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ABSTRACT CONVEXITY, SOME RELATIONS AND APPLICATIONS

JUAN-VICENTE LLINARES*

Departamento de Fundamentos del Análisis Económico, Facutad de Economía y Empresa, Universidad de Murcia, Campus de Espinardo, E-30100 Murcia, Spain

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The aim of this article is to analyze the relationship between various notions of abstract convexity structures that we find in the literature, in connection with the problem of the existence of continuous selections and fixed points of correspondences. We focus mainly on the notion of *mc*-spaces, which was introduced in [J.V. LLinares (1998). Unified treatment of the problem of the existence of maximal elements in binary relations: a characterization. *Journal of Mathematical Economics*, **29**, 285–302], and its relationship with *c*-spaces [Ch.D. Horvath (1991). Contractibility and generalized convexity. *Journal of Mathematical Analysis and Applications*, **156**, 341–357], simplicial convexity [R. Bielawski (1987). Simplicial convexity and its applications. *Journal of Mathematical Analysis and Applications*, **127**, 155–171], order convexity (used in [Ch.D. Horvath and J.V. LLinares (1996). Maximal elements and fixed points for binary relations on topological ordered spaces. *Journal of Mathematical Economics*, **25**, 291–306]), B'-simplicial convexity and *L*-spaces [H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J.V. LLinares (1998). Abstract convexity and fixed points. *Journal of Mathematical Analysis and Applications*, **222**, 138–150]. Moreover, in the context of *mc*-spaces, a characterization result of nonempty finite intersection, in the line with the Knaster–Kuratowski–Mazurkiewicz Lemma, some consequences of it and some generalizations of Browder's existence of continuous selection and fixed point theorem are presented.

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*E-mail: llinares@um.es

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1 INTRODUCTION

The notion of convexity is a basic mathematical structure that is used to analyze many different problems. In the literature, many papers deal with the problem of generalizing usual convexity from different points of view: c-spaces [7], simplicial convexity [2], geodesic convexity [20], L-convexity [1] or convexity induced by an order [9] are some of the generalizations dealt with.

In general, we can consider two different kinds of generalizations for this notion. On the one hand, there are those that are motivated by concrete problems, (e.g., the existence of continuous selections and fixed points [1,2,7,14,17,25], or nonconvex optimization problems [11,18, 20,21,22], etc.) and, on the other hand, those that are stated from an axiomatic point of view, where the notion of abstract convexity is based on properties of a family of sets (similar to the properties of the convex sets in topological vector spaces) [3,16,19,24].

In the context of abstract convexity, there are some authors who consider different definitions of abstract convexity, asking for additional conditions for the family of subsets that defines the convexity. In [26] for instance a convexity on a topological space, X, as a family of closed subsets $\mathcal{C}$, of X, which contains X as an element and which is closed under arbitrary intersections is considered. Note that this definition of abstract convexity does not generalize the notion of usual convexity (in topological vector spaces).

In this article, we consider some abstract convexities that have been used in the literature to generalize some results on the existence of continuous selections and fixed points to correspondences. In this framework, we focus on an abstract convexity structure called *mc-spaces* (introduced in [14]), which is based on the idea of substituting the segment that joins any pair of points (or the convex hull of a finite set of points) by a set that plays their role, and study the relationship between it and L-convexity [1] which is equivalent to it, simplicial convexity [2], c-spaces [7,8], B'-simplicial convexity [1] and the convexity induced by an order used in [9]. As an application, we present, in the context of *mc*-spaces, a characterization result of nonempty finite intersection, in the line of the Knaster–Kuratowski–Mazurkiewicz (KKM) Lemma, as well as an extension of Browder's result on the existence of continuous selection and fixed point to correspondences with open lower sections.

2 ABSTRACT CONVEXITIES

As previously mentioned, we shall present some particular abstract convexities that appear in the literature, in relation to the problem of the existence of continuous selections and fixed points to correspondences. We first present the general notion of abstract convexity structure.

Definition 1 [10] A family $\mathcal{C}$ of subsets of a set X is an *abstract convexity structure* for X if $\emptyset$ and X belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersections.

The elements of $\mathcal{C}$ are called $\mathcal{C}$-convex (or simply abstract convex) subsets of X and the pair $(X, \mathcal{C})$ is called a convex space. Moreover, the abstract convexity notion allows us to define the notion of the convex hull operator, which is similar to that of the closure operator in topology.

Definition 2 [10,24] If X is a set with an abstract convexity $\mathcal{C}$ and A is a subset of X, then the hull operator generated by a convexity structure $\mathcal{C}$, which we will denote by $C_{\mathcal{C}}$ and call *$\mathcal{C}$-convex hull*, is defined for any subset $A \subseteq X$ by $C_{\mathcal{C}}(A) = \cap\{B \in \mathcal{C}\colon A \subseteq B\}$.

This operator enjoys certain properties that are identical to those of usual convexity: for instance, $C_{\mathcal{C}}(A)$ is the smallest $\mathcal{C}$-convex set that contains set A.

Although there are more abstract convexities (in reference to the fixed point theory) than the ones we present in this article[1], most of them are particular cases of c-spaces or simplicial convexity (see [2] or [17]). We will only focus therefore, on those that are more intuitive.

2.1 *K*-convex Structure

The K-convex structure is based on the idea of considering functions that join pairs of points. That is, the segments used in usual convexity are substituted by an alternative path, previously fixed on X.

[1]Michael's convex structure [15], Komiya convex spaces [12], etc.

Definition 3 [6] A *K-convex structure* on the set X is given by a mapping $K : X \times X \times [0,1] \to X$. Furthermore (X, K) will be called a *K-convex space* and function K a *K-convex function*.

Note that if (X, K) is a K-convex space, it is possible for any pair of points $x, y \in X$ to associate themselves with a subset given by $K(x, y, [0,1]) = \cup\{K(x, y, t): t \in [0, 1]\}$ (in a similar way to the case of the union operation, see [19], or interval spaces, see [22]). Moreover, we can define an abstract convexity on X by considering a family, $\mathcal{C}$, of subsets of X as follows:

$$Z \in \mathcal{C} \Longleftrightarrow \forall x, y \in Z \quad K(x, y, [0, 1]) \subseteq Z.$$

The elements of $\mathcal{C}$ will be called *K-convex sets* and the *K-convex hull operator* associated to this family $\mathcal{C}$ of K-convex sets (see Definition 2) will be denoted by $C_{\mathcal{K}}$.

By imposing different conditions on function K, particular abstract convexity structures are obtained. The following definitions present some of these particular cases.

Definition 4 [13] If X is a topological space, a *K-convex continuous structure* on X is defined by a continuous function $K: X \times X \times [0, 1] \to X$, such that $K(x, y, 0) = x$, and $K(x, y, 1) = y$.

From function K, it is easy to define a family of continuous paths joining pairs of points in X as follows, for any $x, y \in X, K_{xy} : [0, 1] \to X, K_{xy}(t) = K(x, y, t)$.

Obviously a K-convex continuous structure can be defined in any usual convex subset of a topological vector space. Furthermore it can be proven that a K-convex continuous structure on X can be defined if and only if X is a contractible set, although it does not mean that K-convex subsets coincide with contractible subsets (for more details see [13]).

A different case of K-convexity is that of the equiconnected spaces, introduced by Dugundji [4] and Himmelberg [5], which are a particular case of K-convex continuous spaces.

Definition 5 [4,5] A metric space X is *equiconnected* if and only if, there exists a continuous function $K: X \times X \times [0, 1] \to X$ such that for all $x, y \in X$, $K(x, y, 0) = x$, $K(x, y, 1) = y$ and $K(x, x, t) = x$ for any $t \in [0, 1]$.

In general, absolute retract spaces[2] (AR) are equiconnected spaces (see [4]). Furthermore, in the context of metric spaces with finite dimensionality, equiconnected spaces coincide with AR ones.

The following example shows that the notions of K-convex continuous structure and that of equiconnected space are different.

Example 1 Let $X \subset \mathbb{R}^2$ be the following set,

$$X = \bigcup \{(x, x/n), x \in [0, 1]: n \in \mathbb{N}\} \cup [0, 1] \times \{0\}$$

It is clear that X is a star-shape set[3] and function $K : X \times X \times [0, 1] \to X$ can be defined as follows,

$$K(x, y, t) = \begin{cases} (1 - 2t)x + 2tx_0 & t \in [0, 0.5] \\ (2 - 2t)x_0 + (2t - 1)y & t \in [0.5, 1]. \end{cases}$$

Note that function K obviously satisfies the requirements that define a K-convex continuous structure but not, however, an equiconnected structure on X, since it does not satisfy that $K(x, x, t) = x$ for all $t \in [0, 1]$. Moreover it is not equiconnected for any function K we could define, since it is not locally equiconnected (see [4, Theorem 2.4]).

2.2 Order Convexity

If $(X, \leq)$ is a partially ordered set (*poset*), and for all $x, y \in X$ the closed interval is denoted by $[x, y] = \{z \in X: x \leq z \leq y\}$, so that it is possible to define an abstract convexity structure on X, called *order convexity*, by considering the abstract convex sets like $Z \subseteq X$, such that for all $x, y \in Z$, $[x, y] \subseteq Z$ (see [9]).

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[2] A space Y is an absolute retract (AR) whenever Y is metrizable and for any metrizable X and any closed subset $A \subseteq X$, it is verified that each continuous function $f: A \to Y$ is extendable over X.

[3] A subset X of a linear space E is a *star-shape set* if there exists $x_0 \in X$ such that $\forall x \in X$, $\forall t \in [0, 1]$, $tx + (1 - t)x_0 \in X$.

Moreover, if $(X, \leq)$ is a (sup)semilattice and the supremum of (x, y) is denoted by $x \vee y$, then it is possible to consider the abstract convex sets like $Z \subseteq X$, such that for all $x, y \in Z$, $[x, x \vee y] \cup [y, x \vee y] \subseteq Z$.

2.3 *c*-Spaces

We can consider abstract convexities on a set X by associating to any finite family of points in X a subset of X. This subset is, in some sense, the generalized convex hull of these points. This is the case, for instance, of the notion of c-space (or H-space), introduced in [8], which associates an infinitely connected set[4] (C^{∞}) that satisfies some monotonicity conditions to any finite subset of X.

Formally, the notion of c-space is as follows:

Definition 6 [8] If X is a topological space and $\langle X \rangle$ denotes the family of nonempty finite subsets of X, then a *c-structure* on X is given by a nonempty set valued map $\Gamma : \langle X \rangle \to X$ that satisfies:

1. for all $A \in \langle X \rangle$, $\Gamma(A)$ is nonempty and infinitely connected.
2. for all $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma(A) \subseteq \Gamma(B)$.

The pair (X, Γ) is called *c-space*, and a subset $Z \subset X$ is called an *H-set* if and only if, it is satisfied for all $A \in \langle Z \rangle$, $\Gamma(A) \subseteq Z$.

Note that in the context of topological vector spaces this definition includes, as a particular case, the notion of usual convexity. Moreover, it is easy to show that the family of H-sets defines an abstract convexity on X.

Remark 1 In [6,7] it is considered a more restrictive notion of c-structure by assuming that sets $\Gamma(A)$ are contractible instead of being infinitely connected.

2.4 Simplicial Convexity

A different way of introducing an abstract convexity structure from a family of continuous functions is by associating a continuous function

[4] A set A is *infinitely connected* if every continuous function defined on the boundary of a finite dimensional sphere with values in A can be extended to a continuous function on the ball, with values in A (see [8]).

defined on the standard simplex that satisfies some conditions to any finite subset of X (see [2]).

Definition 7 [2] If X is a topological space and Δ_k the k-dimensional simplex, X has a *simplicial convexity* if for each $n \in \mathbb{N}$, and for each $(x_1, x_2, \dots, x_n) \in X^n$, there exists a continuous function $\Phi[x_1, x_2, \dots, x_n] : \Delta_{n-1} \to X$ that satisfies

1. for all $x \in X, \Phi[x](1) = x$,
2. for all $n \geq 2$, for all $(x_1, x_2, \dots, x_n) \in X^n$, for all $(t_1, t_2, \dots, t_n) \in \Delta_{n-1}$, if $t_i = 0$, then $\Phi[x_1, x_2, \dots, x_n](t_1, t_2, \dots, t_n) = \Phi[x_{-i}](t_{-i})$, where x_{-i} denotes that x_i is omitted in $(x_1, x_2, \dots, x_n)$.

Moreover, a subset Z of X is called a *simplicial convex set* if and only if, for all $n \in \mathbb{N}$ and for all $(a_1, a_2, \dots, a_n) \in Z^n$ it is satisfied that

$$\text{for all } u \in \Delta_{n-1} \quad \Phi[a_1, a_2, \dots, a_n](u) \in Z.$$

It is not hard to prove that simplicial convex sets are stable under arbitrary intersections. They therefore define an abstract convexity structure.

2.5 B'-simplicial Convexity

The notion of B'-simplicial convexity [1] is an obvious generalization of the notion of simplicial convexity, since we weaken the conditions required of the continuous function that defines the B'-simplicial convexity. Moreover, the notion of B'-simplicial convexity allows us to connect the notion of c-spaces with that of simplicial convexity as well as with other notions of abstract convexities we will introduce later.

Definition 8 [1] A topological space X has a B'-*simplicial convexity* if for each $n \in \mathbb{N}$, and for each $(x_1, x_2, \dots, x_n) \in X^n$, there exists a continuous function $\Phi[x_1, x_2, \dots, x_n] : \Delta_{n-1} \to X$ satisfying that for all $n \geq 2$, for all $(x_1, x_2, \dots, x_n) \in X^n$, and for all $(t_1, t_2, \dots, t_n) \in \Delta_{n-1}$, if $t_i = 0$, then $\Phi[x_1, x_2, \dots, x_n](t_1, t_2, \dots, t_n) = \Phi[x_{-i}](t_{-i})$.

In this context, a subset Z of X is called a B'-*simplicial convex set* if and only if, for all $n \in \mathbb{N}$ and for all $(a_1, a_2, \dots, a_n) \in Z^n$ it is satisfied that for all $u \in \Delta_{n-1} \Phi[a_1, a_2, \dots, a_n](u) \in Z$.

It is obvious that the family of B'-simplicial convex sets is an abstract convexity, and the convex hull induced by this convexity is a *subsimplicial hull* (notion introduced in [25]). Moreover, the abstract convex sets obtained from a subsimplicial hull are B'-simplicial convex sets.

2.6 *L*-spaces

The notion of *L-space* (see [1]) is a different abstract convexity that appears in the context of existence of continuous selections and fixed points to correspondences and which generalizes the (B') simplicial convexity as well as the notion of *c*-spaces.

Definition 9 [1] An *L-structure* on X is given by a nonempty set-valued map $\Gamma : \langle X\rangle \to X$, such that for every $A \in \langle X\rangle$, say $A = \{a_0, a_1, \ldots, a_n\}$, there exists a continuous function $f^A\colon \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, 1, \ldots, n\}$, $f^A(\Delta_J) \subseteq \Gamma(\{a_i\colon i \in J\})$.

The pair (X, Γ) is then called *L-space* and a subset Z of X, is called an *L-convex set* if for all $A \in \langle Z\rangle$, then $\Gamma(A) \subset Z$.

Clearly, the family of *L*-convex sets defines an abstract convexity structure on *X*. Furthermore, it is obvious that the notion of *G-convex spaces*, used in [17], is a particular case of *L*-spaces since, to define the *G-convex spaces*, it is required that all of the conditions of Definition 9 be satisfied, together with a monotonicity condition on the set-valued map Γ.

2.7 *mc*-Spaces

The notion of *mc-space* is a generalization of *K*-convex continuous structures, which is obtained by relaxing the continuity condition on function *K*. Now the ideal is to associate, for any finite set of points, a family of functions requiring their composition to be a continuous function. The image of this composition generates a set, associated with the finite set of points, in a similar way to the case of *c*-spaces or simplicial convexity. However, in contrast to these cases, no monotonicity condition on the associated sets is now required.

Definition 10 [14] A topological space X is an *mc-space* (or has an *mc-structure*) if for every $A \in \langle X\rangle$, say $A = \{a_0, a_1, \ldots, a_n\}$, there exists a family of elements $\{b_0, b_1, \ldots, b_n\} \subset X$, and a family of functions $P_i^A : X \times [0,1] \to X$, such that for $i = 0, 1, \ldots, n$, $P_i^A(x, 0) = x$,

$P_i^A(x,1)=b_i$, for all $x\in X$, and function $G_A:[0,1]^n\to X$ given by $G_A(t_0,t_1,\ldots,t_{n-1})=P_0^A(\ldots(P_{n-1}^A(P_n^A(b_n,1),t_{n-1}),\ldots,t_0)$, is a continuous function.

Note that the notion of *mc*-space ranges over a wide field of possibilities, since it can appear in completely different contexts. For instance, if X is a nonempty topological space then it is always possible to define an *mc*-structure on it by considering, for any finite subset $A=\{a_0,a_1,\ldots,a_n\}$ of X, and for all $i=0,\ldots,|A|-1$, $b_i=a_0$, and the family of functions as follows, $P_i^A(x,t)=x$, for all $x\in X$ and for all $t\in[0,1)$, while $P_i^A(x,1)=a_0$ for all $x\in X$. Moreover, if X is a usual convex subset of a topological vector space, we can define an *mc*-structure by considering for any finite subset $A=\{a_0,a_1,\ldots,a_n\}$ of X and for all $i=0,\ldots,|A|-1$, $b_i=a_i$ and functions $P_i^A(x,t)=(1-t)x+ta_i$ for all $x\in X$ and for all $t\in[0,1]$. In this case, the image of function G_A, coincides with the usual convex hull of A (for more details see [14]). Moreover, it is easy to prove that *mc*-spaces are also extensions of K-convex continuous spaces.

Given an *mc*-structure, it is possible to define an abstract convexity by considering the family of sets that are stable under function G_A. In order to define this convexity, we need some preliminary concepts.

Definition 11 [14] If X is an *mc*-space, and Z a subset of X, then for all $A\in\langle X\rangle$ such that $A\cap Z\neq\emptyset$, say $A\cap Z=\{a_{i_0},a_{i_1},\ldots,a_{i_m}\}$ $(i_0<i_1<\cdots<i_m)$, we define the *restriction of function* G_A to Z as follows, $G_{A|Z}:[0,1]^m\to X$, $G_{A|Z}(t)=P_{i_0}^A(\ldots P_{i_{m-1}}^A(P_{i_m}^A(a_{i_m},1),t_{i_{m-1}})\ldots,t_{i_0})$ where $P_{i_k}^A$ are the functions associated with the elements $a_{i_k}\in A\cap Z$.

By making use of this notion, we now define *mc-sets*, which generalize usual convex sets.

Definition 12 [14] A subset Z of an *mc*-space X is an *mc-set* if and only if, for any $A\in\langle X\rangle$ such that $A\cap Z\neq\emptyset$, it is satisfied that $G_{A|Z}([0,1]^m)\subseteq Z$, where $m=\mathrm{card}(A\cap Z)-1$.

Since the family of *mc*-sets is stable under arbitrary intersections, it defines an abstract convexity on X. Furthermore, we can define the *mc-hull operator* C_{mc} in the usual way (see Definition 2). So, it is obvious that for all $Z\subset X$, and for all $A\in\langle X\rangle$ such that $A\cap Z\neq\emptyset$, it is satisfied that $G_{A|Z}([0,1]^m)\subseteq C_{mc}(Z)$.

3 RELATIONS BETWEEN THE VARIOUS ABSTRACT CONVEXITIES

Throughout this section, the relationship between the abstract convexity notions introduced in the previous section is analyzed. Since some of them are obvious[5], we will focus on those that are not easily obtained.

We start by showing that any K-convex continuous space is a c-space in which the K-convex sets are H-sets.

PROPOSITION 1 *If (X, K) is a K-convex continuous space, then there exists a nonempty set valued map $\Gamma : \langle X \rangle \to X$ such that (X, Γ) is a c-space and K-convex sets are H-sets.*

Proof If (X, K) is a K-convex continuous space, then we can define the mapping $\Gamma : \langle X \rangle \to X$, by means of the K-convex hull, that is $\Gamma(A) = C_K(A)$. Then by applying Proposition 1.1 in [13], we know that $\Gamma(A)$ is contractible (and therefore C^∞). Moreover, it is easy to prove that for all $A, B \in \langle X \rangle$, if $A \subset B$ then $\Gamma(A) \subseteq \Gamma(B)$, so (X, Γ) is a c-space.

Finally, to show that K-convex sets are H-sets, assume that there exists a K-convex set Z such that $A \in \langle Z \rangle$ and $\Gamma(A) = C_K(A) \not\subseteq Z$. Then, we have that $A \subset C_K(A)$, $A \subset Z$ and that both of them are K-convex sets, so $A \subset Z \cap C_K(A) \subsetneqq C_K(A)$, which is in contradiction with the fact that $C_K(A)$ is the smallest K-convex set containing A. ■

The following result states the relationship between K-convex continuous structures and simplicial convexities.

PROPOSITION 2 *If (X, K) is a topological space with a K-convex continuous structure, then it is possible to define a simplicial convexity on X such that K-convex sets are simplicial convex sets.*

Proof For any $n \in \mathbb{N}$, and for any $(a_1, a_2, \ldots, a_n) \in X^n$, we define the family of functions $\Phi[a_1, a_2, \ldots, a_n]$ as follows,

1. if $n = 1$, $\Phi[a] = K(a, a, 1)$,
2. for $n \geq 2$,

$$\Phi[a_1, a_2, \ldots, a_n](t_1, t_2, \ldots, t_n) = K(\ldots K(K(a_n, a_{n-1}, t_{n-1}), a_{n-2}, t_{n-2}) \ldots), a_1, t_1).$$

[5]For instance, to show that an equiconnected space has a K-convex continuous structure, or that the K-convex continuous structure is a particular case of the mc-space.

It is easy to show that this family of functions defines a simplicial convexity on X that coincides with the one that is obtained from K. ■

The next proposition shows that the order convexity structure (in topological semilattices) is a particular case of the simplicial convexities.

PROPOSITION 3 *If* $(X, \leq)$ *is a topological* (*sup*)*semilattice with path connected intervals*, *then there exists a simplicial convexity on X such that order convex sets are simplicial convex sets.*

Proof If $(X, \leq)$ is a topological (sup)semilattice with path-connected intervals, then we can define a nonempty set valued map $\Gamma : \langle X \rangle \to X$ given by $\Gamma(A) = \cup_{a \in A}[a, \sup A]$. By applying Lemma 2.1 in [9] we know that for any $n \in \mathbb{N}$, any continuous function $g : \partial\Delta_n \to \Gamma(A)$ can be extended to a continuous function $f : \Delta_n \to \Gamma(A)$, so $\Gamma(A)$ is C^∞. Therefore, if we define *hull* $\{A\} = \Gamma(A)$, the family of order convex sets is an abstract convexity such that *hull* $\{A\}$ is C^∞ and, by applying Proposition 1.5 in [2], we obtain the conclusion. ■

The next proposition, which was obtained in [1], states the relationship between c-spaces and simplicial convexities, as well as between c-spaces and B'-simplicial convexities.

PROPOSITION 4 [1]

1. *If* (X, Γ) *is a c-space such that for all* $x \in X$, $x \in \Gamma(\{x\})$, *then X has a simplicial convexity such that H-sets are simplicial convex sets.*
2. *If* (X, Γ) *is a c-space*, *then it is possible to define a* B'*-simplicial convexity such that H-sets are* B'*-simplicial convex sets.*

The next result is immediately obtained from the definition of simplicial convexity.

PROPOSITION 5 *If X is a topological space with a simplicial convexity*, *then this simplicial convexity defines a* B'*-simplicial convexity such that simplicial convex sets are* B'*-simplicial convex sets.*

The next results shows that the notion of L-space is more general than that of B'-simplicial convexity.

PROPOSITION 6 *If X has a* B'*-simplicial convexity*, *then X is an L-space such that* B'*-simplicial convex sets are L-convex sets.*

Proof For every $n \in \mathbb{N}$, let Λ_n be the set of all functions σ: $\{0,1,\ldots,n\} \to \{0,1,\ldots,n\}$. Then for any $A \in \langle X\rangle$, $A = \{a_0, a_1, \ldots, a_n\}$ we define the mapping $\Gamma : \langle X\rangle \to X$ as follows,

$$\Gamma(A) = \bigcup \{\Phi[a_{\sigma(0)}, a_{\sigma(1)}, \ldots, a_{\sigma(n)}](\Delta_n) \colon \sigma \in \Lambda_n\},$$

and function $f^A : \Delta_n \to \Gamma(A)$ by $f^A(\lambda) = \Phi[a_0, a_1, \ldots, a_n](\lambda)\ \forall \lambda \in \Delta_n$. It is clear that function f^A is continuous and satisfies the requirement that for all $J = \{i_0, \ldots, i_m\} \subset \{0, 1, \ldots, n\}$, $(i_0 < i_1 < \cdots < i_m)$ and for all $\lambda \in \Delta_J$,

$$\begin{aligned} f^A(\lambda) &= \Phi[a_0, \ldots, a_n](\lambda) = \Phi[a_{i_0}, \ldots, a_{i_m}](\lambda_{i_0}, \ldots, \lambda_{i_m}) \\ &\subseteq \Phi[a_{i_0}, \ldots, a_{i_m}](\Delta_m) \subseteq \Gamma(\{a_{i_0}, \ldots, a_{i_m}\}), \end{aligned}$$

so $f^A(\Delta_J) \subseteq \Gamma(\{a_{i_0}, \ldots, a_{i_m}\})$.

Finally, if $Z \subseteq X$ is a B'-simplicial convex set, since for any $n \in \mathbb{N}$ and any $(a_0, a_1, \ldots, a_n) \in Z^{n+1}$ we know that $\Phi[a_0, a_1, \ldots, a_n](\Delta_n) \subseteq Z$, it is satisfied that for any $A \in \langle Z\rangle$, $\Gamma(A) \subseteq Z$; therefore Z is an L-convex set. ■

The next propositions show that the notion of *mc*-space and *L*-space are equivalent, in the sense that if we have an *mc*-space, then it is possible to define an *L*-structure such that *mc*-sets are *L*-sets, and conversely, if we have an *L*-space then, it is possible to define an *mc*-structure such that *L*-sets are *mc*-sets.

PROPOSITION 7 *If X is an L-space, then X is an mc-space such that L-convex sets are mc-sets.*

Proof If X has a L-structure, then we can define for all $A \in \langle X\rangle$, $A = \{a_0, a_1, \ldots, a_n\}$ functions P_i^A as follows:

$$\begin{aligned} P_n^A(a_n, 1) &= f^A(e_n), \\ P_{n-1}^A(P_n^A(a_n, 1), t_{n-1}) &= f^A(t_{n-1}e_{n-1} + (1 - t_{n-1})e_n), \\ P_{n-2}^A(P_{n-1}^A(P_n^A(a_n, 1), t_{n-1}), t_{n-2}) &= f^A(t_{n-2}e_{n-2} + (1 - t_{n-2}) \\ &\qquad \times [t_{n-1}e_{n-1} + (1 - t_{n-1})e_n]), \end{aligned}$$

and so on. Moreover, for those values not considered until now, functions P_i^A are defined in such a way that $P_i^A(x,0) = x$ and $P_i^A(x,1) = f^A(e_i)$. Therefore, function $G_A(t_0,t_1,\ldots,t_{n-1}) = f^A(\sum_{i=0}^n \alpha_i e_i)$ where coefficients α_i depend continuously on t_j, $j = 0,1,\ldots,n$. Finally, if Z is a L-convex set, then it is satisfied that for all $A \in \langle Z \rangle$, $\Gamma(A) \subseteq Z$. To see that Z is also an mc-set, we have to prove that for all $A \in \langle X \rangle$, such that $A \cap Z \neq \emptyset$, $A \cap Z = \{a_{i_0},\ldots,a_{i_m}\}$, then $G_{A|Z}([0,1])^m) \subseteq Z$. But if $J = \{i_0,\ldots,i_m\}$ then $\Delta_J = C(\{e_{i_k} : k = 0,\ldots,m\})$ and therefore, by the definition of function G_A, we get $G_{A|Z}([0,1]^m) \subseteq f^A(\Delta_J) \subseteq \Gamma(\{a_{i_k} : k = 0,\ldots,m\}) = \Gamma(A \cap Z) \subseteq Z$. ■

In order to prove the next proposition, we use the following Lemma, which was proven in [14].

LEMMA 1 [14] *If X is an mc-space, for $i = 0,1,\ldots,n$, functions $t_i : \Delta_n \to [0,1]$ are defined by*

$$t_i(\lambda) = \begin{cases} 0 & \text{if } \lambda_i = 0 \\ \dfrac{\lambda_i}{\sum_{j=i}^n \lambda_j} & \text{if } \lambda_i \neq 0 \end{cases}$$

and function $\mathcal{T} : \Delta_n \to [0,1]^n$ is defined by $\mathcal{T}(\lambda) = (t_0(\lambda), t_1(\lambda), \ldots, t_{n-1}(\lambda))$, then for any finite set $A \in \langle X \rangle$, $A = \{a_0, a_1, \ldots, a_n\}$ the composition $f^A = G_A \circ \mathcal{T}$ is a continuous function.

PROPOSITION 8 *If X is an mc-space, then X is an L-space. Moreover, mc-sets are L-convex sets.*

Proof If X is an mc-space then for any finite set $A \in \langle X \rangle$, $A = \{a_0, a_1, \ldots, a_n\}$, there exists a family of functions P_i^A, such that their composition G_A is a continuous function. By applying Lemma 1 the function $f^A : \Delta_n \to X$ defined by $f^A = G_A \circ \mathcal{T}$ is continuous.

In order to obtain an L-convexity in such a way that mc-sets are L-convex sets, we define the nonempty set-valued map $\Gamma : \langle X \rangle \to X$, as follows,

$$\Gamma(B) = \cup\{G_{A|B}([0,1]^m) \colon B \subseteq A, A \in \langle X \rangle\},$$

where $m = \mathrm{card}(A \cap B) - 1$.

Notice that for any $A \in \langle X\rangle$, $A=\{a_0,a_1,\dots,a_n\}$, for all $J=\{i_0,\dots,i_m\}\subseteq \{0,1,\dots,n\}$, $(i_0<i_1<\cdots<i_m)$, and for all $\lambda\in\Delta_J$,

$$f^A(\lambda) = G_A(\mathcal{T}(\lambda)) \in G_{A|\{a_{i_0},\dots,a_{i_m}\}}([0,1]^m) \subseteq \Gamma(\{a_{i_k}: i_k \in J\}),$$

so $f^A(\Delta_J) \subseteq \Gamma(\{a_{i_0},\dots,a_{i_m}\})$.

Moreover, if Z is an *mc*-set, then for all $A \in \langle X\rangle$, such that $A\cap Z \neq \emptyset$, $A\cap Z = \{a_{i_0},\dots,a_{i_m}\}$, then $G_{A|Z}([0,1]^m) \subseteq Z$. Therefore, for any $B \in \langle Z\rangle$, it is satisfied that $\Gamma(B) \subseteq Z$, so Z is an L-convex set. ■

The following example shows that *mc*-sets do not coincide with H-sets.

Example 2 Consider the following subset $X\subseteq \mathbb{R}$; $X=\bigcup_{n=0}^{\infty}[2n,2n+1]$. We want to prove that X is an *mc*-space whose *mc*-sets are not H-sets. To do so, we define the following functions for all $A=\{a_1,\dots,a_n\}\in\langle X\rangle$.

$$P_i^A(x,0) = x, \quad P_i^A(x,t) = \max\{a_i: a_i \in A\} = a^* \ \forall t \in (0,1].$$

It is clear that G_A is a continuous function since it is a constant one,

$$G_A(t_0,t_1,\dots,t_{n-1}) = P_0^A(\dots P_{n-1}^A(P_n^A(a_n,1),t_{n-1})\dots,t_0) = a^*,\ \forall t_i \in [0,1].$$

X is therefore an *mc*-space. Moreover, for all $w \in X$, subsets $Z_w = [w,+\infty)\cap X$ are *mc*-sets, since for every finite subset A of X such that $A\cap Z_w \neq \emptyset$ we know that $a^*\in Z_w$, therefore $G_{A|Z_w}([0,1]^m) = a^* \in Z_w$. However, it is not possible to define a c-structure on X such that Z_w are H-sets for all $w\in X$. By contradiction, if we assume that $\Gamma:\langle X\rangle \to X$ defines a c-structure on X, then it has to be satisfied that for all $A\in\langle X\rangle$, $\Gamma(A)$ has to be contractible set and, therefore, to be included in some interval $[2n,2n+1]$. Moreover, by the monotonicity condition (if $A\subset B$, then $\Gamma(A)\subseteq\Gamma(B)$), this interval has to be the same for every $A\in\langle X\rangle$, since, otherwise, they belong to two different connected components and they would not be contractible sets. Therefore, whenever $w>2n+1$, it is clear that Z_w is not an H-set due to the fact that, in this particular case, for every $A\in\langle Z_w\rangle$, $\Gamma(A)$ is not included in Z_w.

Note that this example also shows that, in general, an *mc*-structure does not induce a B'-simplicial convexity.

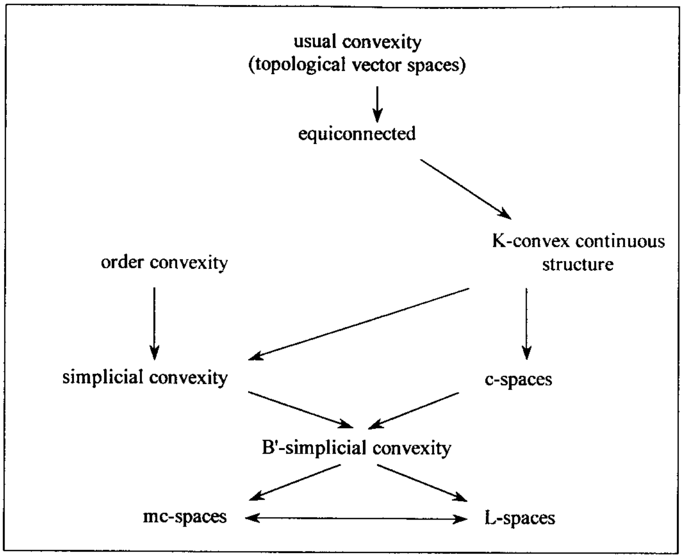

FIGURE 1.

The relations between the various abstract convexities considered in this section can be summarized by the following diagram (Fig. 1).

4 SELECTIONS, FIXED POINT AND KKM RESULTS

This section is devoted to presenting some results on the existence of fixed points, continuous selections to correspondences and KKM results in the context of *mc*-spaces. The first result was obtained in [14] and states the existence of a continuous selection with a fixed point of the *mc*-convex hull of a correspondence defined on *mc*-spaces. Moreover, it gives the key for obtaining the generalization of Browder's result on the existence of continuous selection and fixed point to correspondences with open lower sections. Henceforth, we shall consider Hausdorff topological spaces.

THEOREM 1 [14] *If X is a compact topological mc-space and $\phi : X \to X$ is a nonempty valued correspondence satisfying that*

$$y \in \phi^{-1}(x) \Rightarrow \exists x' \in X : y \in \operatorname{int} \phi^{-1}(x'),$$

then, there exists a nonempty finite subset A of X and a continuous function $f : X \to X$ *such that for all* $x \in X, f(x) \in G_{A|\phi(x)}([0,1]^m)$ *and there exists* $x^* \in X$ *such that* $x^* = f(x^*)$.

The following result is an extension of Browder's theorem to the context of *mc*-spaces.

THEOREM 2 *If X is a compact topological mc-space and* $\phi : X \to X$ *is a nonempty mc-set valued correspondence satisfying one of the following conditions,*

(i) $y \in \phi^{-1}(x) \Rightarrow \exists x' \in X\colon y \in \operatorname{int} \phi^{-1}(x')$,

(ii) *for each* $y \in X$, $\phi^{-1}(y)$ *contains an open subset* O_y *of X such that* $\bigcup_{y \in X} O_y = X$,

then ϕ *has a continuous selection and a fixed point.*

Proof It is easy to show that conditions (i) and (ii) are equivalent if ϕ is a nonempty set valued correspondence, so the conclusion is obtained by applying Theorem 1 and by considering that ϕ has *mc*-set values. ∎

In addition, if ϕ is a nonempty *mc*-set valued correspondence with open lower sections, then we can apply this theorem and ensure the existence of a continuous selection and a fixed point.

Furthermore, if paracompactness on the space X is considered instead of compactness and X has a B'-simplicial convexity, then we can also ensure the existence of a continuous selection to correspondences with open lower sections.

THEOREM 3 [1] *If X is a paracompact topological space, Y is a topological space with a* B'*-simplicial convexity and* $\phi : X \to Y$ *is a correspondence with nonempty* B'*-simplicial convex values and open lower sections* $\phi^{-1}(y) = \{x \in X \mid y \in \phi(x)\}$, *then* ϕ *has a continuous selection.*

The next definition presents the notion of *compressibility*, introduced in [25], which is related to results on the existence of continuous selections to correspondences with open lower sections in spaces with an abstract convexity structure.

Definition 13 [25] A topological space X is *compressible* into a topological space Y (with respect to an operation of hull defined on Y) if for every finite family $\mathcal{A}$ of open sets covering X and every system

$\{y_A: A \in \mathcal{A}\}$ of elements of Y, there exists a continuous function $f: X \to Y$ such that for every $x \in X$, $f(x) \in C(\{y_A: x \in A\})$.

From this definition it is not difficult to prove the following theorem.

THEOREM 4 *A normal topological space is compressible into* (a) *every L-space*, (b) *every mc-space*.

Note that the previous Theorems 1, 2, 3, could not be obtained as a consequence of the results of Wieczorek [25, Theorem 8 and Corollary 9] since we use weaker conditions. Moreover, we also obtain the existence of fixed points (in Theorem 2).

Next results present some generalization of the well-known Knaster–Kuratowski–Mazurkiewicz (KKM) result in the context of *mc*-spaces. First, we show a characterization result of nonempty finite intersection that contains, as a corollary, the classical KKM-Lemma as well as its version in the context of *mc*-spaces and, therefore, in all of the abstract convexity structures that we have introduced in the previous sections.

THEOREM 5 *If X is a topological space and $\{R_i: i = 0, \dots, n\}$ is a family of closed subsets of X, then the following statements are equivalent*:

(i) $\bigcap_{i=0}^{n} R_i \neq \emptyset$;

(ii) *X is an mc-space and there exists a finite set $A = \{x_0, \dots, x_n\} \in \langle X \rangle$ such that for any family $\{i_0, \dots, i_k\} \subseteq \{0, 1, \dots, n\}$ of indices, it is satisfied that $G_{A|\{x_{i_0},\dots,x_{i_k}\}}([0,1]^k) \subseteq \bigcup_{j=0}^{k} R_{i_j}$.*

Proof On the one hand, if X is an *mc*-space, we know that for all $A = \{x_0, \dots, x_n\} \in \langle X \rangle$ there exist functions P_i^A such that their composition G_A is continuous and, from Lemma 1, the function $f = G_A \circ \mathcal{T} : \Delta_n \to X$ is continuous. Moreover, if $J = \{i_0, \dots, i_k\} \subseteq \{0, \dots, n\}$, then, by definition of $\mathcal{T}$ and G_A, we have that

$$f(\Delta_J) \subseteq G_{A|\{x_i: i \in J\}}([0,1]^k) \subseteq \bigcup_{i \in J} R_i.$$

Therefore $f^{-1}(R_i) = F_i$ are closed subset of Δ_n and it is satisfied that $\Delta_J \subseteq \bigcup_{i \in J} F_i$, so we can apply the KKM-Lemma and obtain that there exists $z \in \bigcap_{i=0}^{n} F_i$, so $f(z) \in \bigcap_{i=0}^{n} R_i$.

On the other hand, consider $x^* \in \bigcap_{i=0}^{n} R_i$ and define the *mc*-structure in the following way: for all $A \in \langle X \rangle$, $P_i^A(x,t) = x, \forall x \in X,\ t \in [0,1)$ and $P_i^A(x,1) = x^*$. Then, $G_A(t_0,\ldots,t_{n-1}) = x^*$ is trivially continuous and satisfies that for every subfamily $\{i_0,\ldots,i_k\} \subseteq \{0,\ldots,n\}$ of indices

$$G_{A|\{x_{i_0},\ldots,x_{i_k}\}}([0,1]^k) = x^* \in \bigcup_{j=0}^{k} R_{i_j}.$$

■

In the same line we obtain the generalization of the previous characterization result to the nonfinite case.

THEOREM 6 *If X is a topological space and $\phi : X \to X$ is a nonempty valued correspondence with closed values, and there exists $x_0 \in X$ such that $\phi(x_0)$ is compact, then the following statements are equivalent*:

(i) $\bigcap_{x \in X} \phi(x) \neq \emptyset$;
(ii) *X is an mc-space such that, for all finite subset $A = \{x_0, \ldots, x_n\}$ of X, it is satisfied that for any family $\{i_0, \ldots, i_k\} \subseteq \{0, 1, \ldots, n\}$ of indices, then $G_{A|\{x_{i_0},\ldots,x_{i_k}\}}([0,1]^k) \subseteq \bigcup_{j=0}^{k} \phi(x_{i_j})$.*

As a corollary, we obtain the finite version of the KKM-Lemma in the context of *mc*-spaces.

COROLLARY 1 *If X is an mc-space, $\{R_i\colon i = 0, \ldots, n\}$ is a family of closed subset of X and there exists points $x_0, \ldots, x_n$ of X such that for every family $\{i_0, \ldots, i_k\} \subseteq \{0, 1, \ldots, n\}$ of indices $C_{mc}(\{x_{i_0}, \ldots, x_{i_k}\} \subseteq \bigcup_{j=0}^{k} R_{i_j}$, then $\bigcap_{i=0}^{n} R_i \neq \emptyset$*

To obtain other generalizations of the KKM-Lemma, we use the notion of *KKM-relation* introduced in [25].

Definition 14 [25] Given topological spaces X and Y, Y with an abstract convexity structure, a correspondence $\phi\colon X \to Y$ is a *KKM-relation* on a set $F = (x_1, \ldots, x_n\} \subset X$ with respect to a correspondence $\Theta : X \to Y$ whenever, for every $y_1 \in \Theta(x_1), \ldots, y_n \in \Theta(x_n)$,

$$C(\{y_1, \ldots, y_n\}) \subseteq \bigcup_{i=1}^{n} \phi(x_i).$$

It is said that ϕ is a *KKM-relation* with respect to Θ if it is a KKM-relation on every finite set F.

To simplify the notation, and whenever Θ is the identity function ($\Theta(x) = x$), we shall, henceforth, denote ϕ as a *KKM-mapping* instead of a KKM-relation with respect to identity function.

Corollary 1 can be extended by considering the notion of KKM-mapping.

COROLLARY 2 *If X is an mc-space and $\phi\colon X \to X$ is a KKM-mapping which has nonempty closed values, then the family $\{\phi(x)\colon x \in X\}$ has the finite intersection property.*

As an immediate consequence, we obtain:

COROLLARY 3 *If X is an mc-space and $\phi\colon X \to X$ is a KKM-mapping which has nonempty closed values and if there is x_0 such that $\phi(x_0)$ is compact, then $\bigcap_{x \in X} \phi(x) \neq \emptyset$.*

In the following theorem, we use similar continuity conditions to those considered in [23]. The generalization obtained is as follows:

THEOREM 7 *If X is a compact mc-space and $\phi\colon X \to X$ is a KKM-mapping such that for all $x \in X$ the set $X - \phi(x)$ contains an open subset O_x that satisfies $\bigcup_{x \in X} O_x = X$ if $\bigcap_{x \in X} \phi(x) = \emptyset$, then $\bigcap_{x \in X} \phi(x) \neq \emptyset$.*

Proof By contradiction, assume that $\bigcap_{x \in X} \phi(x) = \emptyset$, that is,

$$X = X - \bigcap_{x \in X} \phi(x) = \bigcup_{x \in X} (X - \phi(x)).$$

Since $X = \bigcup_{x \in X} O_x$ and X is compact, then there exists a finite subcovering and a finite partition of unity $\{\psi_i\}_{i=0}^{n}$ subordinated to it, that is,

$$\psi_i(x) > 0 \quad \text{if and only if, } x \in O_{a_i} \subset X - \phi(a_i),\ i = 0, 1, \ldots, n.$$

and we define function $\Psi : X \to \Delta_n$ by $\Psi(x) = (\psi_0(x), \ldots, \psi_n(x))$.

Moreover, since X is an *mc*-space, if we take $A = \{a_0, \ldots, a_n\}$, then there exists a continuous function $G_A : [0,1]^n \to X$, and by applying Lemma 1 then, the function $f : X \xrightarrow{\Psi} \Delta_n \xrightarrow{\mathcal{T}} [0,1]^n \xrightarrow{G_A} X$, is continuous. Furthermore, if we consider $g = G_A \circ \mathcal{T}$, then function $\Psi \circ g : \Delta_n \to \Delta_n$ is a continuous one defined from a compact convex set into itself. We can therefore apply Brouwer's fixed point theorem and conclude that there exists a fixed point, that is, $\exists y^* \in \Delta_n$: $\Psi(g(y^*)) = y^*$ and by denoting $x^* = g(y^*)$ we obtain that $f(x^*) = x^*$.

Therefore, if we define $J(x^*) = \{i: \psi_i(x^*) > 0\}$, it is satisfied that

$$x^* = G_A(\mathcal{T}(\Psi(x^*))) \in C_{mc}(\{a_i: i \in J(x^*)\}) \subseteq \bigcup_{i \in J(x^*)} \phi(a_i).$$

Furthermore, if $i \in J(x^*)$, then $x^* \in O_{a_i} \subset X - \phi(a_i)$, hence:

$$x^* \in \bigcap_{i \in J(x^*)} (X - \phi(a_i)) = X - \left(\bigcup_{i \in J(x^*)} \phi(a_i) \right),$$

which is a contradiction. ■

The next result is in the same line as those obtained in [25], but by considering *mc*-spaces.

THEOREM 8 *If X is a normal mc-space, $\{x_0, x_1, \ldots, x_n\} \subset X$ and $\phi : X \to X$, and $\Theta : X \to X$ are two correspondences satisfying*

(i) $\Theta : X \to X$, *is upper semicontinuous* (*u.s.c.*), *such that for all* $x \in X$, $x \in \Theta(x)$,
(ii) $\phi : X \to X$, *is a KKM-relation, with respect to* Θ,
(iii) *for every* $x_* \in X$, $\left\{\bigcap_{i=0}^n \overline{\phi(x_i)}\right\} \cap \Theta(x_*) = \emptyset$ *implies that* $\overline{\phi(x_j)} \cap \Theta(x_*) = \emptyset$ *for some j, then there exists $x' \in X$ such that* $\left\{\bigcap_{i=0}^n \overline{\phi(x_i)}\right\} \cap \Theta(x') \neq \emptyset$.

In particular, if $\overline{\phi(x^*)}$ is compact for some $x^* \in X$, then $\bigcap_{x \in X} \overline{\phi(x)} \neq \emptyset$.

Proof By contradiction, suppose that for every $x \in X$, $\left\{\bigcap_{i=0}^n \overline{\phi(x_i)}\right\} \cap \Theta(x) = \emptyset$. By (iii), we know that for every $x \in X$, there exists some j such that $\overline{\phi(x_j)} \cap \Theta(x) = \emptyset$. Therefore, by the u.s.c. of Θ, the sets $X_i = \left\{x: \overline{\phi(x_i)} \cap \Theta(x) = \emptyset\right\}$, for $i = 0, 1, \ldots n$, are open and form an open covering of X. But since X is a normal space, we know that there exists a partition of unity $\{\psi_i\}_{i=0}^n$ subordinated to this covering, so

$$\psi_i(x) > 0 \quad \text{if and only if } x \in X_i, \ i = 0, 1, \ldots, n,$$

and we can define function $\Psi: X \to \Delta_n$ by $\Psi(x) = (\psi_0(x), \ldots, \psi_n(x))$.

Moreover, if we choose $y_0 \in \Theta(x_0), \ldots, y_n \in \Theta(x_n)$ and we take $A = \{y_0, \ldots, y_n)$, since X is an mc-space, then there exists a continuous function $G_A : [0,1]^n \to X$. Therefore, from Lemma 1 the function $f = G_A \circ \mathcal{T} \circ \Psi$,

$$f : X \xrightarrow{\Psi} \Delta_n \xrightarrow{\mathcal{T}} [0,1]^n \xrightarrow{G_A} X,$$

is continuous. Then, by reasoning as we did in the previous theorem, we can conclude that there exists a fixed point of function f, that is, $f(w) = w$. Moreover, this element w satisfies that

$$\begin{aligned} w = f(w) &= G_A(\mathcal{T}(\Psi(w))) \in C_{mc}(\{y_i \colon w \in X_i\}) \\ &= C_{mc}(\{y_i \colon \overline{\phi(x_i)} \cap \Theta(w) = \emptyset)\} \subseteq C_{mc}(\{y_i \colon \phi(x_i) \cap \Theta(w) = \emptyset\} \end{aligned}$$

and from the KKM-relation condition we obtain

$$C_{mc}(\{y_i \colon \phi(x_i) \cap \Theta(w) = \emptyset\} \subseteq \cup\{\phi(x_i) \colon \phi(x_i) \cap \Theta(w) = \emptyset\},$$

which is in contradiction with the condition that correspondence Θ contains the diagonal. ■

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