, are as in Theorem 6.1:

$$(\mathrm{P})\begin{cases} \text{minimize} & f_0(y) \\ \text{subject to} & f_n(y) \leqslant 0 \quad \text{for every } n \in \mathbb{N}. \\ \text{and} & y \in D \end{cases}$$

PROPOSITION 6.5. *Assume that the value of the program* (P) *is a finite real number γ. If there exist n_0 and $\beta > 0$ such that $D \cap \{y / \forall n \leqslant n_0\ f_n(y) \leqslant \beta\} \cap \{y / f_0(y) \leqslant \gamma\}$ is compact, then*

$$\gamma = \sup_{\substack{\lambda_i \geqslant 0 \\ 1 \leqslant i \leqslant m}} \inf_{y \in D} \left(\sum_{i=1}^{m} \lambda_i f_i(y) + f_0(y) \right).$$

Proof. For every $\alpha > 0$ the inequality system $f_0(y) \leqslant \gamma - \alpha$, $f_i(y) \leqslant 0$ is inconsistent. Thus there exist $\lambda_i \geqslant 0$, $0 \leqslant i \leqslant m$, and $\varepsilon > 0$ such that $\forall y \in D$ $\sum_{i=1}^{m} \lambda_i f_i(y) + \lambda_0 (f_0(y) - \gamma + \alpha) > \varepsilon$ (Theorem 6.1). The program (P) being consistent, $\lambda_0 \neq 0$ and $\lambda_0^{-1} \sum_{i=1}^{m} \lambda_i f_i(y) + f_0(y) \geqslant \gamma - \alpha$ whence

$$\sup_{\substack{\lambda_i \geqslant 0 \\ 1 \leqslant i \leqslant m}} \inf_{y \in D} \sum_{i=1}^{m} \lambda_i f_i(y) + f_0(y) \geqslant \gamma.$$

The reverse inequality is obvious. Q.E.D.

Remark 6.6. Following Duffin [14–16] we can say that a program (P) satisfying the condition of Proposition 6.5 is *canonically closed.* Duffin gives the above proposition for a finite number of constraints. Proposition 6.5 generalizes to ininite dimensional spaces Y the main result of Karney [25, Theorem 4.5]. Combining Lemma 6.3 and Proposition 6.5 we essentially obtain the same results as Karney [23, Theorem 2.1] and [25, Proposition 2.3 and Theorem 4.5], and generalize [24, Theorem 2].

THEOREM 6.7. *We assume that D is convex and that $\forall n \in \mathbb{N}$ f_n is finite and convex on D. Also we suppose that $\forall y \in D$ $\exists \alpha(y)$ such that $\forall n \in \mathbb{N}$ $|f_n(y)| \leqslant \alpha(y)$. Then the following assertions are equivalent:*

(i) $\forall \varepsilon > 0 \ \exists y \in D$ *such that* $\forall n \in \mathbb{N} \ f_n(y) \leqslant \varepsilon$;

(ii) $\forall \lambda_i \geqslant 0 \ \sum_{i=1}^{\infty} \lambda_i \leqslant 1 \ \forall \varepsilon > 0 \ \exists y \in D$ *such that* $\sum_{i=1}^{\infty} \lambda_i f_i(y) \leqslant \varepsilon$.

Moreover assume that Y is normed, that the previous assumptions hold on $D + W_0$ *for* $W_0 \in \mathfrak{W}$ (see Section III), *and that the family* $f_n(\cdot)$ $(n \in \mathbb{N})$ *is equicontinuous on* $D + W_0$. *Then*

(iii) $\forall \lambda_i \geqslant 0, \sum_{i=1}^{\infty} \lambda_i \leqslant 1, \exists y \in D$ *such that* $\sum_{i=1}^{\infty} \lambda_i f_i(y) < 0$ *implies that*:

(iv) $\forall W \in \mathfrak{W} \ \exists y \in D + W$ *such that* $\forall n \in \mathbb{N} \ f_n(y) < 0$.

Proof. The first assertion is the translation of Theorem 4.4 for $f(n, y) = f_n(y)$.

Condition (iii) implies that the strict inequality (6) holds in Theorem 4.8. Thus we get (iv). Q.E.D.

Remark 6.8. Corollary 6.2 can also be regarded as a direct consequence of Theorem 6.7.

We will not examine in this section the classical case, where the set D is assumed to be compact, and the numerous consequences of Corollary 4.5. We will study this situation in the last section.

Less well known are the results that can be obtained when C is compact. For instance, we straightforwardly derive the following result from Theorem 4.2.

THEOREM 6.9. *Assume that C is compact, that* $\forall x \in C \ f(x, \cdot)$ *is convex on the convex set D, and that* $\forall y \in D \ f(\cdot, y)$ *is finite, u.s.c., and bounded from below on C. Then the following assertions are equivalent*:

(i) $\exists y \in D$ *such that* $\forall x \in C \ f(x, y) < 0$;

(ii) $\exists \alpha < 0 \ \forall \mu \in \mathfrak{M}_+ \ \|\mu\| = 1 \ \exists y \in D$ *such that* $\int_C f(x, y) \, d\mu(x) \leqslant \alpha$.

Let us consider the program (Q):

$$(\mathrm{Q}) \begin{cases} \text{minimize} & f(y) \\ \text{subject to} & \forall x \in C \quad f(x, y) \leqslant 0 \\ y \in D. & \end{cases}$$

We suppose that the functions $f(x, y)$ satisfy the assumptions of Theorem 6.9, and similarly that $f(y)$ is convex and finite.

PROPOSITION 6.10. *Assume that the value* γ *of the program* (Q) *is finite; then*

$$\gamma = \max_{\mu \in \mathfrak{M}_+} \inf_{y \in D} \left(\int_C f(x, y) \, d\mu(x) + f(y) \right)$$

provided that the following condition holds:

(i) $\exists \ y \in D$ *such that* $\forall x \in C \ f(x, y) < 0$.

Proof. Let us consider the compact set $C' = C \cup \{x_0\}$, where x_0 is any point which does not belong to C, $\{x_0\}$ being a neighborhood of x_0. We set $f(x_0, y) = f(y) - \gamma$. Then the inequality system $\forall x \in C' \ f(x, y) < 0$ is inconsistent. By Theorem 4.2 there exists $\mu \in \mathfrak{N}$ such that $\forall y \in D$ $\int_C f(x, y) \, d\mu(x) + \mu_0(f(y) - \gamma) \geqslant 0$. Now μ_0 is strictly positive; otherwise Theorem 4.2 shows that (i) cannot be true. Considering $\mu_0^{-1}\mu = \nu$ we get $\|\nu\| = \mu_0^{-1}$ and $\forall y \in D \ \int_C f(x, y) \, d\nu(x) + f(y) \geqslant \gamma$. On the other hand for every $\varepsilon > 0$ the system $\forall x \in C' \ f(x, y) < \varepsilon$ is consistent. Applying Theorem 4.2 to $f - \varepsilon$ it follows that $\max_{\mu \in \mathfrak{N}} \inf_{y \in D} (\int_C f(x, y) \, d\mu(x) + \mu_0(f(y) - \gamma)) \leqslant 0$. For every positive measure with $\mu_0 > 0$ it follows that $\inf_{y \in D} (\int_C f(x, y) \, d\mu(x) + f(y)) \leqslant \gamma$ and γ is attained for $\mu = \nu$. Q.E.D.

Remark 6.11. When C is finite Proposition 6.10 essentially is the Ky Fan–Glicksberg–Hoffman Theorem 2 [40]. Also it is noteworthy to point out that, thanks to the Helly-type theorem of Klee [28] as recalled by Borwein [4], the case *Y is finite dimensional* can be reduced to a finite number of inequalities. More precisely when $f(x, y)$ is quasi-convex in y for every $x \in C$, and u.s.c. (of the two variables), $f(\cdot)$ is quasi-convex and u.s.c., D is closed and convex, then ν in Proposition 6.10 has a finite support. Moreover (i) is equivalent to: for every set of $n + 1$ points $\{x_0, x_1, ..., x_n\}$ in C there exists $y \in D$ such that $f(x_i, y) < 0$, $i = 0, 1, ..., n$ (see [4, Theorems 3.1 and 4.1]).

We finish this section by applying Corollary 5.7. We have introduced the set $B_0 = \{(u, t) \, / \, \exists \mu \in \mathfrak{S}$ such that $\sup_{y \in D}(\langle u, y \rangle - K(\mu, y)) \leqslant t\}$. Setting $C = \mathbb{N}$ and $\mathfrak{S} = \{\mu \, / \, \mu \in \mathfrak{M}_+^K \ \|\mu\| = 1\}$ the set B_0 becomes

$$B_0 = \left\{ (u, t) \, / \, \exists \lambda_i \geqslant 0, 1 \leqslant i \leqslant m, \sum_{i=1}^{m} \lambda_i = 1, \left(\sum_{i=1}^{m} \lambda_i f_i + \psi_D \right)^* (u) \leqslant t \right\}.$$

THEOREM 6.12. *Assume that D is a closed convex subset of the normed TVS Y, that the functions* f_i *are l.s.c., convex and proper on D. Assume also that*

$$\inf_{y \in D} \sup_{\substack{\lambda_i \geqslant 0, m \\ \sum_{i=1}^{m} \lambda_i = 1}} \sum_{i=1}^{m} \lambda_i f_i(y) < +\infty,$$

that there exists $\bar{y} \in D$ *such that* $\forall n \in \mathbb{N} \ f_n$ *is continuous at* $\bar{y}$ *and that*

$$B_0 = \left\{ (u, t) \, / \, \forall m \, \forall \lambda_i \geqslant 0, 1 \leqslant i \leqslant m, \sum_{i=1}^{m} \lambda_i = 1 \text{ such that } \min_{\substack{u_1, u_2, \dots, u_{m+1} \\ u_1 + u_2 + \cdots + u_{m+1} = u}} \left(\sum_{i=1}^{n} \lambda_i f_i^*(\lambda_i^{-1} u_i) + \sup_{y \in D} \langle u_{m+1}, y \rangle \right) \leqslant t \right\} \text{ is weak*-closed.}$$

Then the following assertions are equivalent:

(i) $\forall \varepsilon > 0\ \exists y \in D$ *such that* $\forall n \in \mathbb{N}\ f_n(y) \leqslant \varepsilon$;

(ii) $\forall m\ \forall \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m \lambda_i = 1,\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\sum_{i=1}^m \lambda_i f_i(y) \leqslant \varepsilon$.

Proof. With the continuity assumption at $\bar{y}$, it is easy to check that B_0 has the given form (see [56, Theorem 20]). All the hypotheses of Corollary 5.7 are satisfied except those related to Lemma 4.7. Here it is not necessary to use Lemma 4.7 because $K(\mu, \cdot) = \sum_{i=1}^m \lambda_i f_i(\cdot)$ is l.s.c. Q.E.D.

VII. Application to Infinite Systems of Linear Inequalities

Of course a linear inequality being in particular a convex one, the results of the previous section apply.

Throughout this section we suppose that Y is a locally convex TVS whose the topological dual is U (see Section III). A linear inequality is of the type $\langle u_i, y\rangle \leqslant b_i$, where $u_i \in U$ and $b_i \in \mathbb{R}$.

Theorem 6.1 becomes:

THEOREM 7.1. *Assume that D is closed and convex. If there exist $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that $D \cap \{y \,/\, \forall n \leqslant n_0\ \langle u_n, y\rangle \leqslant b_n + \varepsilon\}$ is compact, then the following assertions are equivalent*:

(i) $\exists y \in D$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant b_n$;

(ii) $\forall m\ \forall \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\sum_{i=1}^m \lambda_i(\langle u_n, y\rangle - b_n) \leqslant \varepsilon$.

When Y is finite dimensional the compactness condition may be weakened to $D \cap \{y \,/\, \forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant b_n + \varepsilon\}$ is compact (Lemma 6.3).

The following result is a consequence of Corollary 6.2.

COROLLARY 7.2. *Assume that D is closed and convex and that the u_n are contained in a weak*-compact set of U. Assume also that there exists $b \in \mathbb{R}$ such that $\forall n \in \mathbb{N}\ |b_n| \leqslant b$. If there exists $\varepsilon > 0$ such that $D \cap \{y \,/\, \forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant b_n + \varepsilon\}$ is compact, then the following assertions are equivalent*:

(i) $\exists y \in D$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant b_n$;

(ii) $\forall \lambda_i \geqslant 0,\ \sum_{i=1}^\infty \lambda_i \leqslant 1,\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\sum_{i=1}^\infty \lambda_i(\langle u_i, y\rangle - b_i) \leqslant \varepsilon$.

We also have

(iii) $\forall \lambda_i \geqslant 0,\ \sum_{i=1}^\infty \lambda_i \leqslant 1,\ \exists y \in D$ *such that* $\sum_{i=1}^\infty \lambda_i(\langle u_i, y\rangle - b_i) < 0$ *implies that*

(iv) $\forall W \in \mathfrak{W}\ \exists y \in D + W$ *such that* $\forall n \in \mathbb{N}\ f_n(y) < 0$ ($\mathfrak{W}$ *is a basis of convex 0-neighborhoods*).

Proof. It results of Theorem 6.7, because the bounded sets in the dual are equicontinuous. Q.E.D.

We suppose that, as at the end of the previous section, $\mathfrak{S} = \{\mu \in \mathfrak{M}_+^K \,/\, \|\mu\| = 1\}$. Then the set B_0 is equal to $\{(u, t) \,/\, \exists m\ \exists \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m \lambda_i = 1$ such that $(\sum_{i=1}^m \lambda_i b_i + \sup_{y \in D} \langle u - \sum_{i=1}^m \lambda_i u_i, y\rangle) \leqslant t\}$. When D is a cone $B_0 = \{(u, t) \,/\, \exists m\ \exists \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m \lambda_i = 1$ such that $\sum_{i=1}^m \lambda_i u_i - u \in D^0$ and $\sum_{i=1}^m \lambda_i b_i \leqslant t\}$.

For every $\mu \in \mathfrak{S}$ we introduce the linear application $T(\mu) = (\sum_{i=1}^m \mu_i u_i, \sum_{i=1}^m \mu_i b_i)$, so that $B_0 = T(\mathfrak{S}) + (-D^0) \times \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the nonnegative real numbers.

THEOREM 7.3. *Assume that D is a closed convex cone of a normed TVS. If there exists $b > 0$ such that $\forall n \in \mathbb{N}\ |b_n| \leqslant b$ and if $T(\mathfrak{S}) + (-D^0) \times \mathbb{R}_+$ is closed, then the following assertions are equivalent*:

(i) $\forall \varepsilon > 0\ \exists y \in D$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant b_n + \varepsilon$;

(ii) $\forall m\ \forall \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m \lambda_i = 1,\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\sum_{i=1}^m \lambda_i(\langle u_i, y\rangle - b_i) \leqslant \varepsilon$.

Proof. In order to apply Theorem 6.12 it remains to verify that $-\underline{l} < +\infty$. It is true since for $y = 0$ one has $-\underline{l} \leqslant b$. Q.E.D.

Setting $F = \bigcup_{n \in \mathbb{N}} (u_n, b_n)$ we have $T(\mathfrak{S}) = \mathrm{co}(F)$, where co denotes the convex hull (see [7, Chap. II, Sect. 2, Proposition 8]).

COROLLARY 7.4. *Assume that D is a closed convex cone of a Banach space Y. If* co (F) *is weak*-compact (which is the case if F is weak*-compact), then the conclusion of Theorem 7.3 holds.*

Proof. One has $T(\mathfrak{S}) = \mathrm{co}\,(F)$, which is compact, thus $T(\mathfrak{S}) + (-D^0) \times \mathbb{R}_+$ is closed. From the compactness we also deduce that $\forall n \in \mathbb{N}\ |b_n| \leqslant b$. (The dual of a Banach space is weak* complete [7, Chap. IV, Sect. 2, Corollary 2, Proposition 1]; thus co (F) is weak*-compact when F is weak*-compact [57, Chap. II, 4.3, p. 50]). Q.E.D.

Using Theorem 2.14 we obtain another type of assumption on co (F).

THEOREM 7.5. *Assume that D is a closed convex set and that the interior of* co (F) *is nonempty (for a compatible topology). If moreover there* *exists* $(\bar{u}, \bar{b}) \in \operatorname{int} \operatorname{co}(F)$ *such that* $\inf_{y\in D}\langle \bar{u}, y\rangle > -\infty$, *then the following assertions are equivalent*:

(i) $\exists y \in D$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant b_n$;

(ii) $\forall m\ \forall \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m \lambda_i = 1,\ \forall \varepsilon > 0\ \exists y \in D$ *such that* $\sum_{i=1}^m \lambda_i(\langle u_i, y\rangle - b_i) \leqslant \varepsilon$.

Proof. It suffices to prove (ii) implies (i). Assume (ii); it entails that

$$\sup_{(u,s)\in \operatorname{co}(F)} \inf_{(y,t)\in D\times\{-1\}} (\langle u, y\rangle + st) \leqslant 0.$$

From 2.14

$$\min_{(y,t)\in D\times\{-1\}} \sup_{(u,s)\in \operatorname{co}(F)} (\langle u, y\rangle + st) \leqslant 0,$$

whence there exists $y_0 \in D$ satisfying (i). Q.E.D.

If D has a nonempty interior we get the following "symmetrical" result.

THEOREM 7.6. *Assume that* $\operatorname{co}(F \cup \{0\})$ *is closed and that D is convex, contains* 0, *and has a nonempty interior (for a compatible topology). If there exist* $a \in \mathbb{R}$, $t_0 > 1$ *and* $y_0 \in \operatorname{int} D$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, y_0\rangle - t_0 b_n \leqslant a$, *then the conclusion of Theorem* 7.3 *holds.*

Proof. We only have to prove that (ii) implies (i). Condition (ii) means that

$$\sup_{(u,b)\in \operatorname{co}(F)} \inf_{(y,t)\in D\times\mathbb{R}'} (\langle u, y\rangle + bt) \leqslant 0,$$

where $\mathbb{R}' =]-\infty, -1]$. Applying Theorem 2.14 to $-(\langle u, y\rangle + bt)$ we get

$$\inf_{(y,t)\in D\times\mathbb{R}'} \sup_{(u,b)\in \operatorname{co}(F)} (\langle u, y\rangle + bt) \leqslant 0$$

provided that this last expression is not $-\infty$. To ensure that, we may assume that $0 \in \operatorname{co}(F)$, which does not change the problem.

It follows that $\forall \varepsilon > 0\ \exists (\bar{y}, \bar{t}) \in D \times \mathbb{R}'$ such that $\forall n \in \mathbb{N}$ $\langle u_n, \bar{y}\rangle + \bar{t} b_n \leqslant \varepsilon$, which implies (i) after dividing by $-\bar{t}$, since $-\bar{t}^{-1}\bar{y} \in D$. Q.E.D.

Remark 7.7. If D is a cone with vertex 0, we can replace, in the above proof, $\mathbb{R}'$ by $]-\infty, \alpha]$ with $\alpha < 0$. Thus the interior point $(y_0, -t_0)$ of $D \times \mathbb{R}'$ used in the assumptions of Theorem 7.6 may be replaced by $(y_0, -\beta)$ with $\beta > -\alpha$.

We finish this section by studying the following program (Π).

(Π) minimize $\langle u_0, y\rangle$ subject to $y \in D$ and $\forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant b_n$.

PROPOSITION 7.8. *The set D being convex, assume that the value* γ *of* (Π) *is finite. Then*

$$\gamma = \sup_{\substack{\lambda_i \geqslant 0, m \\ 1\leqslant i\leqslant m}} \inf_{y\in D} \left(\langle u_0, y\rangle + \sum_{i=1}^m \lambda_i(\langle u_i, y\rangle - b_i)\right)$$

in the following cases:

(i) *D is closed and* $\exists n_0 \in \mathbb{N}\ \exists \varepsilon > 0$ *such that* $D \cap \{y \,/\, \forall n \leqslant n_0\ \langle u_n, y\rangle \leqslant b_n + \varepsilon\} \cap \{y \,/\, \langle u_0, y\rangle \leqslant \gamma\}$ *is compact.*

(ii) *D is closed,* $\operatorname{co}(F)$ *has a nonempty interior and* $\exists(\bar{u}, \bar{b}) \in \operatorname{int} \operatorname{co}(F)$ *such that* $-\infty < \inf_{y\in D}\langle \bar{u}, y\rangle$.

(iii) *D is a cone, Y is a Banach space,* $\operatorname{co}(F)$ *is weak*-compact, and* $\exists \varepsilon_0 > 0\ \exists \bar{y} \in D$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, \bar{y}\rangle - b_n \leqslant -\varepsilon_0$.

(iv) *D has a nonempty interior,* $0 \in D$, $\operatorname{co}(F \cup \{0\} \cup (u_0, \gamma))$ *is closed*; $\exists a \in \mathbb{R}\ \exists t_0 > 1$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, y_0\rangle - t_0 b_n \leqslant a$; $\exists \varepsilon_0 > 0\ \exists \bar{y} \in D$ *such that* $\forall n \in \mathbb{N}\ \langle u_n, \bar{y}\rangle - b_n \leqslant -\varepsilon_0$.

Proof. The first two assertions are proved as in Proposition 6.5. With the assumptions (iii) or (iv), we can observe that for every $\alpha \in \mathbb{R}$, $0 < \alpha \leqslant 2^{-1}\varepsilon_0$, the inequality system $\langle u_0, y\rangle \leqslant \gamma - \alpha$, $\langle u_i, y > \leqslant b_i - \alpha$ is cinconsistent. Thus there exist $\lambda_i \geqslant 0$, $1 \leqslant i \leqslant m$, $\sum_{i=1}^m \lambda_i = 1$, and $\bar{\varepsilon} > 0$ such that $\forall y \in D\ \sum_{i=1}^m \lambda_i(\langle u_i, y\rangle - b_i + \alpha) + \lambda_0(\langle u_0, y\rangle - \gamma + \alpha)\rangle > \bar{\varepsilon}$. (Theorems 7.3 and 7.6).

If $\lambda_0 = 0$ then it follows that $\forall y \in D\ \sum_{i=1}^m \lambda_i(\langle u_i, y\rangle - b_i)\rangle > \bar{\varepsilon} - \alpha \geqslant -2^{-1}\varepsilon_0$, which is absurd for $\bar{y}$. Thus $\lambda_0 \neq 0$ and after dividing by λ_0 we get our result. Q.E.D.

Remark 7.9. When F is weak*-compact it is clear that the following assertions are equivalent:

(i) $\exists \varepsilon_0 > 0\ \exists \bar{y} \in D$ *such that* $\forall i \in \mathbb{N}\ \langle u_i, \bar{y}\rangle - b_i \leqslant -\varepsilon_0$;

(ii) $\exists \bar{y} \in D$ such that $\forall i \in \mathbb{N}\ \langle u_i, \bar{y}\rangle - b_i < 0$.

Taking into account this observation, Proposition 7.8(iii) generalizes to Banach spaces the result of Duffin and Karlovitz [17, p. 126]. It is noteworthy that the Duffin–Karlovitz proof [17, p. 128] is nothing else than a direct proof of the closedness of B_0.

On the other hand the first assertion of the above proposition generalizes to infinite dimensional spaces Y the main result of Karney [23, Theorem 2.1]. When Y is finite dimensional, in order that (i) be satisfied it suffices that $\operatorname{rc}(D) \cap \{y \,/\, \forall n \in \mathbb{N}\ \langle u_n, y\rangle \leqslant 0\} \cap \{y/\langle u_0, y\rangle \leqslant 0\}$ be reduced to $\{0\}$ ($\operatorname{rc}(D)$ is the recession cone of D) (see Lemma 6.3 and [55, Theorem 8.4]). The value of (Π) is $-\infty$ if there exists

$y \in \mathrm{rc}(D) \cap \{y \,/\, \forall n \in N\ \langle u_n, y\rangle \leqslant 0\}$ such that $\langle u_0, y\rangle < 0$. Thus condition (i) actually reduces to $\mathrm{rc}(D) \cap \{y \,/\, \forall n \in \mathbb{N}\ \ \langle u_n, y\rangle \leqslant 0\} \cap \{y \,/\, \langle u_0, y\rangle = 0\} = \{0\}$. Hence Proposition 7.8(i) is equivalent to Theorem 2.4 of Karney [23].

At last, let us say that we have not envisaged in this section the well-known results relying on the properties of the (closed) conical hull of F; see [37, 12], and a recent survey by Borwein [6].

VIII. Mazur–Orlicz-Type Theorems

This section mainly relies on the result of Ky Fan (Corollary 4.5). Ky Fan himself has already drawn a large number of applications from this result [34–36]. Here we would like to focus our attention on the Mazur–Orlicz and moment theorems.

First let us show that the Mazur–Orlicz theorem [47, 59] can easily be deduced from Corollary 4.5.

Theorem 8.1 (Mazur and Orlicz [47]). *Let E be a vector space, T a nonempty set, p a sublinear functional on E, d a functional on T, and φ a mapping from T into E. Then the following statements are equivalent*:

(i) *There exists a linear functional v on E such that $\forall x \in E$ $v(x) \leqslant p(x)$ and $\forall t \in T$ $v(\varphi(t)) \leqslant d(t)$;*

(ii) $\forall m\ \ \forall \lambda_i \geqslant 0,\ \ 1 \leqslant i \leqslant m,\ \ \forall t_i \in T,\ \ 1 \leqslant i \leqslant m,\ \ \sum_{i=1}^m \lambda_k d(t_k) \leqslant p(\sum_{i=1}^m \lambda_k \varphi(t_k))$.

Proof. Let us set $f(t, v) = d(t) - v(\varphi(t))$. The vector space E is endowed with its finest locally convex topology [1, p. 168]; then v is continuous and p is also continuous [1, p. 169].

The condition $\forall x \in E\ v(x) - p(x) \leqslant 0$ means that $p^*(v) \leqslant 0$. Thus we can restate (i) as "$\exists v \in \{v \,/\, p^*(v) \leqslant 0\}$ such that $\forall t \in T\ f(t, v) \leqslant 0$." Since p is continuous then $\{v \,/\, p^*(v) \leqslant 0\}$ is weak*-compact [49]. By Corollary 4.5(i) is equivalent to:

(iii) $\forall m\ \forall \lambda_k \geqslant 0,\ 1 \leqslant k \leqslant m,\ \sum_{k=1}^m \lambda_k = 1,\ \exists v$ satisfying $p^*(v) \leqslant 0$ and $\sum_{k=1}^m \lambda_k d(t_k) \leqslant v(\sum_{k=1}^m \lambda_k \varphi(t_k))$.

It is clear that (iii) implies (ii). Now p being positively homogenous one has $p(x) = \max p^*(v) \leqslant 0\ (\langle x, v\rangle)$ [43, Theorem 6.8.7]. Consequently $\sum_{k=1}^m \lambda_k d(t_k) \leqslant p(\sum_{k=1}^m \lambda_k \varphi(t_k))$ implies that there exist v satisfying $v(\sum_{k=1}^m \lambda_k \varphi(t_k)) = p(\sum_{k=1}^m \lambda_k \varphi(t_k))$ and $p^*(v) \leqslant 0$ showing that (ii) implies (iii). Q.E.D.

Along the same line it is possible to obtain numerous "moment" theorems.

Theorem 8.2 (Bittner [3]). *The following assertions are equivalent*:

(i) *There exists a linear functional satisfying $\forall x \in E$ $f(x) \leqslant p(x)$ and $f(x_i) \geqslant \alpha_i$ $(i \in I)$, where p is a convex functional and I an arbitrary set.*

(ii) $\forall m\ \ \forall \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m \lambda_i \alpha_i \leqslant P(\sum_{i=1}^m \lambda_i x_i)$, *where P is the sublinear hull of p.*

Proof. As previously we endow E with its finest locally convex topology, so that p is continuous. Then (i) is equivalent to:

(iii) $\forall m\ \forall \lambda_i \geqslant 0,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^m \lambda_i = 1,\ \exists f$ satisfying $p^*(f) \leqslant 0$ and $\sum_{i=1}^m \lambda_i \alpha_i \leqslant f(\sum_{i=1}^m \lambda_i x_i)$. Now (iii) is equivalent to (ii) if P is the sublinear hull of p which is defined by $P(x) = \inf_{\mu > 0} \mu^{-1} p(\mu x)$ (see [43, Theorems 6.8.7 and 6.8.9]). Q.E.D.

The following "moment" theorem is also closely related to the Mazur–Orlicz theorem. See also [10, 58, 41] for some other results in the same spirit.

Theorem 8.3 (Landsberg and Schirotzek [41]). *Let E and F be two vector spaces paired by the bilinear form $\langle \cdot, \cdot \rangle$. Let P be a cone in E and let T be a nonvoid set. Further let φ be a map from T into E and f a functional on T. Assume that W is a convex weak*-compact subset of F containing 0. Then the following statements are equivalent*:

(i) $\exists v \in W$ *such that* $\forall x \in P\ \langle x, v\rangle \geqslant 0$ *and* $\forall t \in T\ \langle \varphi(t), v\rangle = f(t)$;

(ii) $\forall m\ \ \forall \lambda_k,\ \ 1 \leqslant k \leqslant m,$ *such that* $\sum_{k=1}^m \lambda_k \varphi(t_k) \in W^0 + P$ *then* $\sum_{k=1}^m \lambda_k f(t_k) + 1 \geqslant 0$.

Proof. The vector space E is topologized by the MacKey topology $\mathfrak{T}(E, F)$ while F is endowed with the weak* topology $\sigma(E, F)$. Then W being compact, Corollary 4.5 implies that (i) is equivalent to:

(iii) $\forall m\ \forall \lambda_k,\ 1 \leqslant k \leqslant m,\ \exists v \in W \cap P^0$ such that

$$\sum_{k=1}^m \lambda_k \langle \varphi(t_k), v\rangle - \sum_{k=1}^m \lambda_k f(t_k) \leqslant 0.$$

It is clear that (iii) implies (ii) because if $\sum_{k=1}^m \lambda_k \varphi(t_k) \in P + W^0 \subset (W \cap P^0)^0$ one has $\sum_{k=1}^m \lambda_k \langle \varphi(t_k), v\rangle \geqslant -1$.

Let us show that (ii) implies (iii). We have for any $z \in E$ $\min_{v \in W \cap P^0} \langle v, z\rangle \leqslant 0 \leqslant \max_{v \in W \cap P^0} \langle v, z\rangle$. If $\min_{v \in W \cap P^0} \langle v, z\rangle < 0$, then there exist $\alpha > 0$ and $v_0 \in W \cap P^0$ such that $\min_{v \in W \cap P^0} \langle v, \alpha z\rangle = -1 = \langle v_0, \alpha z\rangle$. For every λ_k let us set $z = \sum_{k=1}^m \lambda_k \varphi(t_k)$; we have $\alpha z \in (W \cap P^0)^0 = \overline{W^0 + P}$ and $\langle v_0, \alpha z\rangle = -1$. Now $W^0 + P$ has a nonempty interior since W^0 is a 0-neighborhood and for every ε sufficiently small $(\alpha - \varepsilon) z \in W^0 + P$ implying by (ii) that $(\alpha-\varepsilon)\sum_{k=1}^{m}\lambda_k f(t_k)\geqslant -1$. Thus $\alpha\sum_{k=1}^{m}\lambda_k f(t_k)\geqslant -1=\alpha\sum_{k=1}^{m}\lambda_k\langle\varphi(t_k),v_0\rangle$ entailing (iii).

Assume now that $\min_{v\in W\cap P^0}\langle v,z\rangle=0$, then for every real number β $\beta z\in\overline{W^0+P}$. It follows from (ii) that $\sum_{k=1}^{m}\lambda_k f(t_k)\geqslant 0$. Thus there exists $v_0\in W\cap P^0$ such that $\sum_{k=1}^{m}\lambda_k\langle\varphi(t_k),v_0\rangle=0$ and (iii) is satisfied. Q.E.D.

Remark 8.4. In the above theorem, it is not necessary that P be a cone. When P is any subset of E, it suffices to replace in (ii) "$\sum_{k=1}^{m}\lambda_k\varphi(t_k)\in W^0+P$" by "$\sum_{k=1}^{m}\lambda_k\varphi(t_k)\in \operatorname{co}(W^0\cup P)$," where co denotes the convex hull.

In the preceding theorem, when we do not have a compact subset W it is possible to obtain a similar result by taking a cone P with a nonempty interior so that P^0 has a compact basis. For instance let us prove a result of Ky Fan [34, Theorem 17].

THEOREM 8.5 (Ky Fan [34]). *Let E be a locally convex* TVS *and F its topological dual. Let P be a convex cone with* $\operatorname{int}(P)\neq\varnothing$, *$x_i$ $(i\in I)$ be a family of vectors of E and α_i $(i\in I)$ be a family of real numbers. Assume that there exists $x_{i_0}\in\operatorname{int}(P)$ with $\alpha_{i_0}>0$. Then the following assertions are equivalent*:

(i) $\exists v\in F$ *such that* $\forall x\in P$ $\langle v,x\rangle\geqslant 0$ *and* $\forall i\in I$ $\langle v,x_i\rangle=\alpha_i$

(ii) $\forall m$ $\forall\lambda_k$, $1\leqslant k\leqslant m$, *such that* $\sum_{k=1}^{m}\lambda_k\alpha_k=0$ *the linear combination* $\sum_{k=1}^{m}\lambda_k x_k$ *is not in* $\operatorname{int}(P)$.

Proof. Let us set $H_{i_0}=\{v\,/\,\langle v,x_{i_0}\rangle\leqslant\alpha_{i_0}\}$. Since $0\in\operatorname{int}(P-x_{i_0})$ the set $H=P^0\cap H_{i_0}$ is weak*-compact. In (i), $v\in H$; therefore (i) is equivalent to (see the previous proof):

(iii) $\forall m\,\forall\lambda_i$, $1\leqslant i\leqslant m$, $\exists v\in H$ such that $\sum_{i=1}^{m}\lambda_i\langle x_i,v\rangle\leqslant\sum_{i=1}^{m}\lambda_i\alpha_i$.

It is clear that (i) implies (ii) since $v\neq 0$ and $\langle v,x\rangle>0$ for $x\in\operatorname{int}P$ [7, Chap. II, Sect. 2, Corollary 1, Proposition 17].

To prove the equivalence of the three assertions it remains to show that (ii) implies (iii). Assume that (ii) holds; then (iii) is satisfied with $v=0$ if $\sum_{i=1}^{m}\lambda_i\alpha_i\geqslant 0$. Thus we consider $a=\sum_{i=1}^{m}\lambda_i\alpha_i<0$ and we set $(1-\bar\lambda)a+\bar\lambda\alpha_{i_0}=0$ $(\bar\lambda=a(a-\alpha_{i_0})^{-1})$. Applying (ii) it follows that $\bar x=(1-\bar\lambda)\sum_{i=1}^{m}\lambda_i x_i+\bar\lambda x_{i_0}\notin\operatorname{int}(P)$. Thus we can separate $\bar x$ and $\operatorname{int}(P)$, which implies that there exists v_0 satisfying $\langle v_0,\bar x\rangle\leqslant 0$ and $\forall x\in P$ $\langle v_0,x\rangle\geqslant 0$. Setting $\bar v=\alpha_{i_0}v_0(\langle v_0,x_{i_0}\rangle)^{-1}$ we check that (iii) is fulfilled. Q.E.D.

Let us finish by showing how to obtain a result in the same spirit as the previous results, using Theorem 7.5.

For any set C we denote by C^c the conical hull of C (i.e., $C^c=\bigcup_{\lambda\geqslant 0}\lambda C$) and by C^i the open conical hull of the interior of C (*i.e.*, $C^i=\bigcup_{\lambda>0}\lambda\operatorname{int}C$).

THEOREM 8.6. *Let E be a locally convex* TVS, *and F its topological dual, and let C be a convex subset of E. Consider a family x_i $(i\in I)$ of vectors in E and $\alpha_i\in\mathbb{R}$ $(i\in I)$. Assume that there exists $i_0\in I$ such that $x_{i_0}\in\operatorname{int}C$ and $\alpha_{i_0}<0$. Then the following assertions are equivalent*:

(i) $\exists v\in F$ *such that* $\forall x\in C$ $\langle x,v\rangle\geqslant 0$ *and* $\forall i\in I$ $\langle x_i,v\rangle\leqslant\alpha_i$;

(ii) $\forall m\in\mathbb{N}$ $\forall\lambda_i\geqslant 0$, $1\leqslant i\leqslant m$, *such that* $\sum_{i=1}^{m}\lambda_i\alpha_i\leqslant 0$ *then* $(\sum_{i=1}^{m}\lambda_i x_i-C^c)\cap C^i=\varnothing$;

(iii) $\forall m\in\mathbb{N}$ $\forall\lambda_i\geqslant 0$, $1\leqslant i\leqslant m$, $\forall z\in C^c$ $\exists v\in F$ *such that* $\forall x\in C$ $\langle x,v\rangle\geqslant 0$ *and* $\sum_{i=1}^{m}\lambda_i(\langle x_i,v\rangle)-\langle z,v\rangle\leqslant\sum_{i=1}^{m}\lambda_i\alpha_i$.

Proof. Let us introduce the set $C^{\cdot}=\{v\mid\forall x\in C\ \langle x,v\rangle\geqslant 0\}$. We can reexpress (i) as $\exists v\in C^{\cdot}$ such that $\langle -x,v\rangle\leqslant 0$ and $\langle x_i,v\rangle\leqslant\alpha_i$. Setting $F=(-C\times\{0\})\cup(\bigcup_{i\in I}(x_i,\alpha_i))$ and assuming that $-x_{i_0}+W$ is contained in $\operatorname{int}(-C)$ (W being a 0-neighborhood), one can verify that $(-x_{i_0}+2^{-1}W)\times(]2^{-1}\alpha_{i_0},0[)$ is contained in $\operatorname{co}(F)$. Then Theorem 7.5 applies. It follows that (i) is equivalent to (see remark 6.4):

(iii′) $\forall m\in\mathbb{N}$ $\forall\lambda_i\geqslant 0$, $1\leqslant i\leqslant m$, $\sum_{i=1}^{m}\lambda_i=1$, $\exists v\in C^{\cdot}$ such that

$$\sum_{\substack{i=1\\ i\in I}}^{p}\lambda_i\langle x_i,v\rangle-\sum_{k=p+1}^{m}\lambda_k\langle x^k,v\rangle\leqslant\sum_{\substack{i=1\\ i\in I}}^{p}\lambda_i\alpha_i \qquad (\text{where } x^k\in C).$$

It is easy to see that (iii)′ is equivalent to (iii).

Now (i) implies (ii) because, as in Theorem 8.5, $v\neq 0$ and $\forall x\in C$ $\langle x,v\rangle\geqslant 0$ implies $\langle x,v\rangle>0$ for $x\in\operatorname{int}C$ [7, Chap. II, Sect. 2, Proposition 17].

As above it remains to prove that (ii) implies (iii). Assume (ii). If $\sum_{i=1}^{m}\lambda_i\alpha_i\geqslant 0$ then $v=0$ satisfies (iii). If $\sum_{i=1}^{m}\lambda_i\alpha_i=a<0$ then $x_{i_0}+\sum_{i=1}^{m}\lambda_i x_i-C^c$ does not meet C^i. Separating C^i and $x_{i_0}+\sum_{i=1}^{m}\lambda_i x_i-z$ we find v_0 such that $\forall x\in C^i$ $\langle v_0,x\rangle>0$ and $\langle v_0,x_{i_0}+\sum_{i=1}^{m}\lambda_i x_i-z\rangle\leqslant 0$. But one has $\langle x_{i_0},v_0\rangle>0$; we therefore get $\langle v_0,\sum_{i=1}^{m}\lambda_i x_i-z\rangle<0$, and μv_0 for μ sufficiently large satisfies (iii). Q.E.D.

Note added in proof. Since this paper was submitted, I have become aware of the following recent works which are more or less concerned with some parts of my survey, and anyway bring some related additional references.

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