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Separate weak*-continuity of the triple product in dual real JB*-triples

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Abstract. We prove that, if E is a real JB*-triple having a predual E_*, then E_* is the unique predual of E and the triple product on E is separately $\sigma(E, E_*)-$continuous.

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1 Introduction

In last years, a special category of complex Banach spaces, called JB*-triples, has focused the attention of many researchers. Historically, JB*-triples arose in the study of bounded symmetric domains in complex Banach spaces (see [L] and [K1]) and it has been shown by Kaup [K2], that every such domain is biholomorphic to the open unit ball of a JB*-triple. Every C*-algebra is a JB*-triple in the triple product $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$ and every JB*-algebra is a JB*-triple in the triple product $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. In the context of Functional Analysis, JB*-triples arise in a natural way in the solution of the contractive projection problem for C*-algebras, concretely, the range of such a projection is a JB*-triple for a suitable triple product (see [S], [K3] and [FR1]).

We refer to ([R], [Ru] and [CM]) for recent surveys and to [U] for the general theory of JB*-triples.

Recently, a theory of real JB*-triples has been developed (see [D] , [CDRV], [DR], [BC] , [IKR], [K4] and [CGR]) extending to the real context

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many results in (complex) JB*-triples. However, the extension to the real case, of the important result proved by Barton-Timoney [BT] assuring that if E is a JB*-triple which is a dual Banach space, then E has a unique predual and the triple product on E is separately weak*-continuous, was an open problem which explicitly appears in the papers [IKR] and [CGR]. In this paper we solve this problem, so extending the above mentioned result of Barton-Timoney.

Isidro-Kaup-Rodríguez [IKR] introduce the concept of real JBW*-triple (as a real form of a complex JBW*-triple) and they have shown [IKR, Theorem 4.4] that every real JB*-triple E is a real JBW*-triple if and only if E has a predual in such a way that the triple product is separately weak*-continuous. From this, using our main result, we conclude that every dual real JB*-triple is a real JBW*-triple.

JB*-algebras and JB-algebras are real JB*-triples. If they have a unit, then the Jordan product is uniquely determined by the triple product (see [U, Proposition 19.13]) and the unit. Therefore our main result also gives the known separately weak*-continuity of the product in dual JB*-algebras and JB-algebras.

The proof of our main result (Theorem 2.11) follows several steps.

In a first step we prove that the dual of a real JB*-triple is well-framed, (Lemma 2.2). As a consequence the predual of every dual real JB*-triple, say E, is unique and every isometric bijection of E is weak*-continuous.

Once it is proved that the Peirce projections on E and the operators $L(e, e)$ and $Q(e)$ are weak*-continuous for all tripotents e in E (Proposition 2.4), it can be concluded that if E has a distinguished unitary element, then the triple product is separately weak*-continuous (Proposition 2.7).

Finally, starting from the existence of complete tripotents in dual real JB*-triples and using Peirce decomposition, we conclude the proof.

2 Main result

We recall that a complex JB*-triple is a complex Banach space B with a continuous triple product $\{.,.,.\} : B \times B \times B \to B$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, satisfying:

1. (Jordan Identity) $L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x,L(b,a)y,z\} + \{x,y,L(a,b)z\}$ for all a,b,c,x,y,z in B, where $L(a,b)x := \{a,b,x\}$;
2. The map $L(a,a)$ from B to B is an hermitian operator with spectrum ≥ 0 for all a in B;
3. $\|\{a,a,a\}\| = \|a\|^3$ for all a in B.

A real Banach space A together with a trilinear map $\{.,.,.\}: A \times A \times A \to A$ is called (see [IKR]) a real JB*-triple if there is a complex JB*-triple B and an $\mathbb{R}$-linear isometry λ from A to B such that $\lambda\{x, y, z\} = \{\lambda x, \lambda y, \lambda z\}$ for all x, y, z in A.

Real JB*-triples are essentially the closed real subtriples of complex JB*-triples and, by [IKR, Proposition 2.2], given a real JB*-triple A there exists a unique complex JB*-triple B and a unique conjugation (conjugate linear and isometric mapping of period 2) τ on B such that $A = B^\tau := \{x \in B : \tau(x) = x\}$. In fact, B is the complexification of the vector space A, with triple product extending in a natural way the triple product of A and a suitable norm.

The class of real JB*-triples includes all JB-algebras [H], all real C*-algebras [Go], and all J*B-algebras [A].

Real JB*-triples are Jordan triples. So, given a tripotent e ($\{e, e, e\} = e$) in a real JB*-triple A, there exists a decomposition of A into the eigenspaces of $L(e, e)$, known as the Peirce decomposition;

$$A = A_0(e) \oplus A_1(e) \oplus A_2(e)$$

where $A_k(e) = \{x \in A : L(e, e)x = \frac{k}{2}x\}$ for $k = 0, 1, 2$.

$A_k(e)$ is called the Peirce k-space of e. Peirce k-spaces satisfy the following multiplication rules:

1. $\{A_i(e), A_j(e), A_k(e)\} \subseteq A_{i-j+k}(e)$, where $i, j, k = 0, 1, 2$ and $A_l(e) = 0$ for $l \neq 0, 1, 2$.
2. $\{A_0(e), A_2(e), A\} = \{A_2(e), A_0(e), A\} = 0$.

These rules are known as Peirce arithmetic. In particular, Peirce k-spaces are subtriples.

The projection $P_k(e)$ of A onto $A_k(e)$ is called the Peirce k-projection of e. These projections are given by

$$P_2(e) = Q(e)^2;$$
$$P_1(e) = 2(L(e, e) - Q(e)^2);$$
$$P_0(e) = Id_A - 2L(e, e) + Q(e)^2;$$

where $Q(e)x = \{e, x, e\}$.

If X is a dual Banach space with predual X_*, we will denote by w^* the $\sigma(X, X_*)$ topology in X. The next Lemma summarizes some important results on well-framed Banach spaces, relevant to our purpose. We refer to [G] for a detailed presentation of the well-framed property and the proof of the next Lemma.

Lemma 2.1 *[G, Th. 15 and Th. 16]*

Let X be a real or complex Banach space, then

1. *If X is well-framed, then X is the unique predual of X^*. Furthermore, every isometric bijection on X^* is w^*-continuous.*
2. *If X is well-framed, then so is any closed linear subspace of X.*

In the proof of [BT, Theorem 2.1] it is shown that the dual B^* of a complex JB*-triple B is well-framed. The next lemma shows that this is still true for real JB*-triples.

Lemma 2.2 *The dual of a real JB*-triple is well-framed.*

Proof. Let A be a real JB*-triple and suppose that B is a complex JB*-triple such that $A = B^\tau$, where τ is a conjugation on B. Then $\tau^* : B^* \to B^*$ defined by $(\tau^* f)x := \overline{f\tau(x)}$, for all f in B^* and x in B, is a conjugation on B^*. Furthermore, the map $f \mapsto f\,|_{B^\tau}$ is an isometric bijection between $(B^*)^{\tau^*}$ and $(B^\tau)^*$, hence $A^* = (B^*)^{\tau^*}$ is a real subspace of B^*. It is known [IR, Lemma 1.4] that if X is a well-framed complex Banach space, then its underlying real Banach space $X_{\mathbb{R}}$ is well-framed, too. Hence $(B^*)_{\mathbb{R}}$ is well-framed, so A^* is well-framed too by Lemma 2.1, 2. □

The following proposition is a first application of Godefroy´s theory of well-framed Banach spaces to dual real JB*-triples (that is real JB*-triples which are dual Banach spaces).

Proposition 2.3 *Let E be a real JB*-triple with a predual E_*. Then*

1. *E_* is the unique predual of E and every isometric bijection on E is w^*-continuous.*
2. *The operator $L(a,b) - L(b,a)$ on E is w^*-continuous for all a, b in E.*

Proof. 1. By the above Lemma, E^* is well-framed. Since E_* is a subspace of E^*, Lemma 2.1 gives the first assertion.
2. Let a, b in E. It is known [IKR, proposition 2.5] that $\exp(t(L(a,b) - L(b,a)))$ is an isometric bijection on E, for all t in $\mathbb{R}$. Hence by the first assertion it is w^*-continuous. Now the operator

$$L(a,b) - L(b,a) = \lim_{t\to 0} \frac{\exp\left(t\left(L(a,b) - L(b,a)\right)\right) - Id_E}{t}$$

is w^*-continuous, because the set of all w^*-continuous operators on E is norm-closed in the Banach space of all bounded linear operators on E. □

We recall that if Y is a w^*-closed subspace of a Banach dual space X, then Y is a Banach dual space (with predual $X_*/Y_\circ$). Furthermore $\sigma(Y, Y_*)$ and $\sigma(X, X_*)\,|_Y$ are the same topology on Y. On the other hand if e is a tripotent in a complex JB*-triple B, then e is a tripotent in the subtriple

$B_2(e)$ such that $L(e,e)$ is the identity map on $B_2(e)$ (i.e. e is a unitary element in $B_2(e)$). Therefore $B_2(e)$ is a unital JB*-algebra with product $x \circ y = \{x,e,y\}$ and involution $x^* = Q(e)x$ ([BKU, Theorem 2.2] and [KU, Theorem 3.7], see also [U, Proposition 19.13]).

The next proposition shows that the triple product in a dual real JB*-triple is separately w^*-continuous if we fix the same tripotent in two variables.

Proposition 2.4 *Let E be a dual real JB*-triple and e a tripotent in E. Then the Peirce projections, $L(e,e)$ and $Q(e)$ are w^*-continuous operators on E.*

Proof. Let B a complex JB*-triple and τ a conjugation on B such that $E = B^\tau$. First we observe that every tripotent in E is a tripotent in B and the restrictions to E of Peirce projections on B are the Peirce projections on E.

For every $\varepsilon \in \mathbb{C}$ let $S_\varepsilon := S_\varepsilon(e) = \sum_{k=0}^{2} \varepsilon^k P_k(e)$. Then S_ε is an isometric automorphism of B if $|\varepsilon| = 1$ by [FR2, Lemma 1.1]. Then $S_{\pm 1}$ are isometries of E and hence w^*-continuous. Therefore $P_1(e) = (S_1 - S_{-1})/2$ is w^*-continuous and the subtriple $E_2(e) + E_0(e)$ is w^*-closed in E. But S_i and $P_0(e) - P_2(e)$ have the same restriction to $E_2(e) + E_0(e)$. This implies that $P_0(e), P_2(e)$ and $L(e,e) = P_2(e) + \frac{1}{2}P_1(e)$ are w^*-continuous. The restriction of $Q(e)$ to the w^*-closed subtriple $E_2(e)$ is isometric and hence is w^*-continuous on E. $\square$

Following [IKR] a real JBW*-triple is a real JB*-triple E such that $E = B^\tau$ for a dual complex JB*-triple (JBW*-triple) B and a conjugation τ on B.

From [IKR, Theorem 4.4] E is a real JBW*-triple if an only if E has a predual E_* in such a way that the triple product is separately w^*-continuous. In this paper we prove that every dual real JB*-triple is a real JBW*-triple. Concretely we will prove that in every dual real JB*-triple the triple product is separately w^*-continuous.

The following Proposition is a first approach to our purpose.

Proposition 2.5 *Let E be a dual real JB*-triple. Suppose that for every a in E and for every $\varepsilon > 0$, there exists a family $\{e_1, ..., e_n\}$ of pairwise orthogonal tripotents and $\lambda_1, ..., \lambda_n$ in $\mathbb{R}$, such that $\left\| a - \sum_{i=1}^{n} \lambda_i e_i \right\| < \varepsilon$. Then the triple product of E is separately w^*-continuous.*

Proof. Let $a \in E$ and $1 > \varepsilon > 0$. Then by hypothesis, there exists a family $\{e_1, ..., e_n\}$ of pairwise orthogonal tripotents and $\lambda_1, ..., \lambda_n$ in $\mathbb{R}$, such that

$$\|a - a_n\| < \frac{\varepsilon}{2(1 + \|a\|)}$$

for $a_n := \sum_{i=1}^{n} \lambda_i e_i$. By orthogonality $L(a_n, a_n) = \sum_{i=1}^{n} \lambda_i^2 L(e_i, e_i)$. Hence $L(a_n, a_n)$ is w^*-continuous by Proposition 2.4. Since

$$\begin{aligned}\|L(a,a) - L(a_n,a)\| &= \|L(a-a_n,a)\| \leq \|a-a_n\| \, \|a\| \\ &< \frac{\varepsilon}{2(1+\|a\|)}(1+\|a\|)\end{aligned}$$

and

$$\begin{aligned}\|L(a_n,a_n) - L(a_n,a)\| &= \|L(a_n,a_n-a)\| \leq \|a-a_n\| \, \|a_n\| \\ &< \frac{\varepsilon}{2(1+\|a\|)}(1+\|a\|)\end{aligned}$$

(where we have used that $\|a-a_n\| < \varepsilon \Rightarrow \|a_n\| < \varepsilon + \|a\| < 1 + \|a\|$). It follows that

$$\begin{aligned}\|L(a,a) - L(a_n,a_n)\| &\leq \|L(a,a) - L(a_n,a)\| \\ &\quad + \|L(a_n,a_n) - L(a_n,a)\| \\ &< \varepsilon.\end{aligned}$$

This implies that $L(a,a)$ is in the norm closure of the set of all w^*-continuous operators and hence is w^*-continuous for all a in E. In particular $L(a,b) + L(b,a) = L(a+b,a+b) - L(a,a) - L(b,b)$ is w^*-continuous. Now by using Proposition 2.3, 2., we have $L(a,b)$ is w^*-continuous for all a, b in E.

It is known [IKR, Lemma 3.6] that e_i, e_j are orthogonal tripotents if and only if $e_i \pm e_j$ are tripotents. Therefore, if e_i, e_j are orthogonal tripotents, then $Q(e_i + e_j)$ is w^*-continuous by Proposition 2.4. Thus

$$\begin{aligned}Q(a_n, a_n) &= \sum_{i,j=1}^{n} \lambda_i \lambda_j Q(e_i, e_j) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \lambda_i \lambda_j (Q(e_i + e_j) - Q(e_i) - Q(e_j))\end{aligned}$$

is w^*-continuous.

Again

$$\begin{aligned}\|Q(a) - Q(a_n,a)\| &= \|Q(a-a_n,a)\| \leq \|a-a_n\| \, \|a\| \\ &< \frac{\varepsilon}{2(1+\|a\|)}(1+\|a\|)\end{aligned}$$

and

$$\begin{aligned}\|Q(a_n) - Q(a_n,a)\| &= \|Q(a_n,a_n-a)\| \leq \|a-a_n\| \, \|a_n\| \\ &< \frac{\varepsilon}{2(1+\|a\|)}(1+\|a\|),\end{aligned}$$

so

$$\|Q(a) - Q(a_n)\| < \varepsilon.$$

Hence $Q(a)$ is w^*-continuous for all a in E. Finally

$$Q(a,b) = \frac{1}{2}(Q(a+b) - Q(a) - Q(b))$$

is w^*-continuous for all a, b in E. □

If E is a real JB*-triple there exists a complex JB*-triple B and a conjugation τ on B such that $E = B^\tau$. Let e be a tripotent in E, as we have commented before, $B_2(e)$ is a JB*-algebra. Therefore $A(e) := \{x \in E_2(e) : Q(e)x = x\}$ is a JB-algebra as a closed subalgebra of the JB-algebra $\{x \in B_2(e) : Q(e)x = x^* = x\}$.

We are going to show that if E is a dual real JB*-triple and e is a tripotent in E, then every element in $A(e)$ can be approximated by finite linear combinations of pairwise orthogonal tripotents. An argument similar to that in the proof of Proposition 2.5 then shows that $L(a,b)$ and $Q(a,b)$ are w^*-continuous for all a, b in $A(e)$.

Proposition 2.6 *Let E be a dual real JB*-triple and e a tripotent in E. Then $L(a,b)$ and $Q(a,b)$ are w^*-continuous for all a, b in $A(e)$.*

Proof. $A(e)$ is a JB-algebra and since $Q(e)$ is w^*-continuous, then $A(e)$ is w^*-closed in E. Therefore $A(e)$ is a JBW-algebra [H, Theorem 4.4.16]. Again by [H, Lemma 4.1.11] if $a \in A(e)$, then the w^*-closure of the subalgebra generated by a, $W(a)$, is isometrically isomorphic to a monotone complete $C(X)$ where X is a compact Hausdorff space and for all $\varepsilon > 0$, there exist pairwise orthogonal idempotents $e_1, ..., e_n$ in $W(a)$ and $\lambda_1, ..., \lambda_n$ in $\mathbb{R}$ such that $\left\| a - \sum_{i=1}^{n} \lambda_i e_i \right\| < \varepsilon$ [H, Proposition 4.2.3]. In fact $e_1, ..., e_n$ are pairwise orthogonal tripotents in E because $e_i \pm e_j$ are tripotents in E for all $i \neq j$. Finally we proceed as in the proof of Proposition 2.5. □

The following result is one of the keys in the proof of our main result and gives the separate w^*-continuity of the triple product in the case that the dual real JB*-triple E has a unitary element u, i.e. $L(u,u) = Id_E$ ($E = E_2(u)$).

Proposition 2.7 *Let E be a dual real JB*-triple with a unitary element. Then the triple product is separately w^*-continuous.*

Proof. Since E is a real JB*-triple, there exists a complex JB*-triple B and a conjugation τ on B such that $E = B^\tau$. Let u be a unitary element in

E. Then u is a unitary element in B. So B is a complex JB*-triple with a unitary element u. By [BKU, Theorem 2.2] and [KU, Theorem 3.7], (see also [U, Proposition 19.13]) it follows that B is a unital JB*-algebra with product $x \circ y := \{x, u, y\}$, involution $x^* := \{u, x, u\} = Q(u)\, x$ and unit u.

Put $A := \{x \in E : Q(u)\, x = x\}$ and $D := \{x \in E : Q(u)\, x = -x\}$. Then $E = A \oplus D$, because $Q(u)$ is an involution on E.

For $M_a := L(a, u)$ the identities

$$L(a, b) = M_a M_{b^*} - M_{b^*} M_a + M_{a \circ b^*},$$

$$Q(a, b) = (M_a M_b + M_b M_a - M_{a \circ b}) Q(u)$$

for all $a, b \in E$ imply that only the w^*-continuity of every M_a, $a \in E$, has to be shown. For $a = a^*$ this follows from Proposition 2.6 and for $a^* = -a$ from the identity

$$2M_a = L(a, u) - L(u, a)$$

and Proposition 2.3, 2. □

Proposition 2.8 *Let E be a dual real JB*-triple and e a tripotent in E. Then $L(a, b)$ and $Q(a, b)$ are w^*-continuous operators on E for all a, b in $E_2(e)$.*

Proof. $E_2(e)$ is a dual real JB*-triple with a unitary element e. Then by Proposition 2.7 the triple product is separately w^*-continuous on $E_2(e)$. Therefore $E_2(e)$ is a real JBW*-triple [IKR, Theorem 4.4].

From [IKR, Proof of Theorem 4.8] it can be concluded that for all $a \in E_2(e)$ and $\varepsilon > 0$, there exist a family of pairwise orthogonal tripotents $\{e_1, ..., e_n\}$ in $E_2(e)$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that $\left\| a - \sum_{i=1}^{n} \lambda_i e_i \right\| < \varepsilon$. As in the proof of Proposition 2.5, we conclude that $L(a, b)$ and $Q(a, b)$ are w^*-continuous operators on E for all a, b in $E_2(e)$. □

The next two lemmas are needed in the proof of the Main Theorem below. We are inspired in some results of Friedman and Russo [FR2, Propositions 1 and 2] for their proofs.

Lemma 2.9 *Let E be a real JB*-triple, $f \in E^*$ and e a tripotent in E such that $\|f P_2(e)\| = \|f\|$. Then $f = f P_2(e)$.*

Proof. The same proof as in [FR2, Proposition 1] runs here. □

Lemma 2.10 *Let E be a dual real JB*-triple. Then*

1. *If $f \in E_*$, there exists a tripotent e in E such that $f = f P_2(e)$.*

2. *A net $\{x_\alpha\}$ converges to zero in the w^*-topology if and only if $\{P_2(u)x_\alpha\}$ converges to zero in the w^*-topology for every tripotent u in E.*

Proof. 1. Let us suppose $\|f\| = 1$, then the set $S = \{x \in E : f(x) = \|x\| = \|f\| = 1\}$ is nonempty convex and w^*-compact. Therefore there exists an extreme point of the closed unit ball of E, e, such that e is in S, too. Hence e is a tripotent [IKR, Lemma 3.3], and $f(e) = \|f\| = 1$. We have $f = fP_2(e)$ by Lemma 2.9.

2. ($\Rightarrow$) Straightforward because $P_2(u)$ is w^*-continuous (for every tripotent u in E) by Proposition 2.4.

   ($\Leftarrow$) Suppose that $P_2(u)x_\alpha \overset{w^*}{\to} 0$, for every tripotent u in E. Let $f \in E_*$. From the first assertion, there exists a tripotent e in E such that $f = fP_2(e)$. By hypothesis $P_2(e)x_\alpha \overset{w^*}{\to} 0$. Therefore $fP_2(e)x_\alpha = f(x_\alpha) \to 0$. □

A tripotent e in a Jordan triple A is called complete if $A_0(e) = 0$. In a dual real JB*-triple E we have many complete tripotents because by [IKR, Lemma 3.3], the complete tripotents in E are exactly the extreme points of the closed unit ball of E, B_E. Banach-Alaoglu´s and Krein-Millman´s theorems give that B_E is the w^*-closed convex hull of its extreme points.

Having disposed of these preliminary steps we can now prove the Main Theorem.

Theorem 2.11 *Let E be a dual real JB*-triple. Then the triple product is separately w^*-continuous, i.e., E is a real JBW*-triple.*

Proof. We first prove that $L(a,b)$ is w^*-continuous for all $a, b \in E$.

Let e be a complete tripotent in E. If we fix $a \in E_1(e)$ and $b \in E_2(e)$, using Peirce arithmetic, it is easy to check that

$$L(a,b) = L(a,b)P_2(e) \text{ and}$$
$$L(b,a) = L(b,a)P_1(e).$$

From Proposition 2.3, 2., $L(a,b) - L(b,a)$ is w^*-continuous and by Proposition 2.4, $P_2(e)$ and $P_1(e)$ are w^*-continuous. Therefore

$$\begin{aligned} L(a,b) &= L(a,b)P_2(e) - L(b,a)P_1(e)P_2(e) \\ &= (L(a,b) - L(b,a))P_2(e) \end{aligned}$$

is w^*-continuous. (3.1)

In a similar way $L(b,a) = -(L(a,b) - L(b,a))P_1(e)$ is w^*-continuous. (3.2)

Now if $a \in E$ and $b \in E_2(e)$, then $a = a_1 + a_2$ where $a_i \in E_i(e)$ for $i = 1, 2$. Since

$$L(a,b) = L(a_1, b) + L(a_2, b)$$
$$L(b,a) = L(b, a_1) + L(b, a_2)$$

we can conclude by (3.1), (3.2) and Proposition 2.8 that $L(a,b)$ and $L(b,a)$
(3.3) are w^*-continuous for all $a \in E$ and $b \in E_2(e)$.

Let $a \in E$, by applying Jordan identity, we have

$$L(a, L(e,e)a) = -L(e,a)L(a,e) + L(\{e,a,a\}, e) + L(a,e)L(e,a).$$

(3.4) Thus by (3.3), $L(a, L(e,e)a)$ is w^*-continuous because $e \in E_2(e)$.

From

$$\begin{aligned} L(a, L(e,e)a) &= L(a_1 + a_2, L(e,e)a_1 + a_2) \\ &= L\left(a_1 + a_2, \frac{1}{2}a_1 + a_2\right) \\ &= \frac{1}{2}L(a_1, a_1) + \frac{1}{2}L(a_2, a_1) + L(a_1, a_2) + L(a_2, a_2) \end{aligned}$$

we deduce that

$$\begin{aligned} &L(a_1, a_1) \\ &= 2\left(L(a, L(e,e)a) - \frac{1}{2}L(a_2, a_1) - L(a_1, a_2) - L(a_2, a_2)\right) \end{aligned}$$

is w^*-continuous which follows from (3.3) and (3.4). We have proved that
(3.5) $L(a_1, a_1)$ is w^*-continuous for all $a_1 \in E_1(e)$.

Finally since $E = E_1(e) \oplus E_2(e)$, and $L(.,.)$ is bilinear, by (3.3) and
(3.6) (3.5) we can conclude that $L(a,b)$ is w^*-continuous for all $a, b \in E$.

The last part of the proof is devoted to prove that $Q(a,b)$ is w^*-continuous for all $a, b \in E$.

We fix $a, b \in E$. It is easy to check that $Q(b)Q(a) = 2L(b,a)L(b,a) - L(\{b,a,b\}, a)$. Thus $Q(b)Q(a)$ is w^*-continuous for all $a, b \in E$ by (3.6). In particular $Q(u)Q(a)$ is w^*-continuous for every tripotent u in E. So (using Proposition 2.4) $P_2(u)Q(a) = Q(u)Q(u)Q(a)$ is w^*-continuous
(3.7) for every tripotent u in E.

Now by Lemma 2.10, 2., $Q(a)$ is w^*-continuous if and only if $P_2(u)Q(a)$ is w^*-continuous for every tripotent u in E. Hence, using (3.7), we conclude the proof. □

Edwards [E, Theorems 3.2 and 3.4] has shown that the complexification of a JB-algebra, J, is a JBW*-algebra if and only if J is a JBW-algebra. This result now is a consequence of our main result and [IKR, Theorem 4.4].

Corollary 2.12 *Let J be a JB-algebra. Then J is a JBW-algebra if and only if its complexification is a JBW*-algebra.*

Proof. Consider $B = J \oplus iJ$ (the complexification of J) as JB*-triple, τ the natural involution on B and $J = B^{\tau}$. □

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