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Direct Sums of Cartan Factors.

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In [Pe2] we envisaged the orbit of the origin in the unit ball of a direct sum of two complex Banach spaces (endowed with a suitable norm), with respect to the group of holomorphic automorphisms, and we obtained some general results. As a special case we considered the class of p-norms, and we proved that the most interesting case is when p equals 2. For $p = 2$ we succeeded in giving some information about the orbit of the origin when one of the spaces is either a Hilbert space or a commutative C^*-algebra with identity. In this paper we consider the case when one of the spaces is a Cartan factor. The reason for considering Cartan factors is that, as we proved in [Pe2], only spaces in which the orbit of the origin in the unit ball is non-trivial can give rise to a direct sum in which such an orbit is non-trivial: and the unit ball of a Cartan factor is homogeneous.

Our main result can be expressed in the following way: if F is a Cartan factor of type I, II, III or IV and F is not isometric to a Hilbert space, then, given a non-trivial complex Banach space G, no point in the orbit of the origin in the unit ball of the 2-sum of G and F can have non-zero F-coordinate.

In the last section we shall prove some results concerning duality theory for Cartan factors.

1. Preliminaries and notations.

First of all we recall the definition of Cartan factors (see e.g. [Ha]).

If H and K are complex Hilbert spaces, we shall denote by $\mathcal{L}(H, K)$ the Banach space of continuous linear operators from H to K, endowed with the usual «sup» norm; $\mathcal{L}(H, K)$ will be called a *Cartan factor of type* I.

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An anti-linear involutive isometry τ of H will be called a conjugation on H (remark that such a τ enjoys $(\tau x|\tau y)=(y|x)\ \forall x, y \in H$); given τ we define the associated transposition on $\mathcal{L}(H)$ by ${}^tA = \tau A^*\tau$.

LEMMA 1. Given a coniugation τ on H there exists an orthonormal basis $\{\phi_\alpha\}$ of H such that $(\tau\phi|\phi_\alpha)=\overline{(\phi|\phi_\alpha)}\ \forall\phi \in H,\ \forall\alpha$.
With respect to this basis $({}^tA\phi_\alpha|\phi_\beta)=(A\phi_\beta|\phi_\alpha)\ \forall A \in \mathcal{L}(H),\ \forall\alpha,\beta$.

PROOF. For the first assertion it suffices to show that $\exists\{\phi_\alpha\}$ such that $\tau\phi_\alpha=\phi_\alpha\ \forall\alpha$. Given $\phi \in H\setminus\{0\}$ set

$$\phi_1=\begin{cases} i\phi & \text{if } \tau\phi+\phi=0,\\ \tau\phi+\phi & \text{otherwise;}\end{cases}$$

we have $\tau\phi_1=\phi_1$, $\phi_1\neq 0$; since τ preserves orthogonality the conclusion follows at once by a maximality argument.
The second assertion is a direct consequence of the first one. ■

Given τ, the space $\{A \in \mathcal{L}(H): {}^tA=A\}$ will be called a *Cartan factor of type* II, and the space $\{A \in \mathcal{L}(H): {}^tA+A=0\}$ will be called a *Cartan factor of type* III; it is easily checked that they are closed subspaces of $\mathcal{L}(H)$, and hence they are naturally endowed with a Banach space structure.
According to Lemma 1, a Cartan factor of type II (resp. III) is the space $\mathcal{L}^{(s)}_{\{\phi_\alpha\}}(H)$ (resp. $\mathcal{L}^{(a)}_{\{\phi_\alpha\}}(H)$) of symmetric (resp. skew-symmetric) operators with respect to some fixed orthonormal basis $\{\phi_\alpha\}$ of H. Since different choices of the basis give rise to isomorphic Banach spaces, the subscript will be omitted.
A closed subspace $\mathcal{U}$ of $\mathcal{L}(H)$ will be called a *Cartan factor of type* IV if for any A in $\mathcal{U}$ the square of A is a scalar multiple of the identity operator and A^* belongs to $\mathcal{U}$. As well-known (see [Ha]), a Cartan factor of type IV is linearly and topologically isomorphic to a Hilbert space K, and there exists a conjugation τ on K such that the norm of $\psi \in K$ as a point of the Cartan factor is given by

$$\|\psi\|^2_{\mathcal{U}}=(\psi|\psi)+((\psi|\psi)^2-|(\psi|\tau\psi)|^2)^{1/2}.$$

As above, let H and K be complex Hilbert spaces. We shall denote by $\mathcal{L}_0(H,K)$ the closed subspace of $\mathcal{L}(H,K)$ consisting of compact operators; for $\phi \in H$ and $\psi \in K$, an element $\psi\otimes\bar{\phi}$ of $\mathcal{L}_0(H,K)$ is defined by

$$(\psi\otimes\bar{\phi})(\phi_1)=(\phi_1|\phi)\cdot\psi, \qquad \phi_1 \in H.$$

In [Pe1] (extending Schatten's works [Sc1] and [Sc2] from the case of operators on one Hilbert space to the case of operators between two possibly different Hilbert spaces) we defined a subspaces $\mathcal{L}_1(H,K)$ of $\mathcal{L}_0(H,K)$ as the set of those operators A such that $\|A\|_1\equiv \operatorname{tr}((A^*A)^{1/2})$ is finite, and we proved that $\mathcal{L}_1(H,K)$ is a Banach space with respect to the norm $\|\cdot\|_1$. Moreover, we checked that for $A \in \mathcal{L}(H,K)$ and $B \in \mathcal{L}_1(K,H)$ the trace $\operatorname{tr}(AB)$ of AB can be defined, and we established the following.

THEOREM 1. The following isometrical isomorphisms hold:

$$\mathcal{L}_0(H,K)^*\cong\mathcal{L}_1(K,H) \qquad \mathcal{L}_1(H,K)^*\cong\mathcal{L}(K,H)$$

the value of A on B being defined in any case by $\operatorname{tr}(AB)$.

If F and G are complex Banach spaces, and $1\leq p\leq\infty$, we shall denote by $F\oplus_p G$ the direct sum of F and G endowed with the so-called p-norm

$$\|(f,g)\|_p=\begin{cases}(\|f\|^p+\|g\|^p)^{1/p} & \text{if } 1\leq p<\infty,\\ \max\{\|f\|,\|g\|\} & \text{if } p=\infty.\end{cases}$$

2. A few technical lemmas.

According to a theorem proved by Stachó in [St], for any complex Banach space F, the orbit of the origin with respect to the group of holomorphic automorphisms of the open unit ball B of F is given by $B\cap F_0$, where F_0 is a closed linear subspace of F. Moreover the elements of F_0 can be characterized as those points c of F for which there exists a continuous homogeneous polynomial $Q_c: F\to F$ of degree 2 such that

$$\phi(Q_c(a))=\|a\|^2\cdot\overline{\phi(c)}$$

whenever $a \in F$, $\phi \in F^*$ and $\phi(a)=\|a\|\cdot\|\phi\|$; Q_c is uniquely determined by this condition, and it will be referred to as «the polynomial relative to c».
As in [Pe2], all our results will deal with the subspace F_0 and not with the orbit of the origin itself.

LEMMA 2. Let E and F be non-trivial complex Banach spaces and assume there exist a linear isometry $i: E\to F$ and a surjective linear projection $p: F\to i(E)$ such that $\|p\|=1$. Then $p(F_0)\subseteq i(E_0)$.

PROOF. Let $c \in F_0$ and let $Q_c\colon F \to F$ be the polynomial relative to c. Set $c_1 = (i^{-1} \circ p)(c)$ and $Q_1 = i^{-1} \circ p \circ Q_c \circ i\colon E \to E$. We will prove that

$$\phi(Q_1(a)) = \|a\|^2 \cdot \overline{\phi(c_1)}$$

whenever $a \in E$, $\phi \in E^*$ and $\phi(a) = \|a\| \cdot \|\phi\|$, which implies $c_1 \in E_0$ and then the conclusion.

Let us define a linear mapping $j\colon E^* \to F^*$ by the formula

$$j(\phi)(b) = \phi((i^{-1} \circ p)(b)) \quad \forall \phi \in E^*, \ b \in F.$$

It follows from $\|p\| = 1$ that j is an isometry. Now, let $a \in E$, $\phi \in E^*$ be such that $\phi(a) = \|a\| \cdot \|\phi\|$; using the definition of j and the fact that both i and j are isometries, we obtain

$$j(\phi)(i(a)) = \|i(a)\| \cdot \|j(\phi)\|$$

and hence, by the definition of Q_c,

$$j(\phi)(Q_c(i(a))) = \|i(a)\|^2 \cdot \overline{j(\phi)(c)} \Rightarrow$$

$$\Rightarrow \phi((i^{-1} \circ p \circ Q_c \circ i)(a)) = \|a\|^2 \cdot \overline{\phi((i^{-1} \circ p)(c))}$$

i.e.

$$\phi(Q_1(a)) = \|a\|^2 \cdot \overline{\phi(c_1)}$$

and the lemma is proved. ■

We recall a result we proved in [Pe2], which is the basis for all our further investigations.

LEMMA 3. If E is the Banach space $\mathbb{C} \oplus_2 (\mathbb{C} \oplus_\infty \mathbb{C})$, then $E_0 = \mathbb{C} \times \{0\} \times \{0\}$.

Combining Lemmas 2 and 3 we obtain the following.

LEMMA 4. Let L be a non-trivial complex Banach space and assume that there exist a linear isometry $i\colon \mathbb{C} \oplus_\infty \mathbb{C} \to L$ and a linear projection p of L onto $i(\mathbb{C} \oplus_\infty \mathbb{C})$ such that $\|p\| = 1$.

For any non-trivial complex Banach space G, if $(g_0, l_0) \in (G \oplus_2 L)_0$, then $p(l_0) = 0$.

PROOF. Choose $g_1 \in G$, $g_1^* \in G^*$ such that $g_1^*(g_1) = \|g_1\| = \|g_1^*\| = 1$.

Set $F = G \oplus_2 L$ and $E = \mathbb{C} \oplus_2 (\mathbb{C} \oplus_\infty \mathbb{C})$ and define

$$i_1\colon E \to F \qquad p_1\colon F \to i_1(E)$$

by

$$i_1(z_1, z_2, z_3) = (z_1 \cdot g_1, i(z_2, z_3)), \qquad p_1(g, l) = (g_1^*(g) \cdot g_1, p(l)).$$

i_1 and p_1 fulfill the hypothesis of Lemma 2, and hence

$$(i_1^{-1} \circ p_1)(g_0, l_0) \in E_0.$$

It follows from Lemma 3 that

$$(g_1^*(g_0), (i^{-1} \circ p)(l_0)) \in \mathbb{C} \times \{0\} \times \{0\}$$

and hence $(i^{-1} \circ p)(l_0) = 0$, *i.e.* $p(l_0) = 0$. ■

COROLLARY 1. If in Lemma 4 we assume that, for any $l \in L$, i and p can be chosen in such a way that $l \in i(\mathbb{C} \oplus_\infty \mathbb{C})$, then for any non-trivial complex Banach space G we have

$$(G \oplus_2 L)_0 \subseteq G \times \{0\}.$$

PROOF. If $(g_0, l_0) \in (G \oplus_2 L)_0$ and $l_0 \in i(\mathbb{C} \oplus_\infty \mathbb{C})$ then $l_0 = p(l_0) = 0$. ■

LEMMA 5. In order to prove that $(G \oplus_2 L)_0 \subseteq G_0 \times \{0\}$ it suffices to show that $(G \oplus_2 L)_0 \subseteq G \times \{0\}$.

PROOF. Assume $(G \oplus_2 L)_0 \subseteq G \times \{0\}$ and let $(g_0, 0) \in (G \oplus_2 L)_0$; since the projection

$$p\colon G \oplus_2 L \to G \qquad (g, l) \mapsto g$$

has norm 1 then Lemma 2 implies that

$$g_0 = p(g_0, 0) \in G_0$$

and then $(g_0, 0) \in G_0 \times \{0\}$ so that $(G \oplus_2 L)_0 \subseteq G_0 \times \{0\}$. ■

3. Direct sums.

Let H and K be complex Hilbert spaces. If either H or K is 1-dimensional $\mathcal{L}(H, K)$ is Hilbertian, and hence, by Theorem 4 of [Pe2], when a direct sum with 2-norm is performed, homogeneity along $\mathcal{L}(H, K)$ is preserved, *i.e.* for any complex Banach space G

$$(G \oplus_2 \mathcal{L}(H, K))_0 \supseteq \{0\} \times \mathcal{L}(H, K);$$

if this is not the case, the situation is radically modified.

THEOREM 2. If H and K are at least 2-dimensional then for any non-trivial complex Banach space G we have

$$(G \oplus_2 \mathcal{L}(H, K))_0 \subseteq G_0 \times \{0\}.$$

PROOF. Let $F = G \oplus_2 \mathcal{L}(H, K)$, and suppose $(g_0, A) \in F_0$; according to Lemma 5 it is enough to show that $A = 0$. In order to check this, it suffices to prove that for every arbitrary pair of unit vectors $\phi_1 \in H$ and $\psi_1 \in K$, we have $(A\phi_1 | \psi_1) = 0$.

Fix such ϕ_1 and ψ_1 and find $\phi_2 \in H$ and $\psi_2 \in K$ in such a way that $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ are orthonormal pairs.

Now, we define the mappings:

$$i\colon\ \mathbb{C} \oplus_\infty \mathbb{C} \to \mathcal{L}(H, K)$$

$$(x, y) \mapsto x \cdot \psi_1 \otimes \overline{\phi_1} + y \cdot \psi_2 \otimes \overline{\phi_2},$$

$$p\colon\ \mathcal{L}(H, K) \to i(\mathbb{C} \oplus_\infty \mathbb{C})$$

$$B \mapsto (B\phi_1 | \psi_1) \cdot \psi_1 \otimes \overline{\phi_1} + (B\phi_2 | \psi_2) \cdot \psi_2 \otimes \overline{\phi_2}.$$

It is easily checked that i and p fulfill the hypothesis of Lemma 4, and hence $p(A) = 0$; then $(A\phi_1 | \psi_1) = 0$ and the theorem is proved. ∎

COROLLARY 2. If H and K are at least 2-dimensional then for any non-trivial complex Banach space G we have

$$(G \oplus_2 \mathcal{L}_0(H, K))_0 \subseteq G_0 \times \{0\}.$$

PROOF. Since the only property used for $\mathcal{L}(H,K)$ in the above proof is the fact that it contains finite-rank operators, the very same method works for $\mathcal{L}_0(H, K)$. ∎

Let H be a complex Hilbert space. If H has dimension 1, $\mathcal{L}^{(s)}(H)$ has dimension 1 too, and hence it is a Hilbert space. It is easily verified that if $\dim(H) \geq 2$ then $\mathcal{L}^{(s)}(H)$ is not a Hilbert space. Our aim is to prove an analogue of Theorem 2 with $\mathcal{L}^{(s)}(H)$ replacing $\mathcal{L}(H, K)$.

Since for $\dim(H) = 2$ $\mathcal{L}^{(s)}(H)$ is isometrically isomorphic to a three-dimensional type IV Cartan factor (see [Ca-Ve]) we shall not consider this case now. Anyway it will follow from Theorem 5 that the next theorem holds for $\dim(H) = 2$ too.

THEOREM 3. If H is at least 3-dimensional then for any non-trivial complex Banach space G we have

$$(G \oplus_2 \mathcal{L}^{(s)}(H))_0 \subseteq G_0 \times \{0\}.$$

PROOF. As in Theorem 2, we set $F = G \oplus_2 \mathcal{L}^{(s)}(H)$, we suppose $(g_0, A) \in F_0$ and we use Lemma 4 to prove that $A = 0$.

Let $\{\phi_\alpha\}$ be the basis with respect to which symmetry is considered. If, by absurd, $A \neq 0$, we can find two indices α_1, α_2 such that $(A\phi_{\alpha_1} | \phi_{\alpha_2}) \neq 0$. We distinguish the cases $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$.

For $\alpha_1 = \alpha_2$ we choose α_3 different from α_1 and we set

$$i\colon\ \mathbb{C} \oplus_\infty \mathbb{C} \to \mathcal{L}^{(s)}(H)$$

$$(x, y) \mapsto x \cdot \phi_{\alpha_1} \otimes \overline{\phi_{\alpha_1}} + y \cdot \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}},$$

$$p\colon\ \mathcal{L}^{(s)}(H) \to i(\mathbb{C} \oplus_\infty \mathbb{C})$$

$$B \mapsto (B\phi_{\alpha_1} | \phi_{\alpha_1})\, \overline{\phi_{\alpha_1}} \otimes \overline{\phi_{\alpha_1}} + (B\phi_{\alpha_3} | \phi_{\alpha_3})\, \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}}.$$

Lemma 4 applies; we obtain $p(A) = 0 \Rightarrow (A\phi_{\alpha_1} | \phi_{\alpha_1}) = 0$.

For $\alpha_1 \neq \alpha_2$ we choose α_3 different from both of then (we recall that H is at least 3-dimensional) and we set

$$i\colon\ \mathbb{C} \oplus_\infty \mathbb{C} \to \mathcal{L}^{(s)}(H)$$

$$(x, y) \mapsto x \cdot (\phi_{\alpha_1} \otimes \overline{\phi_{\alpha_2}} + \phi_{\alpha_2} \otimes \overline{\phi_{\alpha_1}}) + y \cdot \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}},$$

$$p\colon\ \mathcal{L}^{(s)}(H) \to i(\mathbb{C} \oplus_\infty \mathbb{C})$$

$$B \mapsto (B\phi_{\alpha_1} | \phi_{\alpha_2})(\phi_{\alpha_1} \otimes \overline{\phi_{\alpha_2}} + \phi_{\alpha_2} \otimes \overline{\phi_{\alpha_1}}) + (B\phi_{\alpha_3} | \phi_{\alpha_3})\, \phi_{\alpha_3} \otimes \overline{\phi_{\alpha_3}}$$

and as above we obtain $(B\phi_{\alpha_1} | \phi_{\alpha_2}) = 0$.

In both cases we got a contradiction and hence the theorem is proved. ∎

Let H be a complex Hilbert space. If $\dim(H) = 1$ then $\dim(\mathcal{L}^{(a)}(H)) = 0$, and if $\dim(H) = 2$ then $\dim(\mathcal{L}^{(a)}(H)) = 1$; in both cases $\mathcal{L}^{(a)}(H)$ is a Hilbert space; since for $\dim(H) = 2$ $\mathcal{L}^{(a)}(H) \cong \cong \mathcal{L}(\mathbb{C}, \mathbb{C}^3) \cong \mathbb{C}^3$ (see [Ca-Ve]), $\mathcal{L}^{(a)}(H)$ is a Hilbert space in this case too. It follows that an analogue of Theorems 2 and 3 can hold for $\mathcal{L}^{(a)}(H)$ only if $\dim(H)$ is bigger than 3.

THEOREM 4. If H is at least 4-dimensional then for any non-trivial complex Banach space G we have

$$(G \oplus_2 \mathcal{L}^{(a)}(H))_0 \subseteq G_0 \times \{0\}.$$

PROOF. Let $\{\phi_\alpha\}$ be the orthonormal basis of H with respect to which skew-symmetry in $\mathcal{L}(H)$ is defined, set $F = G \oplus_2 \mathcal{L}^{(a)}(H)$ and suppose $(g_0, A) \in F_0$. As we remarked above, it is enough to show that $A = 0$.

In order to prove this it suffices to check that for an arbitrary fixed pair of indices $\alpha_1 \neq \alpha_2$, it happens that $(B\phi_{\alpha_1} | \phi_{\alpha_2}) = 0$.

Since H is at least 4-dimensional, we can find α_3 and α_4 in such a way that $\alpha_1, \ldots, \alpha_4$ are different from each other. For the sake of simplicity, we set $\phi_i = \phi_{\alpha_i}$.

As above, we define two linear mappings

i: $\mathbb{C} \oplus_\infty \mathbb{C} \to \mathcal{L}^{(a)}(H)$

$$(x, y) \mapsto x(\phi_2 \otimes \overline{\phi_1} - \phi_1 \otimes \overline{\phi_2}) + y(\phi_4 \otimes \overline{\phi_3} - \phi_3 \otimes \overline{\phi_4}),$$

p: $\mathcal{L}^{(a)}(H) \to i(\mathbb{C} \oplus_\infty \mathbb{C})$

$$B \mapsto (B\phi_1 | \phi_2)(\phi_2 \otimes \overline{\phi_1} - \phi_1 \otimes \overline{\phi_2}) + (B\phi_3 | \phi_4)(\phi_4 \otimes \overline{\phi_3} - \phi_3 \otimes \overline{\phi_4}).$$

It is readily verified that the hypothesis of Lemma 4 are fulfilled. It follows that $p(A) = 0$, hence $(B\phi_{\alpha_1} | \phi_{\alpha_2}) = 0$ and the proof is complete. ∎

THEOREM 5. Let $\mathcal{U}$ be a Cartan factor of type IV, and assume that $\mathcal{U}$ is at least 2-dimensional. Then for any non-trivial complex Banach space G we have

$$(G \oplus_2 \mathcal{U})_0 \subseteq G_0 \times \{0\}.$$

PROOF. Since if $\dim(\mathcal{U}) = 2$ then $\mathcal{U} \cong \mathbb{C} \oplus_\infty \mathbb{C}$ (see [Ca-Ve]), the theorem is certainly true in this case (once again Lemma 4 is used).

Using Corollary 1, the general case will be deduced from the following fact: given $x \in \mathcal{U} \setminus \{0\}$ there exists a subspace $\mathcal{V}$ of $\mathcal{U}$ with the property that:

a) $x \in \mathcal{V}$;

b) $\mathcal{V}$ is 2-dimensional;

c) $\mathcal{V}$ is a type IV Cartan factor with respect to the induced norm;

d) there exists a surjective linear projection $p: \mathcal{U} \to \mathcal{V}$ with $\|p\| \leq 1$.

To see this, we represent $\mathcal{U}$ as a Hilbert space K with conjugation τ, as we mentioned in Section 1. We define a 2-subspace $M \subseteq K$ containing x in the following way: if $\tau x \notin \mathbb{C}x$ then M is generated by x and τx; otherwise we remark that $x^\perp$ is τ-invariant, so that we can find $x' \in x^\perp \setminus \{0\}$ such that $\tau x' = x'$, and then M is generated by x and x'. Since M is τ-invariant, it is a type IV Cartan factor with respect to the induced norm. We are left to check property *d*).

We define $p: K \to M$ as the orthogonal projection; our aim is to prove that for $y \in M$ and $z \in M^\perp$ the continuous real function

$$f(t) = \|y + tz\|^2_{\mathcal{U}}$$

has minimum in 0. We can assume that y and z are both non-zero; since M and $M^\perp$ are τ-invariant, $(y | z) = (\tau y | z) = (y | \tau z) = 0$, and then f can be re-written as

$$f(t) = |y|^2 + |z|^2 t^2 + ((|y|^2 + |z|^2 t^2)^2 - |(y | \tau y) + (z | \tau z) t^2|^2)^{1/2}$$

($|w|$ denotes the norm of w in K). If for some point $t_0 \neq 0$ the argument of the square root vanishes in t_0, it is easily verified that f is expressed by

$$f(t) = |y|^2 + |z|^2 t^2$$

and hence it does have minimum in 0. Conversely, suppose the argument of the square root does not vanish in $\mathbb{R} \setminus \{0\}$; then f is differentiable in $\mathbb{R} \setminus \{0\}$; by direct calculation we obtain that $f'(t) = t \cdot q(t)$, where q is a strictly positive continuous function. It follows that f has minimum in 0 in this case too, and hence the proof is complete. ∎

Theorems 2, 3, 4 and 5 of the present paper (together with the accompanying remarks) are summarized by the following.

THEOREM 6. Let G be a non-trivial complex Banach space and let F be a Cartan factor of type I, II, III or IV. The following mutually exclusive possibilities are given:

a) F is a Hilbert space, and

$$(G \oplus_2 F)_0 \supseteq \{0\} \times F;$$

b) F is not a Hilbert space, and

$$(G \oplus_2 F)_0 \subsetneq G_0 \times \{0\}.$$

The above result provides the machinery for the proof of an analogue of Theorem 8 in [Pe2], with the spaces $L^p(\Omega, \mu)$ replaced by Cartan factors.

We shall denote by $\mathcal{C}$ the category of all non-zero Cartan factors of type I, II, III or IV (the morphisms being the linear isometries) and by $\mathcal{S}$ the category whose objects are the Banach spaces obtained from the objects of $\mathcal{C}$ by a finite number of operations of direct sum of the type $\oplus_r$ (with $r \in [1, \infty]$), and whose morphisms are the linear isometries again. (If the collection $\mathcal{B}$ of all Banach spaces were a set, and not only a category, we would have defined $\mathcal{S}$ as the closure of $\mathcal{C}$ in $\mathcal{B}$ with respect to the operations $\oplus_r$.)

If E is an object of $\mathcal{S}$, E is linearly and topologically isomorphic to a product $F_1 \times \ldots \times F_k$ where $F_1, \ldots, F_k$ are objects of $\mathcal{C}$; for $i \in \{1, ..., k\}$ we can think of E as the space built up starting from F_i and adding to it other objects $G_1, \ldots G_h$ of $\mathcal{S}$; that is, we can represent E by

$$(\ldots((F_i \oplus_{r_1} G_1) \oplus_{r_2} G_2) \oplus_{r_3} \ldots) \oplus_{r_h} G_h$$

(or by a similar formula where the sums are not all performed at the right side). In such a case we will say that «$\oplus_{r_1}, \ldots, \oplus_{r_h}$ are, in the order, the direct sums which appear in E *after* F_i».

THEOREM 7. Let E be an object of $\mathcal{S}$, topologically and linearly isomorphic to a product of objects of $\mathcal{C}$, $F_1 \times \ldots \times F_k$. Then

$$E_0 = R_1 \times \ldots \times R_k,$$

where $R_i \subseteq F_i$ is either $\{0\}$ or the whole F_i.

Precisely, R_i is equal to F_i if, and only if, one of the following conditions is fulfilled:

(*a*) F_i is a Hilbert space and after F_i there are first some $\oplus_2$ (possibly none) and then some $\oplus_\infty$ (possibly none);

(*b*) after F_i there are only $\oplus_\infty$ (possibly none).

PROOF. We confine ourselves to a sketch since the argument imitates closely the one presented for Theorem 8 in [Pe2].

Let us consider by simplicity the first coordinate F_1 and prove that if (*a*) or (*b*) are satisfied homogeneity along F_1 is preserved, while if neither (*a*) nor (*b*) are satisfied homogeneity along F_1 is lost.

In case (*a*) the conclusion follows from Theorems 3 and 4 of [Pe2], while case (*b*) is immediately settled.

If neither (*a*) nor (*b*) are fulfilled, one of the following cases occurs:

(*c*) after F_1 there is some $\oplus_p$ with $p \neq 2, \infty$;

(*d*) after F_1 there is a $\oplus_\infty$ followed by a $\oplus_2$;

(*e*) F_1 is not a Hilbert space and after F_1 there is a $\oplus_2$.

In all these cases homogeneity along F_1 is lost: case (*c*) follows from Theorem 2 of [Pe2], case (*d*) from Theorem 6 of [Pe2] and case (*e*) from Theorem 6 above. ■

4. Duality theory for Cartan factors.

Theorem 1 establishes a duality theory for Cartan factors of type I; we will prove that a completely analogous result holds for Cartan factors of type II and III. Afterwards we will consider the case of type IV Cartan factors.

Let H be a non-trivial Hilbert space, let τ be a conjugation on H and let $A \mapsto {}^tA$ be the transposition associated to τ.

We set $\mathcal{L}_i^{(s)}(H) = \mathcal{L}^{(s)}(H) \cap \mathcal{L}_i(H)$ and $\mathcal{L}_i^{(a)}(H) = \mathcal{L}^{(a)}(H) \cap \mathcal{L}_i(H)$ for $i = 0, 1$. We want to prove the following.

THEOREM 8. $\mathcal{L}_i^{(s)}(H)$ and $\mathcal{L}_i^{(a)}(H)$ (for $i = 0, 1$) are complex Banach spaces with respect to the induced norms, and the following isometrical isomorphisms hold:

(1) $\mathcal{L}_0^{(s)}(H)^* \cong \mathcal{L}_1^{(s)}(H)$, (2) $\mathcal{L}_1^{(s)}(H)^* \cong \mathcal{L}^{(s)}(H)$,

(3) $\mathcal{L}_0^{(a)}(H)^* \cong \mathcal{L}_1^{(a)}(H)$, (4) $\mathcal{L}_1^{(a)}(H)^* \cong \mathcal{L}^{(a)}(H)$,

the value of A on B being given in any case by $\operatorname{tr}(AB)$.

For the proof of this theorem we need a few technical preliminaries.

We recall that for $A \in \mathcal{L}(H)$, $[A] \equiv (A^*A)^{1/2}$.

LEMMA 6. $\forall A \in \mathcal{L}(H)$, $[{}^tA] = \tau[A^*]\tau$.

PROOF. Since $(\tau x | \tau y) = (y | x)$ we have

$$(({}^tA)^* x | y) = ((\tau A^* \tau)^* x | y) = (x | \tau A^* \tau y) = (A^* \tau y | \tau x) = (\tau y | A \tau x) =$$

$$= (\tau A \tau x | y) \Rightarrow ({}^tA)^* = \tau A \tau \Rightarrow [{}^tA]^2 = \tau A \tau \tau A^* \tau = (\tau[A^*]\tau)^2$$

and conclusion follows from the fact that $\tau[A^*]\tau \geqslant 0$. ■

LEMMA 7. (*a*) $A \in \mathcal{L}_1(H) \Rightarrow {}^tA \in \mathcal{L}_1(H)$, $\|{}^tA\|_1 = \|A\|_1$, $\operatorname{tr}({}^tA) = \operatorname{tr}(A)$;

(*b*) $A \in \mathcal{L}_0(H) \Rightarrow {}^tA \in \mathcal{L}_0(H)$, $\|{}^tA\| = \|A\|$.

PROOF. (*a*) Let $\{\phi_\alpha\}$ be an orthonormal basis of H; then if $\psi_\alpha = \tau\phi_\alpha$, $\{\psi_\alpha\}$ is an orthonormal basis too, and hence

$$\|{}^tA\|_1 = \operatorname{tr}([{}^tA]) = \sum_\alpha (\tau[A^*]\tau\phi_\alpha \mid \phi_\alpha) = \sum_\alpha (\psi_\alpha \mid [A^*]\psi_\alpha) =$$

$$= \sum_\alpha ([A^*]\psi_\alpha \mid \psi_\alpha) = \operatorname{tr}([A^*]) = \|A^*\|_1 = \|A\|_1 .$$

The first two assertions are proved; as for the third one

$$\operatorname{tr}({}^tA) = \sum_\alpha ({}^tA\phi_\alpha \mid \phi_\alpha) = \sum_\alpha (\tau A^* \tau\phi_\alpha \mid \phi_\alpha) =$$

$$= \sum_\alpha (\tau\phi_\alpha \mid A^*\tau\phi_\alpha) = \sum_\alpha (A\psi_\alpha \mid \psi_\alpha) = \operatorname{tr}(A).$$

(*b*) is obvious. ■

Now, for $A \in \mathcal{L}(H)$ we set $A^s = 1/2\cdot(A + {}^tA)$, $A^a = 1/2\cdot(A - {}^tA)$. It follows from Lemma 7 that if $A \in \mathcal{L}_i(H)$ then A^s, $A^a \in \mathcal{L}_i(H)$ (for $i = 0, 1$).

LEMMA 8. (*a*) Given $A \in \mathcal{L}_1(H)$ we have

$$\operatorname{tr}(AB) = 0 \quad \forall B \in \mathcal{L}_0^{(s)}(H) \Leftrightarrow {}^tA = -A.$$

(*b*) Given $A \in \mathcal{L}_1(H)$ we have $\operatorname{tr}(AB) = 0$ $\forall B \in \mathcal{L}_0^{(a)}(H) \Leftrightarrow {}^tA = A$.

(*c*) Given $A \in \mathcal{L}(H)$ we have $\operatorname{tr}(AB) = 0$ $\forall B \in \mathcal{L}_1^{(s)}(H) \Leftrightarrow {}^tA = -A$.

(*d*) Given $A \in \mathcal{L}(H)$ we have $\operatorname{tr}(AB) = 0$ $\forall B \in \mathcal{L}_1^{(a)}(H) \Leftrightarrow {}^tA = A$.

PROOF. (*a*) $\Leftarrow$.

$$\operatorname{tr}(AB) = -\operatorname{tr}({}^tA\,{}^tB) = -\operatorname{tr}({}^t(BA)) = -\operatorname{tr}(BA) = -\operatorname{tr}(AB).$$

$\Rightarrow$. Suppose $A^s \neq 0$; since $A^s \in \mathcal{L}_1(H)$ and $\mathcal{L}_1(H) \cong \mathcal{L}_0(H)^*$ we can find $B \in \mathcal{L}_0(H)$ with $\operatorname{tr}(A^sB) \neq 0$; but by the hypothesis and by the first implication

$$\operatorname{tr}(A^sB) = \operatorname{tr}(A^sB^s) = \operatorname{tr}(AB^s) = 0$$

and this is absurd.

The proof of (*b*), (*c*), and (*d*) is completely analogous. ■

LEMMA 9. (*a*) $A \in \mathcal{L}^{(s)}(H)$, $B \in \mathcal{L}^{(a)}(H) \Rightarrow \|A + B\| \geq \max\{\|A\|, \|B\|\}$.

(*b*) $A \in \mathcal{L}_1^{(s)}(H)$, $B \in \mathcal{L}_1^{(a)}(H) \Rightarrow \|A + B\|_1 \geq \max\{\|A\|_1, \|B\|_1\}$.

PROOF. (*a*) Since the transposition is an isometry

$$\|A + B\| = \|{}^tA - {}^tB\| = \|{}^t(A - B)\| = \|A - B\| \Rightarrow$$

$$\Rightarrow \|A\| = \frac{1}{2}\|A + B + A - B\| \leq \frac{1}{2}(\|A + B\| + \|A - B\|) = \|A + B\|.$$

The same holds for $\|B\|$.

(*b*) As the transposition is an isometry for $\|\cdot\|_1$ too, the proof works as above. ■

PROOF OF THEOREM 8. Lemma 7 implies that $\mathcal{L}_i^{(s)}(H)$ and $\mathcal{L}_i^{(a)}(H)$ are closed subspaces of $\mathcal{L}_i(H)$ for $i = 0, 1$, and hence the first assertion is abvious.

(1) By Theorem 1, Lemma 8 and the Hahn-Banach theorem there exists a one-to-one mapping α from $\mathcal{L}_0^{(s)}(H)^*$ onto $\mathcal{L}_1^{(s)}(H)$ such that

$$\phi(B) = \operatorname{tr}(\alpha(\phi)B) \quad \forall \phi \in \mathcal{L}_0^{(s)}(H)^*,\ B \in \mathcal{L}_0^{(s)}(H).$$

α is obviously a linear isomorphism. We are left to prove that α is an isometry, i.e.

$$\|A\|_1 = \sup_{B \in \mathcal{L}_0^{(s)}(H)} \frac{|\operatorname{tr}(AB)|}{\|B\|} \quad \forall A \in \mathcal{L}_1^{(s)}(H).$$

By Theorem 1, since $\mathcal{L}_0^{(s)}(H) \subseteq \mathcal{L}_0(H)$, inequality $\geq$ is obvious.

As for the converse, using part (*a*) of Lemma 9,

$$\|A\|_1 = \sup_{B \in \mathcal{L}_0(H)} \frac{|\operatorname{tr}(A(B^s + B^a))|}{\|B^s + B^a\|} \leq \sup_{B \in \mathcal{L}_0(H)} \frac{|\operatorname{tr}(AB^s)|}{\|B^s\|} =$$

$$= \sup_{B \in \mathcal{L}_0^{(s)}(H)} \frac{|\operatorname{tr}(AB)|}{\|B\|}$$

(2) As above, it suffices to prove that

$$\|A\| \leq \sup_{B \in \mathcal{L}_1^{(s)}(H)} \frac{|\operatorname{tr}(AB)|}{\|B\|_1} \quad \forall A \in \mathcal{L}^{(s)}(H).$$

The proof works as above, using part (*b*) of Lemma 9.

(3) and (4) are completely analogous. ■

Now we turn to the case of type IV Cartan factors.

As we mentioned in Section 1, if $\mathcal{U}$ is a type IV Cartan factor then $\mathcal{U}$ is linearly and topologically isomorphic to a Hilbert space K; for a suitable conjugation τ on K the $\mathcal{U}$-norm is given by

$$\|x\|_{\mathcal{U}} = ((x|x) + ((x|x)^2 - |(x|\tau x)|^2)^{1/2})^{1/2} \qquad (x \in K).$$

Since K is a reflexive space $\mathcal{U}$ is reflexive too; moreover the dual space $\mathcal{U}^*$ of $\mathcal{U}$ is linearly and topologically isomorphic to the dual space $\overline{K}$ of K; we recall that $\overline{K}$ is the Hilbert space which coincides with K as a real Hilbert space, in which multiplication by complex number and inner product are given by

$$\lambda \cdot_{\overline{K}} x = \overline{\lambda} \cdot_K x \qquad (\lambda \in \mathbb{C},\ x \in K),$$

$$(x|y)_{\overline{K}} = (y|x)_K \qquad (x, y \in K).$$

We remark that τ is a conjugation on $\overline{K}$ too.

The $\mathcal{U}^*$-norm on $\overline{K}$ is given by

$$\|y\|_{\mathcal{U}^*} = \sup_{x \in K \setminus \{0\}} \frac{|(x|y)|}{\|x\|_{\mathcal{U}}} \qquad (y \in \overline{K}).$$

In order to determine completely the dual space of $\mathcal{U}$ we only have to compute explicitly this norm.

THEOREM 9. $\|y\|^2_{\mathcal{U}^*} = 1/2 \cdot ((y|y) + |(y|\tau y)|)$.

PROOF. The theorem is certainly true if $\dim \mathcal{U} = 1$.

Now, assume $\dim \mathcal{U} = 2$ and let $\{\phi_1, \phi_2\}$ be an orthonormal basis of K such that $\tau\phi_i = \phi_i$ for $i = 1, 2$. An isometrical isomorphism

$$j\colon\ \mathcal{U} \to \mathbb{C} \oplus_\infty \mathbb{C}$$

is explicitly given by

$$j(z_1\phi_1 + z_2\phi_2) = (z_1 + iz_2,\ z_1 - iz_2).$$

Since $(\mathbb{C} \oplus_\infty \mathbb{C})^* = \mathbb{C} \oplus_1 \mathbb{C}$,

$$j^*\colon\ \mathbb{C} \oplus_1 \mathbb{C} \to \mathcal{U}^*$$

is an isometrical isomorphism. By direct computation we deduce from this that the $\mathcal{U}^*$-norm on $\overline{K}$ has the required expression.

Now, assume that $\dim \mathcal{U} \geq 3$; for $y \in \overline{K} \setminus \{0\}$ we define M as the subspace of K generated by y and τy and we consider the orthogonal projection p of K onto M. During the proof of Theorem 5 it was checked

that $\|p\| = 1$, *i.e.*

$$\|p(x)\|_{\mathcal{U}} \leq \|x\|_{\mathcal{U}} \qquad \forall x \in K.$$

It follows that

$$\sup_{x \in K \setminus \{0\}} \frac{|(x|y)|}{\|x\|_{\mathcal{U}}} = \sup_{x \in M \setminus \{0\}} \frac{|(x|y)|}{\|x\|_{\mathcal{U}}}.$$

Everything reduces to the 2-dimensional (or, possibly, 1-dimensional) case, and hence the theorem holds in the general case too. ■

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