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Fixed point theorems in locally G-convex spaces

Sehie Park *

Department of Mathematics, Seoul National University, Seoul 151-742, Korea

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1. Introduction

It is well-known that the Brouwer fixed point theorem, the Sperner lemma, the Knaster–Kuratowski–Mazurkiewicz theorem (simply, the KKM principle), and many results in nonlinear analysis are equivalent. In particular, it was shown in [13] that the KKM principle implies the Brouwer theorem. In this paper, we show that the KKM theorem implies far-reaching generalizations of the Brouwer theorem including well-known fixed point theorems due to Schauder, Tychonoff, Kakutani, Himmelberg, and many others. For the literature, see [17,26].

In a recent work, Tarafdar [34] obtained a fixed point theorem for a continuous compact multimap $T : X \multimap X$ with closed H-convex values, where X is a locally H-convex uniform space. In the present paper, we show that his theorem holds for u.s.c. maps instead of continuous maps and for the class of generalized convex spaces (or G-convex spaces) containing properly that of H-spaces and many other types of spaces. Our main result (Theorem 2) is applied to various fixed point theorems for LG-spaces, LC-spaces, hyperconvex spaces, and normed vector spaces.

Section 2 deals with a new KKM theorem for generalized convex spaces. In Section 3, we obtain our main fixed point theorem for LG-spaces and some of its simple consequences, especially, a generalization of the Himmelberg theorem. Section 4 deals with lower semicontinuous multimaps and Φ-maps defined on paracompact LC-spaces. In fact, the selection theorems due to Ben-El-Mechaiekh and Oudadess [3] and Park [22] are used to deduce new fixed point theorems for such multimaps. In Section 5, our

* Fax: +0822-887-4694.

E-mail address: shpark@math.snu.ac.kr (S. Park).

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main result is applied to hyperconvex spaces. In Section 6, we show that some new versions of selection theorems and fixed point theorems can be deduced for normed vector spaces. Section 7 deals with historical remarks.

2. Generalized convex spaces

For topological spaces X and Y, a *multimap* or a *map* $T: X \multimap Y$ is a function from X into the set of nonempty subsets of Y. A map $T: X \multimap Y$ is *upper semicontinuous* (*u.s.c.*) if for each open subset G of Y, the set $\{x \in X: Tx \subset G\}$ is open in X; *lower semicontinuous* (*l.s.c.*) if for each closed subset F of Y, the set $\{x \in X: Tx \subset F\}$ is closed in X; *continuous* if it is u.s.c. and l.s.c.; and *compact* if the range $T(X) = \{y \in Y: y \in Tx \text{ for some } x \in X\}$ is contained in a compact subset of Y.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n any n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) = (X, X; \Gamma)$. For a G-convex space $(X, D; \Gamma)$ with $X \supset D$, a subset C of X is said to be Γ-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. For details on G-convex spaces, see [27–30], where basic theory was extensively developed.

There are a lot of examples of G-convex spaces:

If $X = D$ is a convex subset of a vector space and each Γ_A is the convex hull of $A \in \langle X \rangle$ equipped with the Euclidean topology, then $(X; \Gamma)$ becomes a *convex space* in the sense of Lassonde [14]. Note that any convex subset of a topological vector space is a convex space, but not conversely.

If $X = D$ and each Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$) and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then (X, Γ) becomes a *C-space* (or an *H-space*) due to Horvath [8,9].

The other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, and so on. For the literature, see [27–30]. Recently, we gave new examples of G-convex spaces and, simultaneously, showed that some abstract convexities of other authors are simple particular examples of our G-convexity (see [24]). Such examples are L-spaces of Ben-El-Mechaiekh et al. [2], continuous images of G-convex spaces, Verma's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, generalized H-spaces of Stachó, and *mc*-spaces of Llinares. Moreover, Ben-El-Mechaiekh et al. [2] gave examples of G-convex spaces $(X; \Gamma)$ as follows: B'-simplicial convexity, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

Futhermore, any hyperbolic space X in the sense of Kirk [12] and Reich-Shafrir [32] is a G-convex space, since the closed convex hull of any $A \in \langle X \rangle$ is contractible [30, p. 542]. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic (see [32]).

Now, we deduce a KKM theorem for G-convex spaces.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

The following result is well known:

The KKM Principle. *Let D be the set of vertices of Δ_n and $F : D \multimap \Delta_n$ be a KKM map with closed* (*resp. open*) *values* (*that is*, $\mathrm{co}\, A \subset F(A)$ *for each* $A \in \langle D \rangle$). *Then* $\bigcap_{a \in D} F(a) \neq \emptyset$.

The closed-valued version is due to [13]. The open-valued version is a simple consequence of the closed-valued version in view of Shih [33, Theorem 1]. For the history of generalizations and applications of the open-valued version of the KKM principle, see [29].

The following is a KKM theorem for G-convex spaces:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap with closed* (*resp. open*) *values. Suppose that F is a KKM map. Then*
(i) $\{F(a)\}_{a \in D}$ *has the finite intersection property*; *and*
(ii) *if $\bigcap_{a \in N} \overline{F(a)}$ is contained in a compact subset K of X for some $N \in \langle D \rangle$, then we have $\bigcap_{a \in D} \overline{F(a)} \neq \emptyset$.*

Proof. Let $A = \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$. Then there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A$ such that, for any $0 \le i_0 < i_1 < \cdots < i_k \le n$, we have

$$\phi_A(\mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_A(\Delta_n).$$

Since F is a KKM map, it follows that

$$\mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \phi_A^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_A(\Delta_n))$$

$$\subset \bigcup_{j=0}^{k} \phi_A^{-1}(F(a_{i_j}) \cap \phi_A(\Delta_n)).$$

Since $F(a_{i_j}) \cap \phi_A(\Delta_n)$ is closed (resp. open) in the compact subset $\phi_A(\Delta_n)$ of Γ_A, $\phi_A^{-1}(F(a_{i_j}) \cap \phi_A(\Delta_n))$ is closed (resp. open) in Δ_n. Note that $e_i \multimap \phi_A^{-1}(F(a_i) \cap \phi_A(\Delta_n))$ is a KKM map. Hence, by the KKM principle, we have

$$\bigcap_{i=0}^{n} \phi_A^{-1}(F(a_i) \cap \phi_A(\Delta_n)) \neq \emptyset,$$

which readily implies $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$. This completes the proof of (i).

Note that (ii) follows immediately from (i). $\square$

For $X=\Delta_n$, if D is the set of vertices of Δ_n and $\Gamma=\mathrm{co}$, the convex hull, Theorem 1 reduces to the KKM principle [13]. If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 1 extends Fan's KKM lemma [6].

3. Locally G-convex spaces

From now on, we assume for simplicity that all topological spaces are Hausdorff and that D is a subset of X.

A G-convex space $(X,D;\Gamma)$ is called an *LG-space* (or a *locally G-convex space*) if $(X,\mathscr{U})$ is a uniform space such that D is dense in X and if there exists a basis $\{V_\lambda\}_{\lambda\in I}$ for the uniformity $\mathscr{U}$ such that for each $\lambda\in I$, $\{x\in X\colon C\cap V_\lambda[x]\neq\emptyset\}$ is Γ-convex whenever $C\subset X$ is Γ-convex, where

$$V_\lambda[x]=\{x'\in X\colon (x,x')\in V_\lambda\}.$$

For a C-space $(X;\Gamma)$, an LG-space reduces to an LC-space [8,9] (or a locally C-convex space [34]). Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an LC-space $(X;\Gamma)$ with $\Gamma_A=\mathrm{co}\,A$ for $A\in\langle X\rangle$. For other examples, see [8,34].

A G-convex space $(X;\Gamma)$ is called an *LG-metric space* if X is equipped with a metric d such that for any $\varepsilon>0$, the set $\{x\in X\colon d(x,C)<\varepsilon\}$ is Γ-convex whenever $C\subset X$ is Γ-convex and open balls are Γ-convex. This concept generalizes that of LC-metric spaces due to Horvath [8].

The following is our main result:

Theorem 2. *Let $(X,D;\Gamma)$ be an LG-space and $T:X\multimap X$ a compact u.s.c. multimap with closed Γ-convex values. Then T has a fixed point $x_0\in X$; that is, $x_0\in Tx_0$.*

Proof. We may assume that V_λ is always closed for $\lambda\in I$. Let $V\in\{V_\lambda\}_{\lambda\in I}$. Since the open members of $\mathscr{U}$ form a basis and $V\in\mathscr{U}$, there exists an open member W of $\mathscr{U}$ such that $W\subset V$. Note that for each $x\in X$, $W[x]$ is an open neighborhood of x. Since $K=\overline{T(X)}$ is compact and D is dense in X, there exists an $M=\{y_1,\dots,y_n\}\in\langle D\rangle$ such that $K\subset\bigcup_{y\in M}W[y]$.

For each $y_i\in M$, let $F(y_i):=\{x\in X\colon Tx\cap V[y_i]=\emptyset\}$. Since T is u.s.c., each $F(y_i)$ is open. Moreover, since $T(X)\subset K\subset\bigcup_{i=1}^n V[y_i]$, we have

$$\bigcap_{i=1}^n F(y_i)\subset\left\{x\in X\colon Tx\cap\bigcup_{i=1}^n V[y_i]=\emptyset\right\}=\emptyset.$$

We will apply Theorem 1 to the G-convex space $(X,M;\Gamma)$. Since the conclusion of Theorem 1 does not hold, $F:M\multimap X$ cannot be a KKM map; that is, there exist an $N\in\langle M\rangle$ and an $x_V\in\Gamma_N$ such that $x_V\notin F(N)=\bigcup_{y\in N}F(y)$. Hence, $Tx_V\cap V[y]\neq\emptyset$ for all $y\in N$, and

$$N\subset L:=\{y\in X\colon Tx_V\cap V[y]\neq\emptyset\}.$$

Since Tx_V is Γ-convex and $(X,D;\Gamma)$ is an LG-space, L is Γ-convex. Therefore, $x_V\in\Gamma_N\subset L$ and hence $Tx_V\cap V[x_V]\neq\emptyset$.

So, for each basis element V, there exist $x_V, y_V \in X$ such that $y_V \in Tx_V$ and $y_V \in V[x_V]$. Since $T(X)$ is relatively compact, we may assume that y_V converges to some $x_0 \in K$. Then x_V also converges to x_0. Since T is u.s.c. with closed values, the graph of T is closed in $X \times \overline{T(X)}$, and hence we have $x_0 \in Tx_0$. This completes our proof. □

Remark. Note that, in the above proof, if $\Gamma_N \subset D$ for each $N \in \langle D \rangle$, it is sufficient to assume that T has Γ-convex values on D, not necessarily on the whole X.

In order to give an example of Theorem 2, we introduce a notion due to Himmelberg [7]:

A nonempty subset Y of a topological vector space E is said to be *almost convex* if for any neighborhood V of the origin 0 in E and for any finite set $\{y_1, y_2, \dots, y_n\} \subset Y$, there exists a finite set $\{z_1, z_2, \dots, z_n\} \subset Y$ such that, for each $i = 1, 2, \dots, n$, $z_i - y_i \in V$ and $\text{co}\{z_1, z_2, \dots, z_n\} \subset Y$.

We give a new example of G-convex spaces:

Lemma 1. *Let X be a subset of a topological vector space E. If X has an almost convex subset Y, then X has a G-convex structure.*

Proof. Choose a neighborhood V of the origin 0 of E. For any $A = \{y_1, y_2, \dots, y_n\} \in \langle Y \rangle$, there exists a $B = \{z_1, z_2, \dots, z_n\} \in \langle Y \rangle$ such that $z_i - y_i \in V$ for all $i = 1, 2, \dots, n$ and $\text{co}\, B \subset Y \subset X$. Define $\Gamma_A := \text{co}\, B$ as above. Then $(X, Y; \Gamma)$ becomes a G-convex space. □

From Remark of Theorem 2 and Lemma 1, we have the following:

Corollary 1. *Let X be a subset of a locally convex Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap X$ be a compact u.s.c. multimap with closed values such that Ty is convex for all $y \in Y$. Then T has a fixed point.*

Proof. By Lemma 1, we have a G-convex space $(X, Y; \Gamma)$. Since Y is dense in X and E is locally convex, $(X, Y; \Gamma)$ becomes an LG-space. Now, in view of Remark with $D = Y$, we have the conclusion from Theorem 1. □

For a single-valued map, Theorem 2 reduces to the following:

Theorem 3. *Let $(X, D; \Gamma)$ be an LG-space such that singletons are Γ-convex. Then any compact continuous function $f : X \to X$ has a fixed point.*

For metrizable spaces, Theorem 1 reduces to the following:

Theorem 4. *Let $(X; \Gamma)$ be an LG-metric space, and $T : X \multimap X$ a compact u.s.c. map with closed Γ-convex values. Then T has a fixed point.*

4. Locally C-convex spaces

In this section, we are mainly concerned with C-spaces $(X,D;\Gamma)$; that is, each Γ_A is infinitely connected for $A\in\langle D\rangle$. In this case, we can define an LC-space as in Section 3.

The following is due to Ben-El-Mechaiekh and Oudadess [3, Corollary 6(A)(iii)]:

Lemma 2. *Let X be a paracompact space, $Z\subset X$ with* $\dim_X Z\le 0$, $B\subset X$ *countable,* (Y,Γ) *a complete LC-metric space such that* $\Gamma_{\{y\}}=\{y\}$ *for all* $y\in Y$, *and* $T:X\multimap Y$ *a l.s.c. map such that* Tx *is closed for* $x\notin B$ *and* $\overline{Tx}$ *is* Γ*-convex for* $x\notin Z$. *Then* T *has a continuous selection* $f:X\to Y$; *that is,* $fx\in Tx$ for all $x\in X$.

For simplicity, we consider only the case $B=\emptyset$ as in [3,11, Theorem 3].

From Theorem 3 and Lemma 2, we have the following:

Theorem 5. *Let* $(X;\Gamma)$ *be a paracompact LC-space such that* $\Gamma_{\{x\}}=\{x\}$ *for all* $x\in X$, Y *a compact LC-metric subset of* X, *and* $Z\subset X$ *with* $\dim_X Z\le 0$. Let $T:X\multimap Y$ *be a l.s.c. map with closed values such that* Tx *is* Γ*-convex for* $x\notin Z$. *Then* T *has a fixed point.*

Proof. By Lemma 2, T has a continuous selection $f:X\to Y$. Now, by applying Theorem 3, f has a fixed point $x_0\in Y$; that is, $x_0=fx_0\in Tx_0$. □

Let X be a topological space and $(Y,D;\Gamma)$ a G-convex space. A multimap $T:X\multimap Y$ is called a Φ-*map* provided that there is a (companion) multimap $S:X\multimap D$ satisfying
(a) for each $x\in X$, $M\in\langle Sx\rangle$ implies $\Gamma_M\subset Tx$; and
(b) $X=\bigcup\{\operatorname{Int} S^- y:\ y\in D\}$.

Even for C-spaces, this concept is more general than that of Horvath [8, Definition 4.1(a)].

We need the following result [22, Theorem 8]:

Lemma 3. *Let X be a paracompact space,* $(Y,D;\Gamma)$ *a C-space, and* $T:X\multimap Y$ *a* Φ*-map. Then* T *has a continuous selection.*

From Lemma 3 and Theorem 3, we have the following:

Theorem 6. *Let* $(X,D;\Gamma)$ *be a paracompact LC-space such that singletons are* Γ*-convex. Then any compact* Φ*-map* $T:X\multimap X$ *has a fixed point.*

Proof. By Lemma 3, T has a continuous selection $f:X\to X$. Since $\overline{f(X)}\subset\overline{T(X)}$ and T is compact, f is also compact. Therefore, by Theorem 3, f has a fixed point $x_0=fx_0\in Tx_0$. □

5. Hyperconvex spaces

A metric space (H,d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H about which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$. For any nonempty bounded subset A of H, its *convex hull* $\operatorname{co} A$ is defined by

$$\operatorname{co} A = \bigcap \{B\colon B \text{ is a closed ball containing } A\}.$$

A subset C of H is said to be *convex* if $\operatorname{co} N \subset C$ for each $N \in \langle C \rangle$.

For details for hyperconvex spaces, see [10] and references therein.

It is known that the space $\mathbb{C}(E)$ of all continuous real functions on a Stonian space E (that is, an extremally disconnected compact Hausdorff space) with the usual norm is hyperconvex, and that every hyperconvex real Banach space is a space $\mathbb{C}(E)$ for some Stonian space E. The spaces $(\mathbf{R}^n, \|\cdot\|_\infty)$, l^∞, and L^∞ are concrete examples of hyperconvex spaces.

Moreover, we note the following:

(1) Horvath [9, Theorem 9] showed that any hyperconvex space (H,d) is a complete LC-metric space (H,Γ), where $\Gamma_N := \operatorname{co} N$ for each $N \in \langle H \rangle$. He also gave other examples of LC-metric spaces.

(2) Khamsi [10, Theorem 4] obtained a KKM theorem for hyperconvex spaces, which is a particular form of Theorem 1.

From Theorems 4 and 5, we have the following:

Theorem 7. *Let (H,d) be a hyperconvex space and $T\colon H \multimap H$ a compact map with closed convex values. If T is either u.s.c. or l.s.c., then T has a fixed point.*

For a single-valued map $T = f\colon H \to H$, Theorem 7 reduces to [21, Theorem 7], which extends earlier works in [5, Lemma 3; 5, Theorem 6]. The u.s.c. case of Theorem 7 was given as Park [25, Theorem 7] with three Corollaries by different method.

Since any hyperconvex space is an LC-metric space, Theorem 6 reduces to the following form of the Fan–Browder theorem:

Theorem 8. *Let (H,d) be a hyperconvex space and $T\colon H \multimap H$ a compact map such that*

(1) *for each $x \in H$, Tx is convex*; *and*

(2) $H = \bigcup \{\operatorname{Int} T^- y\colon y \in H\}$.

Then T has a fixed point.

For the case H itself is compact, Theorem 8 reduces to [20, Theorem 3].

Note that there are a large number of applications of the Fan–Browder theorem for convex spaces (see [17,23] and references therein). Analogously, there would be useful applications of Theorem 8.

6. Normed vector spaces

For normed vector spaces, we have the following form of Lemma 2:

Theorem 9. *Let X be a paracompact space, $Z \subset X$ with* $\dim_X Z \leq 0$, *$B \subset X$ countable, Y a normed vector space, and $T : X \multimap Y$ a l.s.c. map such that Tx is complete for $x \notin B$ and Tx is convex for $x \notin Z$. Then T has a continuous selection.*

Proof. Without loss of generality, we may assume that Y is complete (for the conditions on T remains unchanged in the completion of Y). Now, by applying Lemma 2, we have the conclusion. □

For $B = Z = \emptyset$, Theorem 9 reduces to the following due to Zheng [38, Theorem 2.4]:

Corollary 2. *Let X be a paracompact space, Y a normed vector space, and $T : X \multimap Y$ a l.s.c. map with complete convex values. Then T has a continuous selection.*

More precisely, we can deduce the following result from Corollary 2:

Corollary 3. *Let X be a paracompact space, Y a normed vector space, and $T : X \multimap Y$ a map with complete convex values. Then T is l.s.c. if and only if for each given $x \in X$ and $g \in T(x)$, there exists a continuous selection s for T such that $s(x) = g$.*

If Y is a Banach space, Corollary 4 reduces to Chen [4, Theorem 2.4], whose proof can be easily modified to that of Corollary 3 by applying Corollary 2.

The following form of the Schauder fixed point theorem is a simple consequence of Corollary 1 or Theorems 2–4.

Lemma 4. *Let X be an almost convex subset of a normed vector space and $f : X \to X$ a compact continuous map. Then f has a fixed point $x_0 \in X$; that is, $x_0 = fx_0$.*

From Lemma 4 and Theorem 9, we have the following:

Theorem 10. *Let X be an almost convex subset of a normed vector space, $Z \subset X$* with $\dim_X Z \leq 0$, *$B \subset X$ countable, and $T : X \multimap X$ a compact l.s.c. map such that Tx is closed for $x \notin B$ and Tx is convex for $x \notin Z$. Then T has a fixed point.*

Proof. Note that Tx is compact and hence complete for each $x \notin B$. Applying Theorem 9, T has a continuous selection $f : X \to X$. Since $f(X) \subset T(X)$ and T is compact, so is f. Therefore, by Lemma 4, f has a fixed point $x_0 = fx_0 \in Tx_0 \subset X$. This completes our proof. □

For $B = Z = \emptyset$, Theorem 10 reduces to the following generalization of Zheng [38, Theorem 2.5]:

Corollary 4. *Let X be an almost convex subset of a normed vector space, and $T: X \multimap X$ a compact l.s.c. map with closed convex values. Then T has a fixed point.*

Remark. Zhang [37, Lemma 2.1] obtained a particular form of Corollary 4 for a nonempty closed convex subset X of a Banach space.

7. Historical remarks

1. For a paracompact LC-space $(X;\Gamma)$, Theorem 2 reduces to Ben-El-Mechaiekh et al. [2, Corollary 4.7] with a different proof (see also [25, Theorem 5]).
2. Historically, well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Hukuhara, Bohnenblust-Karlin, Fan and Glicksberg, and Himmelberg are all simple consequences of Corollary 1. For the literature, see [17,26].
3. Particular forms of Theorem 4 were obtained by Rassias [31] for a compact convex subset of a metrizable topological vector space and by Park [15, Corollary 13.2] for a metric compact convex space.
4. The study of C-spaces was initiated by Horvath [8,9]. Note that [8, Corollary 4.4] is a particular form of Theorem 3 for LC-spaces and that [8, Corollary 4.5] is a consequence of Theorem 5. The origin of Φ-maps goes back to the works of Fan and Browder. For the literature, see [16,17,23,27–29].
5. Our work in this paper is motivated by Tarafdar [34, Theorem 2.1], which is a C-space version of Theorem 2 for continuous (u.s.c. and l.s.c.) multimaps on $X=D$. Further, he applied his result to locally strongly convex spaces, locally strongly contractible uniform spaces, and locally group contractible commutative topological groups. In all of his results, continuity of multimaps can now be weakened to u.s.c. After this paper was submitted, Watson [35, Theorem 4.1] published a particular form of Theorem 2, and Wu and Li [36, Theorem 2] showed a particular form of Theorem for LC-spaces under some superfluous restrictions.
6. The notion of hyperconvex spaces was introduced by Aronszajn and Panitchpakdi [1]. Until recently, the study of such spaces was mainly devoted to their relationship with nonexpansive maps. However, Khamsi [10] applied his KKM theorem to prove an analogue of Ky Fan's best approximation theorem extending the Brouwer and the Schauder fixed point theorems. See also [5].
7. Motivated by [10], the present author [20] obtained a Ky Fan-type matching theorem for open covers, a coincidence theorem, a Fan-Browder-type fixed-point theorem, a Brouwer–Schauder–Rothe-type fixed-point theorem, and other results on hyperconvex spaces. Since hyperconvex spaces are C-spaces, some of the works in [10,20,21] are consequences of theory of G-convex spaces [27–30].
8. Lemma 2, Theorem 9, Corollaries 3 and 4 are generalizations of the well-known selection theorems of Michael. For the literature, see [3].
9. Far-reaching generalizations of Lemma 4 for a convex set X are known; see [17–19,26]. Note that many of well-known textbooks state the original form of Schauder's theorem in 1930, where X is a nonempty closed convex subset of a Banach space.

10. Since the Brouwer fixed point theorem is equivalent to the KKM principle, it is clear that Theorems 1–4, Corollary 1, and Lemma 4 are also equivalent formulations of the Brouwer theorem.

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