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A Minimax Inequality for Vector-Valued Mappings

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Abstract—This paper presents a minimax inequality for vector-valued mappings in Hausdorff topological vector spaces with pointed closed convex cones.

Keywords—Minimax inequalities, Vector-valued mapping, Generalized cone-convexity.

Let X and Y be two nonempty sets and f be a scalar-valued function on the product space $X \times Y$. The following minimax equality:

$$\min_{x\in X} \max_{y\in Y} f(x,y) = \max_{y\in Y} \min_{x\in X} f(x,y) \tag{1}$$

was extensively investigated in the literature of optimization, such as [1]. See [2] for a survey. It is obvious that (1) holds if and only if

$$\min_{x\in X} \max_{y\in Y} f(x,y) \le \max_{y\in Y} \min_{x\in X} f(x,y). \tag{2}$$

In recent years, a number of authors focussed their attention on minimax problems of vector-valued mappings. They gave some versions of (2) when f is a vector-valued mapping, such as Nieuwenhuis [3], Ferro [4–6], Tanaka [7,8], and Shi and Lin [9]. In this paper, we establish a new minimax inequality for vector-valued mappings in Hausdorff topological vector spaces with closed convex pointed cones.

First we give some notations and definitions as follows

Let Z be a topological vector space. We denote by Z^* the topological dual space of Z and 0_Z the zero element in Z. For a subset C of Z, $\operatorname{int} C$ and $\operatorname{cl} C$ denote its topological interior and closure, respectively. Denote

$$\operatorname{cone} C := \{\lambda c : \lambda \ge 0,\ \forall\, c \in C\},$$
$$C^+ := \{f \in Y^* : f(c) \ge 0,\ \forall\, c \in C\},$$

and

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$$C^{+i} := \{f \in Y^* : f(c) > 0, \ \forall\, c \in C \setminus \{0_Z\}\}.$$

Recall that a base B of a cone C is a convex subset of C such that

$$0_Z \notin \operatorname{cl} B \quad \text{and} \quad C = \operatorname{cone} B.$$

It is obvious that a cone C is pointed (i.e., $C \cap (-C) = \{0_Z\}$) if C has a base. Moreover, if C is a nonempty closed convex pointed cone in Z, then $C^{+i} \neq \emptyset$ if and only if C has a base.

DEFINITION 1. *(See [10,11].) Let V be a nonempty subset of Z and C a closed convex pointed cone in Z with* $\operatorname{int} C \neq \emptyset$.

(i) *A point $z \in V$ is said to be* **a C-maximal point of** V *if $V \cap (z + C) = \{z\}$;*
(ii) *a point $z \in V$ is said to be* **a weakly C-minimal point of** V *if $V \cap (z - \operatorname{int} C) = \emptyset$;*
(iii) *a point $z \in V$ is said to be* **a Benson properly C-minimal point of** V *if*

$$(-C) \cap \operatorname{cl}\operatorname{cone}(V + C - z) = \{0_Z\}.$$

We denote $\operatorname{Max} V$, $\operatorname{Min}_w V$, and $\operatorname{Min}_p V$ the set of all the C-maximal points of V, the set of all the weakly C-minimal points of V and the set of all the Benson properly C-minimal points of V, respectively.

Similarly, we can define $\operatorname{Max}_w V$, $\operatorname{Min} V$, and $\operatorname{Max}_p V$.

It is easy to show that

$$\operatorname{Min}_p V \subset \operatorname{Min} V \subset \operatorname{Min}_w V \quad \text{and} \quad \operatorname{Max}_p V \subset \operatorname{Max} V \subset \operatorname{Max}_w V. \tag{3}$$

DEFINITION 2. *(See [4].) Let X be a nonempty convex subset of a real vector space E and Z an ordered topological vector space with a pointed convex cone C. A vector-valued mapping $f : X \to Z$ is said to be*

(i) **C-convex** *if for any $x, y \in X$ and $\lambda \in [0,1]$,*

$$f(\lambda x + (1-\lambda)y) \in \lambda f(x) + (1-\lambda) f(y) - C;$$

(ii) **properly quasi-C-convex** *if for any $x, y \in X$,*

$$\textit{either } f(\lambda x + (1-\lambda)y) \in f(x) - C \quad \textit{or} \quad f(\lambda x + (1-\lambda)y) \in f(y) - C.$$

It should be mentioned that a C-convex mapping is not necessarily properly quasi-C-convex. Conversely a properly quasi-C-convex mapping is not necessarily C-convex. See [4] for a detailed discussion. When $Z = R$ and $C = R_+ := \{r \in R : r \geq 0\}$, the C-convexity and the properly quasi-C-convexity reduce to the ordinary convexity and quasiconvexity, respectively.

Our main result is the following minimax theorem.

THEOREM 1. *Let E_1, E_2, and Z be Hausdorff topological vector spaces, X and Y a nonempty compact convex subset of E_1 and E_2, respectively. Let C be a closed convex pointed cone in Z with* $\operatorname{int} C \neq \emptyset$ *and have a compact base. Let $f : X \times Y \to Z$ be a mapping such that*

(i) *it is continuous;*
(ii) *for each $y \in Y$, $f(\cdot, y)$ is C-convex;*
(iii) *for each $x \in X$, $-f(x, \cdot)$ is properly quasi-C-convex.*

Then

$$\emptyset \neq \operatorname{Min}_p \bigcup_{x \in X} \operatorname{Max} f(x, Y) \subset \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w f(X, y) + Z \setminus (C \setminus \{0_Z\}). \tag{4}$$

If, in addition, the following condition

(iv) *for each $x \in X$, $\operatorname{Min}_p \bigcup_{x \in X} \operatorname{Max} f(x, Y) \subset \operatorname{Max} f(x, Y) - C$,* *is satisfied, then*

$$\mathrm{Min}_p \bigcup_{x\in X} \mathrm{Max}\ f(x,Y) \subset \mathrm{Max} \bigcup_{y\in Y} \mathrm{Min}_w f(X,y) - C. \tag{5}$$

PROOF. Define a set-valued mapping $F : X \to 2^Z$ by

$$F(x) = \mathrm{Max}\ f(x,Y).$$

By Conditions (i),(ii) and [5, Lemma 2.3], F is single-valued, continuous and C-convex. Hence, by the compactness of X, $\bigcup_{x\in X} \mathrm{Max}\ f(x,Y) = F(X)$ is compact. Since the closed convex pointed cone C has a base, we can take a $\varphi \in C^{+i}$. Thus, there exists a $z_0 \in F(X)$ such that

$$\varphi(z_0) = \min\{\varphi(z) : z \in F(X)\}.$$

By [11, Theorem 4.1],

$$z_0 \in \mathrm{Min}_p \bigcup_{x\in X} \mathrm{Max}\ f(x,Y) \neq \emptyset.$$

Let $\bar{z} \in \mathrm{Min}_p \bigcup_{x\in X} \mathrm{Max}\ f(x,Y)$. Thus, there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that

$$\bar{z} = f(\bar{x},\bar{y}) = F(\bar{x}) \in \mathrm{Min}_p F(X). \tag{6}$$

Since F is C-convex, by [11, Theorem 4.2], (6) implies that there exists $h \in C^{+i}$ such that

$$h(\bar{z}) = \min_{x\in X} hF(x). \tag{7}$$

Let $x \in X$. Since $hf(x,\cdot)$ is continuous and Y is compact, there exists a $y_x \in Y$ such that

$$\max_{y\in Y} hf(x,y) = hf(x,y_x). \tag{8}$$

Again by [11, Theorem 4.1], (8), (3), and $h \in C^{+i}$ imply that

$$f(x,y_x) \in \mathrm{Max}_p f(x,Y) \subset \mathrm{Max}\ f(x,Y) = F(x). \tag{9}$$

Combining (7)–(9), we have

$$h(\bar{z}) \le hf(x,y_x) = \max_{y\in Y} hf(x,y). \tag{10}$$

Because x can be any element of X, (10) yields

$$h(\bar{z}) \le \min_{x\in X}\max_{y\in Y} hf(x,y). \tag{11}$$

It is easy to verify that the real-valued function $hf : X \times Y \to R$ has the following properties: it is continuous; $hf(\cdot,y)$ is convex for each $y \in Y$; and $hf(x,\cdot)$ is quasiconcave for each $x \in X$. By the minimax theorem in [12], it follows that

$$\min_{x\in X}\max_{y\in Y} hf(x,y) = \max_{y\in Y}\min_{x\in X} hf(x,y).$$

So, there exist $x_0 \in X$ and $y_0 \in Y$ such that

$$\begin{aligned}\min_{x\in X}\max_{y\in Y} hf(x,y) &= \max_{y\in Y} hf(x_0,y)\\ &= \max_{y\in Y}\min_{x\in X} hf(x,y) = \min_{x\in X} hf(x,y_0) = hf(x_0,y_0),\end{aligned} \tag{12}$$

which together with $h \in C^{+i}$ and (3) yields

$$f(x_0, y_0) \in \mathrm{Min}\, f(X, y_0) \subset \mathrm{Min}_w f(X, y_0), \tag{13}$$

and

$$f(x_0, y_0) \in \mathrm{Max}\, f(x_0, Y) = F(x_0). \tag{14}$$

From (11) and (12), we get

$$h(\bar{z}) \le hf(x_0, y_0),$$

which together with $h \in C^{+i}$ implies

$$\bar{z} \notin f(x_0, y_0) + C \setminus \{0_Z\}. \tag{15}$$

From (15) and (13), we have

$$\begin{aligned} \bar{z} &\in f(x_0, y_0) + Z \setminus (C \setminus \{0_Z\}) \\ &\subset \mathrm{Min}_w f(X, y_0) + Z \setminus (C \{0_Z\}) \\ &\subset \bigcup_{y \in Y} \mathrm{Min}_w f(X, y) + Z \setminus (C \setminus \{0_Z\}). \end{aligned} \tag{16}$$

Because f is continuous and because X and Y are compact, by [6, Lemma 2.1], $\bigcup_{y \in Y} \mathrm{Min}_w f(X, y)$ is compact. Hence, by [13, Lemma 1],

$$\bigcup_{y \in Y} \mathrm{Min}_w f(X, y) \subset \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w f(X, y) - C. \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} \bar{z} &\in \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w f(X, y) - C + Z \setminus (C \setminus \{0_Z\}) \\ &= \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}\ f(X, y) + Z \setminus (C \setminus \{0_Z\}). \end{aligned}$$

Hence, conclusion (4) holds. In addition, if Condition (iv) is satisfied, then

$$\begin{aligned} \bar{z} &\in \mathrm{Max} f(x_0, Y) - C = F(x_0) - C \\ &= f(x_0, y_0) - C \qquad \text{(by (14) and the single-valuedness of } F) \\ &\subset \mathrm{Min}_w f(X, y_0) - C \qquad \text{(by (13))} \\ &\subset \bigcup_{y \in Y} \mathrm{Min}_w f(X, y) - C \subset \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w f(X, y) - C - C \qquad \text{(by (17))} \\ &= \mathrm{Max} \bigcup_{y \in Y} \mathrm{Min}_w f(X, y) - C. \end{aligned}$$

Therefore, conclusion (5) holds. The proof is completed.

REMARK 1. Minimax inequalities (4) and (5) remain true if we replace the Benson properly C-minimal point set $\mathrm{Min}_p(\cdot)$ by the Borwein properly C-minimal point set [14] or the super C-minimal point set [15]. So far as we know, we are first to study the minimax inequalities with the properly C-minimal point sets.

In comparison with those minimax inequalities in [3,5–9], our minimax inequality (4) is new. The proof of Theorem 1 is also different from the others. The minimax inequality (5) is somewhat similar to the one in [8], but the conditions are different.

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