Weak*–closed Jordan ideals of nest algebras

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Nest algebras provide examples of partial Jordan *–triples. If A is a nest algebra and $A_s = A \cap A^*$, where A^* is the set of the adjoints of the operators lying in A, then (A, A_s) forms a partial Jordan *–triple. Any weak*–closed ideal in the nest algebra A is also an ideal in the partial Jordan *–triple (A, A_s). An analysis of the ideal structure of (A, A_s) shows that, for a large class of nest algebras, the converse is also true.

1 Introduction

The work of Kaup and Upmeier [12, 14, 15] and Vigué [30, 31, 32, 33] shows how the holomorphic structure of the open unit ball in a complex Banach space A leads to the existence of a closed subspace A_s of A, known as the symmetric part of A, and a partial triple product $(a, b, c) \mapsto \{a\ b\ c\}$ mapping $A \times A_s \times A$ to A. The purely algebraic properties of the partial triple product, namely the linearity and symmetry in the outer variables, the conjugate linearity in the second variable and the existence of a Jordan triple identity, relate any complex Banach space to the Jordan triple systems studied in the late sixties and in the seventies by Koecher [16], Loos [17] and Meyberg [18]. However, in Jordan triple systems, the triple product is universally, not partially, defined. Complex Banach spaces A which coincide with their symmetric parts A_s, and, thus, possessing a globally defined triple product, are said to be symmetric. Although, the category of symmetric complex Banach spaces or, equivalently, the category of JB*–triples has been widely investigated, little is known about spaces the symmetric parts of which are proper subspaces. In fact, to determine the symmetric part of a non–symmetric space has sometimes proved to be an elusive task [4]. In some cases, the symmetric part is merely the null space. However, interesting examples exist in which A_s is neither A nor $\{0\}$. Relevant examples of non–symmetric spaces can be given by certain norm–closed unital Jordan subalgebras of a JB*–algebra. The symmetric part of such a subalgebra coincides with its self–adjoint part [3]. Consequently, any nest algebra of bounded linear operators defined on a complex Hilbert space is a non–symmetric complex Banach space with non–zero symmetric part [3, 5, 11, 24, 25]. In the subsequent section, we shall show how a nest algebra A with symmetric part A_s, along with its intrinsic partial triple product, provides an example of a purely algebraic structure, that of a partial Jordan *–triple (A, A_s) [26, 27, 28, 29]. It is the ideal structure of nest algebras when perceived as partial Jordan *–triples that will be investigated in this paper. The ideals of nest algebras related to its associative multiplication have been extensively investigated (see, for example, [1, 7, 8, 9, 10, 20, 21, 22]). Whilst it is clear that ideals in the associative sense provide examples of ideals in the partial triple sense, the converse assertion remains in general an open problem. It is the aim of this paper to show that, in a large class of nest algebras, the weak*–closed ideals in the partial triple sense are also weak*–closed ideals in the associative algebra sense.

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The paper is organised as follows. In Section 2, a brief account of those aspects of the theory of nest algebras needed in the sequel is presented, and we show how a nest algebra A with symmetric part A_s naturally arises as a partial Jordan *–triple.

The key tools to find the shape of the weak*–closed ideals in the partial Jordan *–triple (A, A_s), formed by a nest algebra A and its symmetric part A_s, are the rank one operators in the algebra. Therefore, a part of Section 3 is concerned with the characterization of the rank one operators that lie in a given ideal. At the beginning of this section, it is shown that two projections may be associated with each rank one operator in A (Lemma 3.2). This fact has proved to be crucial in the subsequent investigations. The main results of the section are Theorems 3.7 and 3.8, which describe the weak*–closed ideals in (A, A_s), and Theorem 3.12, which completely characterizes them for a wide class of nest algebras.

2   Preliminaries

In this section, it is our aim to show that nest algebras can naturally be seen as partial Jordan *–triples. First, we establish the notation. Let H be a complex Hilbert space, let $B(H)$ be the W*–algebra of bounded linear operators on H, let 0 denote the zero operator and let 1 denote the identity operator.

Recall that a totally ordered family $\mathcal{N}$ of projections in $B(H)$ containing 0 and 1 is said to be a *nest*. If, furthermore, $\mathcal{N}$ is a complete sublattice of the lattice of projections in $B(H)$, then $\mathcal{N}$ is called a *complete nest*. The *nest algebra* A associated with $\mathcal{N}$ is the subalgebra of all operators a in $B(H)$ such that, for all projections p in $\mathcal{N}$,

$$(1-p)ap \ = \ 0 \,.$$

It is well–known that A is a unital weak operator closed subalgebra of $B(H)$ (cf. [5]). It is also a known fact that the symmetric part of any unital norm closed subalgebra of $B(H)$ is its self–adjoint part $A \cap A^*$, where A^* is the set of the adjoints of the operators lying in A, and that every such subalgebra possesses an intrinsic algebraic structure said to be a partial triple product ([3], Corollary 2.9). In consequence, the nest algebra A is naturally endowed with this partial triple product, which is defined as follows. The partial triple product associated to A is a mapping from $A \times A_s \times A$ to A, defined, for all a, c in A and all b in A_s, by

$$\{a\ b\ c\} \ = \ \frac{1}{2}\left(ab^*c + cb^*a\right),$$

where $A_s = A \cap A^*$ is the symmetric part of A.

It is this partial triple product that relates the nest algebra A with the concept of partial Jordan *–triple. A *partial Jordan *–triple* is an algebraic structure (B, B_s) formed by a complex vector space B, a complex vector subspace B_s of B and a mapping $(a, b, c) \mapsto \{a\ b\ c\}$ from $B \times B_s \times B$ to B that is symmetric bilinear in a and c, and conjugate linear in b, and which, furthermore, satisfies the conditions:

(i) $\{B_s\ \ B_s\ \ B_s\} \subseteq B_s$;
(ii) for all elements a of B, and all elements b, c and d of B_s,

$$[D(a, b), D(c, d)] \ = \ D(a, \{b\ c\ d\}) - D(\{a\ d\ c\}, b) \,,$$

where, for all $e \in B$ and $f \in B_s$, the symbol $D(e, f)$ denotes the linear mapping, defined on B by

$$D(e, f)g \ = \ \{e\ f\ g\}$$

(cf. [29]). It is a straightforward computation to show that the nest algebra A, together with the partial triple product defined above, is a partial Jordan *–triple.

We say that a complex subspace J of A is an *ideal* of (A, A_s) if

$$\{J\ A_s\ A\} + \{A\ J \cap A_s\ A\} \subseteq J \,.$$

In the sequel, these ideals will sometimes be referred to as *Jordan ideals*, as opposed to the ideals of the associative product. In the next section we investigate the nature of the Jordan ideals of the nest algebra.

A result due to Ringrose states that each nest is contained in a complete nest which generates the same nest algebra (cf. [24]). Since we shall be concerned mainly with the nest algebras and not with the nests themselves, henceforth only complete nests will be considered.

3 Weak*–closed Jordan ideals of nest algebras

In this section, it is shown that the associative ideals and the partial Jordan *–triple ideals coincide for nest algebras satisfying a condition on the finite rank operators. In the following, (A, A_s) denotes the partial Jordan *–triple associated to a nest algebra A in the manner described above. The main results of the section are Theorems 3.7 and 3.8, which describe the weak*–closed ideals in (A, A_s), and Theorem 3.12, which completely characterizes these ideals for a wide class of nest algebras.

For every projection p in $\mathcal{N}$, define a projection p_-, which also lies in $\mathcal{N}$, by

$$p_- \;=\; \bigvee \{q \in \mathcal{N} : q < p\}\,,$$

if p is non–zero, and p_- is zero, otherwise. The symbol $\vee$ designates the supremum taken in the lattice of projections of $B(H)$. For elements ξ and η of the Hilbert space H, we denote by $e_{\xi,\eta}$ the rank one operator $\nu \mapsto \langle \nu, \xi \rangle \eta$ defined on H.

The following lemma is essentially due to Ringrose [24], and, for that reason, its proof is omited.

Lemma 3.1 *Let a be a rank one operator in $B(H)$. Then, a lies in A if and only if there exist a projection p in $\mathcal{N}$ and elements ξ and η of H, with $p_-\xi$ equal to zero and $p\eta$ equal to η, such that*

$$a \;=\; e_{\xi,\eta}\,.$$

Moreover, the projection p can be chosen to be equal to

$$\bigwedge \{q \in \mathcal{N} : q\eta = \eta\}\,.$$

The symbol $\wedge$ designates the infimum taken in the lattice of projections of $B(H)$.

In the lemma that follows, for each element of H, we define two projections, which will prove to be very useful throughout this section.

Lemma 3.2 *Let ν be an element of H. Define the projections $\hat{p}_\nu$ and p_ν by*

$$\hat{p}_\nu \;=\; \bigvee \{p \in \mathcal{N} : p\nu = 0\}\,,$$
$$p_\nu \;=\; \bigwedge \{p \in \mathcal{N} : p\nu = \nu\}\,.$$

Then, the projections $\hat{p}_\nu$ and p_ν lie in the nest $\mathcal{N}$, the element $p_\nu\nu$ is equal to ν and the element $\hat{p}_\nu\nu$ is equal to zero.

P r o o f. It is clear that, because $\mathcal{N}$ is complete, the projections $\hat{p}_\nu$ and p_ν lie in $\mathcal{N}$. The element ν of H lies in the range of all projections used to define p_ν and, therefore, lies in the intersection of these ranges. Hence $p_\nu\nu$ and ν coincide. Similarly, it can be proved that $\hat{p}_\nu\nu$ is equal to zero. $\square$

The two following results have proved to be crucial in the characterization of the weak*–closed partial Jordan *–triple ideals of nest algebras.

Lemma 3.3 *Let J be a weak*–closed ideal in (A, A_s) and let ξ and η be non–zero elements of H such that the rank one operator $e_{\xi,\eta}$ lies in J. Then, for all elements σ and μ of H with $\hat{p}_\xi \le \hat{p}_\sigma$ and $p_\mu \le p_\eta$, the operators $e_{\xi,\mu}$ and $e_{\sigma,\eta}$ lie in J.*

P r o o f. This proof is divided into two parts. First we prove that the operators $e_{\xi,\mu}$ lie in J, and, secondly, we show that the operators $e_{\sigma,\eta}$ also lie in J.

1. Let $e_{\rho,\mu}$ be an operator in the nest algebra A. Then, the triple product

$$\{e_{\xi,\eta} \, 1 \, e_{\rho,\mu}\} \;=\; \frac{1}{2}\left(\langle\mu,\xi\rangle e_{\rho,\eta} + \langle\eta,\rho\rangle e_{\xi,\mu}\right) \tag{3.1}$$

lies in J. We shall analyze below the two possibilities $p_\mu < p_\eta$ and $p_\mu = p_\eta$.

Suppose that $p_\mu < p_\eta$, and let ρ be an element in the range of $\left(1 - p_\mu\right)$ such that the inner product $\langle\eta,\rho\rangle$ is non–zero. The equation (3.1) shows that $e_{\xi,\mu}$ lies in J.

Let the projection p_μ be equal to the projection p_η. The closed subspace $p_\eta(H)$ can be decomposed into the direct sum

$$p_\eta(H) \;=\; \mathrm{span}\langle\{\eta\}\rangle \oplus (\mathrm{span}\langle\{\eta\}\rangle)^\perp \,,$$

where the orthogonal complement is taken in the Hilbert space $p_\eta(H)$. The element μ has the unique decomposition

$$\mu = \mu_p + \mu_o \,,$$

where μ_p lies in $\mathrm{span}\langle\{\eta\}\rangle$ and μ_o lies in the $\left(\mathrm{span}\langle\{\eta\}\rangle\right)^\perp$. Then,

$$2\{e_{\xi,\eta} \, 1 \, e_{\rho,\mu}\} \;=\; \langle\mu,\xi\rangle e_{\rho,\eta} + \langle\eta,\rho\rangle e_{\xi,\mu_o} + \langle\eta,\rho\rangle e_{\xi,\mu_p} \,.$$

Since e_{ξ,μ_p} lies in J, the operator

$$a \;=\; \langle\mu,\xi\rangle e_{\rho,\eta} + \langle\eta,\rho\rangle e_{\xi,\mu_o} \tag{3.2}$$

must also lie in J. Let ω be such that the operator $e_{\xi,\omega}$ lies in A. Then, the triple product

$$2\{a \, 1 \, e_{\xi,\omega}\} \;=\; \langle\eta,\xi\rangle\langle\mu,\xi\rangle e_{\rho,\omega} + \langle\mu_o,\xi\rangle\langle\eta,\rho\rangle e_{\xi,\omega} + \langle\omega,\rho\rangle\langle\mu,\xi\rangle e_{\xi,\eta} + \langle\omega,\xi\rangle\langle\eta,\rho\rangle e_{\xi,\mu_o}$$

lies in J. Therefore, the operator

$$\langle\eta,\xi\rangle\langle\mu,\xi\rangle e_{\rho,\omega} + \langle\mu_o,\xi\rangle\langle\eta,\rho\rangle e_{\xi,\omega} + \langle\omega,\xi\rangle\langle\eta,\rho\rangle e_{\xi,\mu_o}$$

also lies in J. If η is not orthogonal to ξ, let ρ be equal to ξ and let ω be equal to η. Using this in the above expression, we conclude that $e_{\xi,\mu}$ lies in the ideal J. When $\langle\eta,\xi\rangle$ is zero, we consider separately the cases $(p_\eta)_- < p_\eta$ and $(p_\eta)_- = p_\eta$.

Suppose that $(p_\eta)_- < p_\eta$, and let ρ be an element in the range of the projection $(1 - (p_\eta)_-)$ such that the inner product $\langle\eta,\rho\rangle$ is non–zero. If $\langle\mu_o,\xi\rangle$ is non–zero, let ω be equal to μ_o. Then it is clear that $e_{\xi,\mu}$ lies in the Jordan ideal J. If μ_o is orthogonal to ξ, then $\langle\mu,\xi\rangle$ is equal to zero and the equality (3.2) shows that $e_{\xi,\mu}$ lies in J.

Suppose now that the projection $(p_\eta)_-$ is equal to p_η. Then, the subspace $p_\eta(H)$ is the norm–closure of the subspace

$$M \;=\; \mathrm{span}\left\langle \bigcup_{q\in\mathcal{N},\, q<p_\eta} q(H) \right\rangle .$$

Therefore, there exists a sequence (μ_n) in the space M which converges to μ in the norm topology. We shall show that the sequence $\left(e_{\xi,\mu_n}\right)$ converges to $e_{\xi,\mu}$ in the weak*–topology. Observe that, because $p_{\mu_n} < p_\eta$, for all n, the operator e_{ξ,μ_n} lies in J. Let (η_k) and (ω_k) be sequences in H such that the series $\sum_{k=1}^\infty \|\eta_k\|^2$ and $\sum_{k=1}^\infty \|\omega_k\|^2$ converge. Then, using the Schwarz inequality,

$$\left|\sum_{k=1}^m \langle\eta_k,\xi\rangle\langle\mu_n - \mu,\omega_k\rangle\right| \;\leq\; \sum_{k=1}^m \|\eta_k\|\,\|\xi\|\,\|\mu_n - \mu\|\,\|\omega_k\| \;=\; \|\xi\|\,\|\mu_n - \mu\| \sum_{k=1}^m \|\eta_k\|\,\|\omega_k\| \,.$$

Since the series $\sum_{k}^{\infty} \|\eta_k\| \|\omega_k\|$ converges, it follows that the sequence $e_{\xi,\mu_n-\mu}$ converges to zero in the weak*–topology and, hence, that $e_{\xi,\mu}$ lies in J.

2. Let ω be such that $e_{\sigma,\omega}$ lies in A. Then, the triple product

$$\{e_{\xi,\eta} \; 1 \; e_{\sigma,\omega}\} \;=\; \frac{1}{2} \left(\langle\omega,\xi\rangle (e_{\sigma,\eta}) + \langle\eta,\sigma\rangle (e_{\xi,\omega})\right)$$

lies in J. If $\hat{p}_\xi < \hat{p}_\sigma$, then $p_\eta \leq \hat{p}_\sigma$, and the inner product $\langle\eta,\sigma\rangle$ is zero. If we put ω equal to $\hat{p}_\sigma\xi$, then $\langle\omega,\xi\rangle$ is non–zero and $e_{\sigma,\eta}$ lies in J. Now, suppose that the projections $\hat{p}_\sigma$ and $\hat{p}_\xi$ coincide. We study separately the cases where $\hat{p}_\xi < p_\eta$ and $p_\eta \leq \hat{p}_\xi$. In the first case, let ω be equal to $p_\eta\xi$. Then, the inner product $\langle\omega,\xi\rangle$ is non–zero and, because $e_{\xi,\omega}$ lies in J, the operator $e_{\sigma,\eta}$ also lies in this ideal.

In the second case,

$$p_\eta \;\leq\; \hat{p}_\xi \;=\; \hat{p}_\sigma \,,$$

and, therefore, η and σ are orthogonal. Define the projection p in the nest $\mathcal{N}$ by

$$p \;=\; \bigwedge_{q \in \mathcal{N},\, \hat{p}_\xi < q} q \,.$$

If $\hat{p}_\xi < p$, then let ω be equal to $p\xi$. Then, $\langle\omega,\xi\rangle$ is non–zero and $e_{\sigma,\eta}$ lies in the ideal J. If p is equal to $\hat{p}_\xi$, then

$$1 - \hat{p}_\xi \;=\; \bigvee_{q \in \mathcal{N},\, \hat{p}_\xi < q} (1 - q) \,.$$

Therefore, since σ lies in the range of the projection $1 - \hat{p}_\xi$, there exists a sequence (σ_n) in the subspace

$$N \;=\; \operatorname{span}\left\langle \bigcup_{q \in \mathcal{N},\, \hat{p}_\xi < q} (1 - q)(H) \right\rangle$$

of H such that the sequence (σ_n) converges to σ in the norm topology. It is clear that, for each n, there exists a projection q_n in the nest $\mathcal{N}$ with $\hat{p}_\xi < q_n$, such that σ_n lies in the range of the projection $1 - q_n$. Hence, because $\hat{p}_\xi < \hat{p}_{\sigma_n}$, the operator $e_{\sigma_n,\eta}$ lies in J. It is easy to see that the sequence $(e_{\sigma_n,\eta})$ converges to $e_{\sigma,\eta}$ in the weak*–topology. In consequence, $e_{\sigma,\eta}$ lies in J . $\square$

Lemma 3.3 is generalized by the following theorem.

Theorem 3.4 *Let J be a weak*–closed ideal in (A, A_s) and let ξ and η be elements of H such that $e_{\xi,\eta}$ is a rank one operator in J. Then, for all elements σ and μ of H with*

$$\hat{p}_\xi \;\leq\; \hat{p}_\sigma \,, \quad p_\mu \;\leq\; p_\eta \,,$$

the operator $e_{\sigma,\mu}$ lies in J.

P r o o f. Let ω be an element of H such that $e_{\sigma,\omega}$ lies in the nest algebra A. Then, by Lemma 3.3, the triple product

$$\{e_{\xi,\mu} \; 1 \; e_{\sigma,\omega}\} \;=\; \langle\mu,\sigma\rangle e_{\xi,\omega} + \langle\omega,\xi\rangle e_{\sigma,\mu}$$

also lies in J. If ξ and η are such that $\hat{p}_\xi < p_\eta$, then let ω be equal to $p_\eta\xi$. By Lemma 3.3, the operator $e_{\xi,\omega}$ lies in J. Thus, since the inner product $\langle\omega,\xi\rangle$ is non–zero, $e_{\sigma,\mu}$ lies in J. If ξ and η are such that $p_\eta \leq \hat{p}_\xi$, then

$$p_\eta \;\leq\; \hat{p}_\xi \;\leq\; \hat{p}_\sigma \,,$$

and, thus, the inner product $\langle \mu, \sigma \rangle$ is equal to zero. Suppose that $\hat{p}_\xi < \hat{p}_\sigma$, and let ω be equal to $\hat{p}_\sigma \xi$. Then, $\langle \omega, \xi \rangle$ is non–zero and $e_{\sigma,\mu}$ lies in J. If the projection $\hat{p}_\sigma$ is equal to the projection $\hat{p}_\xi$, then define the projection

$$p \;=\; \bigwedge_{q \in \mathcal{N}, \, \hat{p}_\xi < q} q \,.$$

Suppose that $\hat{p}_\xi < p$, and let ω be equal to $p\,\xi$. Hence, since the inner product $\langle \omega, \xi \rangle$ is non–zero, the operator $e_{\sigma,\mu}$ lies in J. Now, suppose that the projections p and $\hat{p}_\xi$ coincide. Then, the element σ of H lies in the range of the projection

$$1 - p \;=\; \bigvee_{q \in \mathcal{N}, \, \hat{p}_\xi < q} (1 - q) \,.$$

Therefore, there exists a sequence (σ_n) and, for each n, a projection q_n in the nest $\mathcal{N}$ satisfying $\hat{p}_\xi < q_n$ such that σ_n lies in the subspace $(1-q_n)(H)$, and the sequence (σ_n) converges to σ in the norm topology. Since, for all n, the operator $e_{\sigma_n,\mu}$ lies in J, it is easy to see that $e_{\sigma,\mu}$ must also lie in J. □

Lemma 3.5 *Let p and $\tilde{p}$ be projections in $\mathcal{N}$, and let B be the set defined by*

$$B \;=\; \{ a \in A : a = pa(1 - \tilde{p}) \} \,.$$

Then, the set B is a weak*–closed ideal in (A, A_s) and the subspace spanned by the rank one operators in B is weak*–dense in B.

P r o o f. Clearly B is a weak*–closed subspace of the nest algebra A. Let a be an operator in B. Then, for all operators b in $A \cap A^*$ and all operators c in A, the triple product $\{a\,b\,c\}$ satisfies

$$
\begin{aligned}
\{a\,b\,c\} \;&=\; \frac{1}{2}\left(ab^*c + cb^*a\right) \\
&=\; \frac{1}{2}\left(pa(1-\tilde{p})b^*c + cb^*pa(1-\tilde{p})\right) \\
&=\; \frac{1}{2}\left(pab^*(1-\tilde{p})c + cpb^*a(1-\tilde{p})\right),
\end{aligned}
$$

since the nest $\mathcal{N}$ is contained in the centre of $A \cap A^*$ (cf. [5]). Then, the triple product

$$
\begin{aligned}
\{a\,b\,c\} \;&=\; \frac{1}{2}\left(pab^*(1-\tilde{p})c(1-\tilde{p}) + pcpb^*a(1-\tilde{p})\right) \\
&=\; \frac{1}{2}\,p\left(ab^*(1-\tilde{p})c + cpb^*a\right)(1-\tilde{p})
\end{aligned}
$$

lies in the subspace B. The remaining part of the definition is proved similarly and, hence, B is a Jordan ideal.

We shall prove that the subspace spanned by the rank one operators in B is weak*–dense in B. Let a be an element of B. Then there exists a net (a_j) in the subspace spanned by the rank one operators in the nest algebra A converging to a in the weak*–topology ([5], Corollary 3.13). Let (η_k) and (ω_k) be sequences in the complex Hilbert space H such that $\sum_{k=1}^{\infty} \|\eta_k\|^2$ and $\sum_{k=1}^{\infty} \|\omega_k\|^2$ converge. Then,

$$\sum_{k=1}^{n} \left\langle \left(pa_j(1-\tilde{p}) - a\right)\eta_k, \omega_k \right\rangle \;=\; \sum_{k=1}^{n} \left\langle p\left(a_j - a\right)(1-\tilde{p})\eta_k, \omega_k \right\rangle,$$

which shows that the net $\left(pa_j(1-\tilde{p})\right)$, defined in the span of the rank one operators in the Jordan ideal B, converges to a in the weak*–topology. □

Lemma 3.6 *Let ξ and η be elements of H such that $e_{\xi,\eta}$ is a non–zero operator in the nest algebra A. Then, for all elements σ and μ of H such that $e_{\sigma,\mu}$ is a non–zero operator in A, the following conditions are equivalent:*

(i) $\hat{p}_\xi \leq \hat{p}_\sigma$ *and* $p_\mu \leq p_\eta$;

(ii) $p_\eta e_{\sigma,\mu}\big(1 - \hat{p}_\xi\big) = e_{\sigma,\mu}$.

P r o o f. Let ξ and η satisfy the conditions of the lemma, and let σ and μ be such that $e_{\sigma,\mu}$ lies in the nest algebra A with

$$\hat{p}_\xi \ \leq \ \hat{p}_\sigma \,, \quad p_\mu \ \leq \ p_\eta \,.$$

Then,

$$p_\eta e_{\sigma,\mu}\big(1 - \hat{p}_\xi\big) \ = \ e_{(1-\hat{p}_\xi)\sigma, p_\eta \mu} \ = \ e_{\sigma,\mu} \,.$$

Conversely, if σ and μ are such that

$$p_\eta e_{\sigma,\mu}\big(1 - \hat{p}_\xi\big) \ = \ e_{\sigma,\mu} \,,$$

then

$$e_{(1-\hat{p}_\xi)\sigma, p_\eta \mu} \ = \ e_{\sigma, p_\eta \mu} + e_{\sigma,(1-p_\eta)\mu} \,,$$
$$e_{-\hat{p}_\xi \sigma, p_\eta \mu} \ = \ e_{\sigma,(1-p_\eta)\mu} \,.$$

Hence, we can conclude that $\hat{p}_\xi \leq \hat{p}_\sigma$, and that $p_\mu \leq p_\eta$. □

Theorem 3.7 *Let $e_{\xi,\eta}$ be a rank one operator in the nest algebra A. Then, the subspace B of A defined by*

$$B \ = \ \big\{a \in A : p_\eta a\big(1 - \hat{p}_\xi\big) = a\big\}$$

is the weak*–closed partial Jordan *–triple ideal generated by $e_{\xi,\eta}$.

P r o o f. By Lemma 3.5, B is a weak*–closed partial Jordan *–triple ideal which coincides with the weak*–closure of the subspace spanned by the rank one operators that belong to B. By Lemma 3.6, a rank one operator $e_{\sigma,\mu}$ lies in B if and only if

$$\hat{p}_\xi \ \leq \ \hat{p}_\sigma \,, \quad p_\mu \ \leq \ p_\eta \,.$$

Theorem 3.4 shows that all of these rank one operators must lie in every weak*–closed Jordan ideal containing $e_{\xi,\eta}$. Hence, B is the least weak*–closed Jordan ideal containing $e_{\xi,\eta}$. □

Let $p \mapsto p'$ be an order homomorphism on $\mathcal{N}$. The homomorphism $p \mapsto p'$ is said to be *left order continuous* if, for all subsets $\mathcal{M}$ of the nest $\mathcal{N}$, the projection $(\bigvee \mathcal{M})'$ is equal to the supremum $\bigvee \mathcal{M}'$.

Theorem 3.8 *Let J be a weak*–closed ideal in (A, A_s) and, for each p in $\mathcal{N}$, let the projection p' be defined by*

$$p' \ = \ \bigvee\big\{p_\eta : e_{\xi,\eta} \in J, \ \hat{p}_\xi < p\big\} \,.$$

Then, a rank one operator a lies in J if and only if, for all projections p in $\mathcal{N}$,

$$(1 - p')ap \ = \ 0 \,,$$

and the mapping $p \mapsto p'$ is a left order continuous homomorphism on the nest $\mathcal{N}$.

P r o o f. Let p and q be projections of the nest $\mathcal{N}$ such that $p < q$. We shall show that $p' \leq q'$. Suppose that there exists an operator $e_{\sigma,\mu}$ in J such that $\hat{p}_\sigma < p$ and $q' < p_\mu$. Then, by Theorem 3.4, all operators $e_{\rho,\omega}$, such that $\hat{p}_\sigma \leq \hat{p}_\rho$ and $p_\omega \leq p_\mu$, lie in J. Hence,

$$q' \;<\; \bigvee \left\{ p_\eta : e_{\xi,\eta} \in J, \; \hat{p}_\xi < q \right\},$$

yielding a contradiction.

Let $p \mapsto p'$ be the order homomorphism defined in the theorem. Suppose that there exists a rank one operator $e_{\sigma,\mu}$ in J such that, for some projection p in the nest $\mathcal{N}$, the operator

$$(1 - p')e_{\sigma,\mu}p \;=\; e_{p\sigma,(1-p')\mu}$$

is non–zero. Then, $p\sigma$ and $(1-p)\mu$ also are non–zero. Therefore, $\hat{p}_\sigma < p$ and $p' < p_\mu$. Hence, there exists an operator $e_{\xi,\eta}$ in J with $p' < p_\eta$ and $\hat{p}_\xi < p$, which contradicts the hypothesis.

Conversely, let $e_{\sigma,\mu}$ be a rank one operator in the nest algebra A satisfying, for all projections p in the nest $\mathcal{N}$, the equality

$$(1 - p')e_{\sigma,\mu}p \;=\; 0 \,.$$

Since $p\sigma$ is non–zero, for all projections p such that $\hat{p}_\sigma < p$, the element $(1 - p')\mu$ is equal to zero and μ lies in $p'(H)$. Suppose that

$$\hat{p}_\sigma \;<\; q \;=\; \bigwedge_{p \in \mathcal{N},\, \hat{p}_\sigma < p} p \,.$$

Then, by Theorem 3.4 and the definition of the mapping $p \mapsto p'$, for all elements ρ of the space $(1 - \hat{p}_\sigma)(H)$ and all elements ω of H such that $p_\omega < q'$, the operators $e_{\rho,\omega}$ lie in J. If $q'_- < q'$, then the projection q' lies in the set

$$\left\{ p_\eta : e_{\xi,\eta} \in J, \; \hat{p}_\xi < q \right\} .$$

Then, by Theorem 3.4 , the operator $e_{\sigma,\mu}$ lies in J. If the projection q'_- coincides with the projection q', then there exists a sequence (ω_n) in the space

$$L \;=\; \operatorname{span} \left\langle \bigcup_{p \in \mathcal{N},\, p < q'} p(H) \right\rangle$$

which converges to μ in the norm topology. Then, (e_{σ,ω_n}) is a sequence of rank one operators in J and $e_{\sigma,\mu}$ is the weak*–limit of this sequence. Therefore, $e_{\sigma,\mu}$ lies in J.

Now, suppose that

$$\hat{p}_\sigma \;=\; \bigwedge_{p \in \mathcal{N},\, \hat{p}_\sigma < p} p \,,$$

and let p be a projection in the nest $\mathcal{N}$ such that $\hat{p}_\sigma < p$. By the definition of the order homomorphism $p \mapsto p'$ and Theorem 3.4, it can be seen that, for all elements ρ with $p \leq \hat{p}_\rho$ and all elements ω such that $p_\omega < p'$, the operator $e_{\rho,\omega}$ lies in J. We shall show that, for all elements ω of H such that the projections p_ω and p' coincide, and all elements ρ satisfying the conditions above, $e_{\rho,\omega}$ also lies in J. If $p'_- < p'$, then the projection p' belongs to the set

$$\left\{ p_\eta : e_{\xi,\eta} \in J, \; \hat{p}_\xi < p \right\}$$

and, in consequence, since $\hat{p}_\sigma < p \leq \hat{p}_\rho$, and $p_\omega = p'$, the operator $e_{\rho,\omega}$ lies in J. Suppose that the projection $(p')_-$ coincides with the projection p'. Then, there exists a sequence (ω_n) in the subspace

$$M \;=\; \operatorname{span} \left\langle \bigcup_{q \in \mathcal{N},\, q < p'} q(H) \right\rangle$$

of H such that (ω_n) converges to ω in the norm topology. Then, for a fixed ρ such that $p \leq \hat{p}_\rho$, the sequence (e_{ρ,ω_n}) converges to the rank one operator $e_{\rho,\omega}$ in the weak*–topology. Since, for all n, the operator e_{ρ,ω_n} lies in J, the operator $e_{\rho,\omega}$ also lies in J. Because σ belongs to $(1 - \hat{p}_\sigma)(H)$, it follows that σ lies in

$$\left(1 - \bigwedge_{p \in \mathcal{N},\, \hat{p}_\sigma < p} p \right)(H) \;=\; \left(\bigvee_{p \in \mathcal{N},\, \hat{p}_\sigma < p} (1 - p) \right)(H)\,.$$

Hence, there exists a sequence (σ_n) in the subspace

$$N \;=\; \mathrm{span}\left\langle \bigcup_{\hat{p}_\sigma < p} (1 - p)(H) \right\rangle$$

of H such that (σ_n) converges to σ in the norm topology. Then, the sequence $(e_{\sigma_n,\mu})$ converges to $e_{\sigma,\mu}$ in the weak*–topology, which shows that $e_{\sigma,\mu}$ lies in J.

Finally, we prove that the order homomorphism is left order continuous. Let $p \mapsto p'$ be the order homomorphism defined in the theorem, let $\mathcal{M}$ be a subset of the nest $\mathcal{N}$ and let the projection q be the supremum of $\mathcal{M}$. Suppose that the projection q lies in $\mathcal{M}$. Since the mapping $p \mapsto p'$ is an order homomorphism, the projection q' coincides with $\bigvee \mathcal{M}'$. Suppose now that the supremum q does not belong to $\mathcal{M}$. Since the projection q does not lie in $\mathcal{M}$, the projection q_- coincides with the projection q. Therefore, we have the equality

$$q' \;=\; \bigvee \left(\bigcup_{p \in \mathcal{N},\, p < q} \left\{ p_\eta : e_{\xi,\eta} \in J,\ \hat{p}_\xi < p \right\} \right)\,.$$

By Theorem 3.4, and because $p \mapsto p'$ is an order homomorphism,

$$q' \;=\; \bigvee \left(\bigcup_{p \in \mathcal{M}} \left\{ p_\eta : e_{\xi,\eta} \in J,\ \hat{p}_\xi < p \right\} \right) \;=\; \bigvee \{ p' : p \in \mathcal{M} \} \;=\; \bigvee \mathcal{M}'\,,$$

which concludes the proof. $\square$

A nest algebra is said to have *property S* if every weak*–closed partial Jordan *–triple ideal J of the algebra satisfies the following condition on the finite rank operators: For all positive integers n, if a is a rank n operator in J, then a may be written as the sum of n rank one operators in J.

The examples below present three types of nest algebras having property S. The first example is a special case of the third example, but we chose to begin with Example 3.9 because the idea of its construction is easily generalized to that of Example 3.10.

Example 3.9 Let A_m be the algebra of $m \times m$ upper triangular complex matrices. This algebra represents the operators defined on a finite dimensional complex Hilbert space whose matrices relative to a fixed orthonormal basis $\{\xi_1, \xi_2, \ldots, \xi_m\}$ are upper triangular. The complete nest $\mathcal{N}$ corresponding to A_m is

$$0 \;<\; p_1 \;<\; \ldots \;<\; p_m \;=\; 1\,,$$

where, for all integers l in the set $\{1, 2, \ldots, m\}$,

$$p_l \;=\; \sum_{i=1}^{l} e_{\xi_i, \xi_i}\,.$$

Let J be a Jordan ideal in A_m and let a be an operator in J of rank n greater than one. Define the order homomorphism $p \mapsto \breve{p}$, on the nest $\mathcal{N}$, by

$$\breve{p} \;=\; \bigwedge \{ q \in \mathcal{N} : (1 - q)ap = 0 \}\,.$$

By [8], Lemma 1.2, there exist n rank one operators e_{ρ_k,ω_k} such that

$$a \;=\; \sum_{k=1}^{n} e_{\rho_k,\omega_k}$$

and

$$(1 - \breve{p})e_{\rho_k,\omega_k}p \;=\; 0\,,$$

for all projections p in the nest $\mathcal{N}$ and for all integers k in the set $\{1,\ldots,n\}$. We shall show that, if t_{ij} is a non–zero entry in the matrix of a, then e_{ξ_j,ξ_i} lies in J. Let b be the operator defined by

$$b \;=\; 2\left\{a \quad 1 \quad e_{\xi_j,\xi_j}\right\}\,.$$

Then, the operator

$$b \;=\; \left(\sum_{k=1}^{n} e_{\rho_k,\omega_k}\right)e_{\xi_j,\xi_j} + e_{\xi_j,\xi_j}\left(\sum_{k=1}^{n} e_{\rho_k,\omega_k}\right) \;=\; \sum_{k=1}^{n}\langle \xi_j,\rho_k\rangle e_{\xi_j,\omega_k} + \sum_{k=1}^{n}\langle \omega_k,\xi_j\rangle e_{\rho_k,\xi_j}$$

lies in J. Let the operator c be defined by

$$c \;=\; 2\left\{b \quad 1 \quad e_{\xi_i,\xi_i}\right\}\,.$$

Then,

$$
\begin{aligned}
c \;=\;& \left(\sum_{k=1}^{n}\langle \xi_j,\rho_k\rangle e_{\xi_j,\omega_k} + \sum_{k=1}^{n}\langle \omega_k,\xi_j\rangle e_{\rho_k,\xi_j}\right)e_{\xi_i,\xi_i} \\
& + e_{\xi_i,\xi_i}\left(\sum_{k=1}^{n}\langle \xi_j,\rho_k\rangle e_{\xi_j,\omega_k} + \sum_{k=1}^{n}\langle \omega_k,\xi_j\rangle e_{\rho_k,\xi_j}\right) \\
=\;& \sum_{k=1}^{n}\langle \xi_j,\rho_k\rangle\langle \xi_i,\xi_j\rangle e_{\xi_i,\omega_k} + \left(\sum_{k=1}^{n}\langle \xi_i,\rho_k\rangle\langle \omega_k,\xi_j\rangle\right)e_{\xi_i,\xi_j} \\
& + \left(\sum_{k=1}^{n}\langle \xi_i,\rho_k\rangle\langle \omega_k,\xi_i\rangle\right)e_{\xi_j,\xi_i} + \sum_{k=1}^{n}\langle \omega_k,\xi_j\rangle\langle \xi_j,\xi_i\rangle e_{\rho_k,\xi_i}\,,
\end{aligned}
$$

and this operator lies in J. Suppose that $i < j$. Since ξ_i and ξ_j are orthogonal and e_{ξ_i,ξ_j} does not belong to the nest algebra,

$$c \;=\; \left(\sum_{k=1}^{n}\langle \xi_j,\rho_k\rangle\langle \omega_k,\xi_i\rangle\right)e_{\xi_j,\xi_i} \;=\; t_{ij}e_{\xi_j,\xi_i}\,,$$

which is an operator in J. Hence, e_{ξ_j,ξ_i} lies in J and, for all integers k in the set $\{1,\ldots,n\}$, the operator $\langle \xi_j,\rho_k\rangle\langle \omega_k,\xi_i\rangle e_{\xi_j,\xi_i}$ also lies in J. If i is equal to j, then

$$
\begin{aligned}
c \;=\;& \sum_{k=1}^{n}\langle \xi_i,\rho_k\rangle e_{\xi_i,\omega_k} + \left(\sum_{k=1}^{n}\langle \xi_i,\rho_k\rangle\langle \omega_k,\xi_i\rangle\right)e_{\xi_i,\xi_i} \\
& + \left(\sum_{k=1}^{n}\langle \xi_i,\rho_k\rangle\langle \omega_k,\xi_i\rangle\right)e_{\xi_i,\xi_i} + \sum_{k=1}^{n}\langle \omega_k,\xi_i\rangle e_{\rho_k,\xi_i}\,.
\end{aligned}
$$

Thus, the operator

$$c \;=\; \sum_{k=1}^{n}\langle \xi_i,\rho_k\rangle e_{\xi_i,\omega_k} + \sum_{k=1}^{n}\langle \omega_k,\xi_i\rangle e_{\rho_k,\xi_i} + 2t_{ii}e_{\xi_i,\xi_i}$$

lies in J. Hence, for i equal to j,

$$c \; = \; b + 2t_{ii}e_{\xi_i,\xi_i} \, .$$

Therefore, since b and c belong to the ideal, the rank one operator e_{ξ_i,ξ_i} must also lie in J. Suppose that there exist i, j and k such that t_{ij} is equal to zero and $\langle \xi_j, \rho_k \rangle \langle \omega_k, \xi_i \rangle e_{\xi_j,\xi_i}$ is non–zero. Since, for all projections p in the nest $\mathcal{N}$, the equality

$$(1 - \breve{p})e_{\rho_k,\omega_k}p \; = \; 0$$

holds, there must be an integer l greater than or equal to i and less than or equal to j, and such that the entry t_{lj} is non–zero. Therefore, e_{ξ_j,ξ_l} lies in the J and, hence, by Theorem 3.4, the operator e_{ξ_j,ξ_i} lies also in this ideal. We may conclude that, for all integers k with $1 \le k \le n$, and all i and j such that $1 \le i \le j \le n$, the operator $\langle \xi_j, \rho_k \rangle \langle \omega_k, \xi_i \rangle e_{\xi_j,\xi_i}$ is either zero or lies in J. In consequence, for all k such that $1 \le k \le n$,

$$e_{\rho_k,\omega_k} \; = \; \sum_{1 \le i \le j \le n} \langle \xi_j, \rho_k \rangle \langle \omega_k, \xi_i \rangle e_{\xi_j,\xi_i}$$

also lies in J. Hence, the nest algebra A_m has property S.

Example 3.10 Let H be an infinite–dimensional separable complex Hilbert space and let (p_l) be an increasing sequence of projections such that p_0 is equal to zero, the space $p_l(H)$ is finite–dimensional, for all positive integers l, and

$$\bigvee_{l=1}^{\infty} p_l \; = \; 1 \, .$$

Let A be the nest algebra associated with the complete nest $\mathcal{N}$ consisting of these projections and the identity operator 1. The operators a in A can be viewed as block upper triangular infinite matrices $\left(t_{ij}\right)_{i,j \ge 1}$, which represent the operators relative to a fixed orthonormal basis (ξ_m). Let J be a weak*–closed partial Jordan *–triple ideal in A and let a be an operator in J of rank n greater than one. Let $\widetilde{\mathcal{N}}$ be the complete nest

$$0 \; < \; q_1 \; < \; \ldots \; < \; q_l \; < \; \ldots \; < \; 1 \, ,$$

where

$$q_l \; = \; \sum_{i=1}^{l} e_{\xi_i,\xi_i} \, .$$

It is clear that the nest $\mathcal{N}$ is contained in the nest $\widetilde{\mathcal{N}}$ and, thus, the nest algebra A contains the nest algebra associated with $\widetilde{\mathcal{N}}$. Define the order homomorphism $q \mapsto \breve{q}$, on $\widetilde{\mathcal{N}}$, by

$$\breve{q} \; = \; \bigwedge \left\{ p \in \widetilde{\mathcal{N}} : (1 - p)aq = 0 \right\} .$$

By [8], Lemma 1.2, there exist n rank one operators e_{ρ_k,ω_k} such that

$$a \; = \; \sum_{k=1}^{n} e_{\rho_k,\omega_k} \, ,$$

and, for all integers k with $1 \le k \le n$, and all projections q in the nest $\widetilde{\mathcal{N}}$,

$$(1 - \breve{q})e_{\rho_k,\omega_k}q \; = \; 0 \, .$$

We shall show that, for all non–zero entries t_{ij} in the matrix of a, the operator e_{ξ_j,ξ_i} lies in J. If i is equal to j or i and j are such that e_{ξ_j,ξ_i} lies in the nest algebra A but e_{ξ_i,ξ_j} does not lie in the algebra, then e_{ξ_j,ξ_i} lies in J, as can be proved in a similar manner to that used in Example 3.9. Let i and j be such that i is not equal to j and the operators e_{ξ_j,ξ_i} and e_{ξ_i,ξ_j} lie in the algebra A, and let c be the operator in J defined by

$$c \;=\; 4\left\{\left\{a\;\;1\;\;e_{\xi_j,\xi_j}\right\}\;\;1\;\;e_{\xi_i,\xi_i}\right\}.$$

Then,

$$c \;=\; \sum_{k=1}^{n}\langle\xi_j,\rho_k\rangle\langle\xi_i,\xi_j\rangle e_{\xi_i,\omega_k} + \sum_{k=1}^{n}\langle\xi_i,\rho_k\rangle\langle\omega_k,\xi_j\rangle e_{\xi_i,\xi_j}$$
$$+ \sum_{k=1}^{n}\langle\xi_i,\rho_k\rangle\langle\omega_k,\xi_i\rangle e_{\xi_j,\xi_i} + \sum_{k=1}^{n}\langle\omega_k,\xi_j\rangle\langle\xi_j,\xi_i\rangle e_{\rho_k,\xi_i}.$$

Hence,

$$c \;=\; \left(\sum_{k=1}^{n}\langle\xi_i,\rho_k\rangle\langle\omega_k,\xi_j\rangle\right)e_{\xi_i,\xi_j} + \left(\sum_{k=1}^{n}\langle\xi_j,\rho_k\rangle\langle\omega_k,\xi_i\rangle\right)e_{\xi_j,\xi_i}.$$

If the entry t_{ji} is equal to zero, it is immediate that e_{ξ_j,ξ_i} lies in J. On the other hand, if the entry t_{ji} is non–zero, let d be the operator defined by

$$d \;=\; 2\left\{c\;\;1\;\;e_{\xi_j,\xi_i}\right\}.$$

Then,

$$d \;=\; \left(\sum_{k=1}^{n}\langle\xi_i,\rho_k\rangle\langle\omega_k,\xi_j\rangle\right)\langle\xi_i,\xi_i\rangle e_{\xi_j,\xi_j} + \left(\sum_{k=1}^{n}\langle\xi_i,\rho_k\rangle\langle\omega_k,\xi_j\rangle\right)\langle\xi_j,\xi_j\rangle e_{\xi_i,\xi_i}$$
$$=\; \left(\sum_{k=1}^{n}\langle\xi_i,\rho_k\rangle\langle\omega_k,\xi_j\rangle\right)e_{\xi_j,\xi_j} + \left(\sum_{k=1}^{n}\langle\xi_i,\rho_k\rangle\langle\omega_k,\xi_j\rangle\right)e_{\xi_i,\xi_i},$$

which lies in J. The triple products $\{d\;1\;e_{\xi_i,\xi_i}\}$ and $\{d\;1\;e_{\xi_j,\xi_j}\}$ yield that the operators e_{ξ_i,ξ_i} and e_{ξ_j,ξ_j} are rank one operators in J. Hence, if i is less than j, then, by Theorem 3.4, the rank one operator e_{ξ_j,ξ_i} also belongs to this ideal. If i is greater than j, then

$$e_{\xi_j,\xi_i} \;=\; 2\left\{e_{\xi_j,\xi_i}\;\;1\;\;e_{\xi_j,\xi_j}\right\}$$

and it follows that e_{ξ_j,ξ_i} lies in J. Suppose that there exist i, j and k such that the entry t_{ij} is equal to zero and the operator $\langle\xi_j,\rho_k\rangle\langle\omega_k.\xi_i\rangle e_{\xi_i,\xi_i}$ is non–zero. Reasoning similar to that used in Example 3.9, but now applied to the nest $\widetilde{\mathcal{N}}$, shows that e_{ξ_j,ξ_i} belongs to J. This concludes the proof that, for all integers k such that $1\le k\le n$, the rank one operator e_{ρ_k,ω_k} lies in J. Hence, the nest algebra A has property S.

Example 3.11 Let H be an m–dimensional Hilbert space and let $\mathcal{N}$ be a nest consisting of the projections $0,p_1,\ldots,p_l,1$ on H such that

$$0 \;<\; p_1 \;<\; \ldots \;<\; p_l \;<\; 1.$$

The nest algebra associated with $\mathcal{N}$ is the algebra of $m\times m$ complex matrices a with block diagonals such that the entries of a are zero below the diagonal. The blocks in the diagonal are defined according to the projections in the nest. A process similar to that of Example 3.10 can be used to show that this algebra also has property S.

The following theorem asserts that, for nest algebras having property S, the ideals of the associative product (cf. [8]) coincide with the Jordan ideals.

Theorem 3.12 *Let $\mathcal{N}$ be a complete nest of projections on the complex Hilbert space H, let A be the nest algebra associated with $\mathcal{N}$, having property S, and let (A, A_s), where $A_s = A \cap A^*$, be the corresponding partial Jordan *–triple. Let J be a weak*–closed subspace of A. Then, the following conditions are equivalent:*

(i) *J is a weak*–closed partial Jordan *–triple ideal of A;*

(ii) *there exists a left order continuous homomorphism $p \mapsto p'$ on $\mathcal{N}$ such that*

$$J \; = \; \{a \in A : (1 - p')ap = 0, \; for\ all\ p \in \mathcal{N}\}\,.$$

P r o o f. Let J be a weak*–closed partial Jordan *–triple ideal in the nest algebra A and let $p \mapsto p'$ be the order homomorphism of Theorem 3.8. Let B be the set defined by

$$B \; = \; \{b \in A : (1 - p')bp = 0, \; for\ all\ p \in \mathcal{N}\}\,.$$

By Lemma 3.5, the set B is a weak*–closed Jordan ideal in (A, A_s). Furthermore, B is a bi–module over A under operator multiplication. Let a be an operator in the nest algebra A and let b be an operator in B, then

$$(1 - p')abp \; = \; (1 - p')a((1 - p') + p')bp \; = \; 0\,,$$

and

$$(1 - p')bap \; = \; (1 - p')b((1 - p) + p)ap \; = \; 0\,.$$

Hence, for all positive integers n, a rank n operator in the weak*–closed Jordan ideal B can be written as a sum of n rank one operators in this set (cf. [8]). By Theorem 3.8, the rank one operators in the ideals J and B are the same. Therefore, every rank n operator in B can be expressed as a sum of n rank one operators in J. Let a be an operator in B. It is known that there exists a net (c_j) of finite rank operators in the nest algebra A such that the net (c_j) converges to 1 in the weak*–topology (cf. [6]).

$$\{a\ 1\ c_j\} \; = \; \frac{1}{2}\left(c_j a + a c_j\right)$$

defines a net of finite rank operators in the weak*–closed Jordan ideal J, converging to a in the weak*–topology. Thus a is the weak*–limit of a net of finite rank operators in J, and, hence, a lies in J.

Conversely, let c be a rank n operator in the weak*–closed Jordan ideal J. Since the nest algebra A has property S, the operator c may be written as the sum of n rank one operators in J. Hence, by Theorem 3.8, all finite rank operators in J belong to the weak*–closed Jordan ideal B. Let a be an operator in J and let (c_j) be the net above. Then,

$$\{a\ 1\ c_j\} \; = \; \frac{1}{2}\left(c_j a + a c_j\right)$$

defines a net of finite rank operators in J converging to a in the weak*–topology. Therefore a is the weak*–limit of a net in the weak*–closed Jordan ideal B, and therefore a lies in B.

Reasoning similar to the proof of Lemma 3.5 can be used to show that condition (ii) implies condition (i).   □

Corollary 3.13 *Let A be a nest algebra having property S and let (A, A_s), where $A_s = A \cap A^*$, be the corresponding partial Jordan *–triple. Then, the weak*–closed Jordan ideals of (A, A_s) are also weakly closed.*

P r o o f. This result is an immediate consequence of Theorem 3.12 and [8], Corollary 1.6.   □

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