Acta Math. Hungar.
83 (1–2) (1999), 115–123.

A TWO FUNCTION METAMINIMAX THEOREM

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Abstract. A generalization of Simons's metaminimax theorem to a metaminimax theorem involving two functions is given.

1. Introduction

The purpose of this note is to generalize the metaminimax theorem by S. Simons in [19] to a metaminimax theorem involving two functions. As mentioned by S. Simons, a metaminimax theorem is really a device for obtaining minimax theorems rather than a minimax theorem in its own right.

Since von Neumann proved the first minimax theorem in 1928, various generalizations of von Neumann's result have appeared and may be considered as three types based on the hypotheses of the theorem: (1) Quantitative minimax theorems as Fan [3], König [13], Neumann [15], Irle [9], Lin–Quan [14], Kindler [11], and Simons [16]. (2) Topological minimax theorems as Wu [26], Tuy [24], [25], Stachó [22] and Komornik [12], Geraghty–Lin [5], Kindler–Trost [10] and Cheng–Lin [1]. (3) Mixed type minimax theorems as Terkelsen [23], Geraghty–Lin [4], [6] and [7], Kindler [11] and Simons [17]. See [20] for an excellent survey on minimax theorems.

As shown by Simons in [19], Theorem 1 in [19] unifies all the results mentioned above. Theorem 1 in this note extends Theorem 1 in [19] to a metaminimax theorem involving two functions. As examples of application of Theorem 1, we give a two function version of a Komornik minimax theorem [12] in Corollary 7 and a two function minimax theorem in Theorem 8.

Let X and Y be topological spaces. Let R denote the set of real numbers and let f, g be two real-valued functions defined on $X \times Y$. Let $f^* = \inf_Y \sup_X f(x,y)$ and $f_* = \sup_X \inf_Y f(x,y)$. The notation $f \leqq g$ means that $f(x,y) \leqq g(x,y)$ for all $(x,y) \in X \times Y$. For each $x \in X$, let $L_f^\beta(x) = \{ y \in Y : f(x,y) \leqq \beta \}$ and $L_f^\beta(W) = \cap_{x \in W} L_f^\beta(x)$ for any finite subset W in X.

DEFINITIONS. 1. [18] We say that the sets H_0 and H_1 are joined by a set H if $H \cap H_0 \neq \emptyset$, $H \cap H_1 \neq \emptyset$ and $H \subset H_0 \cup H_1$. Let $\mathcal{H}$ be a family of sets. We say $\mathcal{H}$ is pseudoconnected if for $H_0, H_1, H \in \mathcal{H}$ and H_0 and H_1 are joined by H, then $H_0 \cap H_1 \neq \emptyset$.

2. Let W be a subset of X. We say that W is a good set for (g_*, f) in

X if (i) W is finite. (ii) $L_f^{g_*}(x) \cap L_f^{g_*}(W) \neq \emptyset$ for all x in X. It is clear that if W is a good set for (g_*, g) in X then W is a good set for (g_*, f) in X when $f \leqq g$.

2. Main results

THEOREM 1. *Let X and Y be topological spaces, and $f, g : X \times Y \to R$. Suppose that for any $\beta > g_*$ and any good set W for (g_*, f) in X, the following conditions are satisfied.*

(i) *$f \leqq g$;*

(ii)(a) *There exists $\beta_0 > g_*$ such that $L_f^{\beta_0}(x)$ is nonempty and compact for all $x \in X$;*

(b) *There exists $x_0 \in X$ such that $L_f^{g_*}(x_0)$ is nonempty and compact;*

(iii) *The family $\{L_f^{\beta}(x) \cap L_f^{\beta}(W)\}_{x \in X}$ is pseudoconnected;*

(iv) *For any x_0, $x_1 \in X$, there exists $x \in X$ such that $L_f^{\beta}(x_0)$ and $L_g^{\beta}(x_1)$ are joined by $L_f^{\beta}(x) \cap L_f^{\beta}(W)$;*

(v) *$f(x, \cdot)$ is lower semicontinuous for all $x \in X$.*

Then

$$(*) \qquad \inf_{y \in Y} \sup_{x \in X} f(x, y) \leqq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

PROOF. Let $x \in X$. If $\mu > g_*$ in R, then there exists y in Y such that $\mu \geqq g(x, y)$. This implies that $y \in L_g^{\mu}(x) \subset L_f^{\mu}(x)$. Hence

$$(1) \qquad L_f^{\mu}(x) \neq \emptyset \quad \text{for all} \quad \mu > g_*.$$

Since $f(x, \cdot)$ is lower semicontinuous, $L_f^{\mu}(x)$ is closed. Also, for any finitely many $\mu_1, \ldots, \mu_n$ with $\mu_i > g_*$ where $i = 1, 2, \ldots, n$, $\min_{1 \leqq i \leqq n}\{\mu_i\} > g_*$. Hence by (1), we have $\bigcap_{i=1}^{n} L_f^{\mu_i}(x) \neq \emptyset$. So $\{L_f^{\mu}(x) : \mu > g_*\}$ is a family of nonempty closed sets with finite intersection property. Hence $\{L_f^{\mu}(x) \cap L_f^{\beta_0}(x) : \mu > g_*\}$ is a family of nonempty closed sets with the finite intersection property. This family is contained in $L_f^{\beta_0}(x)$ which is compact by (ii)(a). Therefore, $\bigcap_{\mu > g_*} L_f^{\mu}(x) \neq \emptyset$. It follows that

$$(2) \qquad \emptyset \neq L_f^{g_*}(x).$$

Since x is arbitrary in X, (2) shows that the empty set is a good set for (g_*, f) in X.

Next, we show, by induction, that every finite subset W of X is a good set for (g_*, f). Suppose that $n \geqq 1$ and $W \subset X$, $|W| \leqq n-1$ imply that W is a good set for (g_*, f).

Let $V \subset X$ and $|V| = n$. Let $x_0 \in V$ and set $W = V \setminus \{x_0\}$. Then, W is a good set for (g_*, f) in X. Let $x \in X$ and $\beta > g_*$. By (iv), there exists an $x_1 \in X$ such that $L_f^\beta(x_0)$ and $L_g^\beta(x)$ are joined by $L_f^\beta(x_1) \cap L_f^\beta(W)$. That is,

$$L_f^\beta(x_0) \cap \left(L_f^\beta(x_1) \cap L_f^\beta(W)\right) \neq \emptyset, \tag{3}$$

$$L_g^\beta(x) \cap \left((L_f^\beta(x_1) \cap L_f^\beta(W)\right) \neq \emptyset, \tag{4}$$

and

$$L_f^\beta(x_1) \cap L_f^\beta(W) \subset L_f^\beta(x_0) \cup L_g^\beta(x). \tag{5}$$

By (i), (3), (4) and (5),

$$L_f^\beta(x_0) \cap \left(L_f^\beta(x_1) \cap L_f^\beta(W)\right) \neq \emptyset, \qquad L_f^\beta(x) \cap \left(L_f^\beta(x_1) \cap L_f^\beta(W)\right) \neq \emptyset,$$

and

$$L_f^\beta(x_1) \cap L_f^\beta(W) \subset L_f^\beta(x_0) \cup L_f^\beta(x).$$

It follows that $L_f^\beta(x_0) \cap L_f^\beta(W)$ and $L_f^\beta(x) \cap L_f^\beta(W)$ are joined by $L_f^\beta(x_1) \cap L_f^\beta(W)$. By (iii),

$$\left(L_f^\beta(x) \cap L_f^\beta(W)\right) \bigcap \left(L_f^\beta(x_0) \cap L_f^\beta(W)\right) \neq \emptyset.$$

Hence

$$L_f^\beta(x) \bigcap L_f^\beta(V) \neq \emptyset. \tag{6}$$

Since (6) is true for all $\beta > g_*$, by (ii)(a) and using the same argument as above, we see that

$$L_f^{g_*}(x) \bigcap L_f^{g_*}(V) \neq \emptyset. \tag{7}$$

Since x is arbitrary in X, so (7) shows that V is a good set for (g_*, f) in X. Hence, by induction, we conclude that any finite set of X is a good set for (g_*, f) in X.

Now, from (7), $\left\{L_f^{g_*}(x) \cap L_f^{g_*}(x_0)\right\}_{x \in X}$ is a family of nonempty closed sets with finite intersection property and the family is contained in the set

$L_f^{g_*}(x_0)$ which is compact by (ii)(b), hence $\bigcap_{x\in X} L_f^{g_*}(x) \neq \emptyset$. Therefore, there exists y^* in Y such that $f(x,y^*) \leqq g_*$ for all x in X. This implies that

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leqq g_* = \sup_{x\in X}\inf_{y\in Y} g(x,y). \qquad \square$$

COROLLARY 2. *Let X be a topological space and Y a compact topological space. Let $f, g : X \times Y \to R$. Suppose that for any $\beta > g_*$ and any good set W for (g_*, f) in X, the conditions* (i), (iii)–(v) *of Theorem* 1 *are satisfied. Then* $(*)$ *holds.*

PROOF. Since Y is compact and $f(x,\cdot)$ is lower semicontinuous, $L_f^\beta(x)$ is nonempty and compact for all $\beta > g_*$ and for all x in X. Hence the minimax result follows from Theorem 1. $\square$

COROLLARY 3 [19]. *Let X and Y be topological spaces, and $f : X \times Y \to R$. Suppose that for any $\beta > f_*$ and any good set W for (f_*, f) in X, the conditions* (iii) *and* (v) *of Theorem* 1, *as well as the following conditions are satisfied:*

(i) *$L_f^{\beta_0}(x)$ is compact for some $\beta_0 \geqq f_*$ for all $x \in X$, and $L_f^{f_*}(x_0)$ is compact for some x_0 in X;*

(ii) *For any x_0, x_1 in X, there exists $x \in X$ such that $L_f^\beta(x_0)$ and $L_f^\beta(x_1)$ are joined by $L_f^\beta(x) \cap L_f^\beta(W)$.*

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

PROOF. Let $f = g$ in Theorem 1. $\square$

REMARK 1. The condition that W is good for (g_*, f) in Theorem 1 is essential since if we replace it by W is good for (f_*, f) then Theorem 1 is nothing more than Simons' flexible minimax theorem. Notice that W is good for (g_*, f) does not imply W is good for (f_*, f).

3. Applications of the main theorem

LEMMA 4. *Suppose that*

(i) *X is a topological space, $\beta \in R$, x_0, $x_1 \in X$ and there exists a connected subset $C \subset X$ such that $\{x_0, x_1\} \subset C$ and for all x in C*

$$L_f^\beta(x) \subset L_f^\beta(x_0) \cup L_g^\beta(x_1);$$

(ii) *Y is a compact topological space and $Z \subset Y$ and the sets $\{(x,y) \in C \times Z : f(x,y) \leqq \beta\}$ and $\{(x,y) \in C \times Z : g(x,y) \leqq \beta\}$ are closed in $C \times Y$ and $L_g^\beta(x) \cap Z \neq \emptyset$ for all $x \in C$.*

Then there exists $x \in X$ such that $L_f^\beta(x_0)$ and $L_g^\beta(x_1)$ are joined by $L_f^\beta(x) \cap Z$.

PROOF. Suppose $L_f^\beta(x_0) \cap L_g^\beta(x_1) \cap Z \neq \emptyset$, then it is easy to check that, by letting $x = x_0$, $L_f^\beta(x_0)$ and $L_g^\beta(x_1)$ are joined by $L_f^\beta(x) \cap Z$. So we may assume that

$$(L_f^\beta(x_0) \cap Z) \bigcap (L_g^\beta(x_1) \cap Z) = \emptyset. \tag{1}$$

Let

$$C_0 = \{x \in C : L_f^\beta(x) \cap Z \subset L_g^\beta(x_1)\}$$

and

$$C_1 = \{x \in C : L_f^\beta(x) \cap Z \subset L_f^\beta(x_0)\}.$$

By (i) and (1),

$$\begin{cases} C_0 = \{x \in C : L_f^\beta(x) \cap Z \cap L_f^\beta(x_0) = \emptyset\}, \\ C_1 = \{x \in C : L_f^\beta(x) \cap Z \cap L_g^\beta(x_1) = \emptyset\}, \end{cases} \tag{2}$$

and $C_0 \cap C_1 = \emptyset$. If there exists an element $x \in C \setminus (C_0 \cup C_1)$ then $L_f^\beta(x) \cap Z \cap L_f^\beta(x_0) \neq \emptyset$, $L_f^\beta(x) \cap Z \cap L_g^\beta(x_1) \neq \emptyset$ and $L_f^\beta(x) \cap Z \subset L_f^\beta(x_0) \cup L_g^\beta(x_1)$ and we are done. Hence we may assume that $C = C_0 \cup C_1$.

Since $x_0 \in C_1$, $C_1 \neq \emptyset$. If C_0 is empty, then $C = C_1$. Since $x_1 \in C$, this implies that $L_f^\beta(x_1) \cap Z \cap L_g^\beta(x_1) = L_g^\beta(x_1) \cap Z = \emptyset$ which contradicts (ii). Hence C_0 is also nonempty.

For $x \in C$, it is easy to check the following statement:

$$x \in C_0 \Leftrightarrow \quad \text{there exists} \quad y \in L_g^\beta(x_1) \cap Z \quad \text{and} \quad f(x,y) \leqq \beta. \tag{3}$$

Indeed, if $x \in C_0$, then by (ii), there exists an element $y \in L_g^\beta(x) \cap Z \subset L_f^\beta(x) \cap Z$. From the definition of C_0 and (i), we see that $y \in L_g^\beta(x_1)$. Thus $y \in L_g^\beta(x_1) \cap Z$ and $f(x,y) \leqq \beta$. The proof of the other direction is trivial.

Now let $\{x_\lambda\}$ be a net in C_0 with $x_\lambda \to x \in C$. By (3), for each λ, there exists $y_\lambda \in L_g^\beta(x_1) \cap Z$ and $f(x_\lambda, y_\lambda) \leqq \beta$. Since Y is compact, passing to a subnet if necessary, we may assume $y_\lambda \to y$ for some y in Y. Then $(x_\lambda, y_\lambda) \to (x, y)$ and $(x_1, y_\lambda) \to (x_1, y)$. By (ii), we see that $y \in Z$, $f(x,y) \leqq \beta$ and $g(x_1, y) \leqq \beta$. From (3), it follows that $x \in C_0$. Hence C_0 is closed. Similarly, we can prove that C_1 is also closed. But this contradicts the connectedness of C. □

The following is a two function version of Lemma 4 in [18].

LEMMA 5. *Let X and Y be topological spaces, $Z \subset Y$ and $\beta \in R$. Suppose that*

(i) *For x_0, $x_1 \in X$, there exists a connected subset $C \subset X$ such that $\{x_0, x_1\} \subset C$ and for all x in C, $L_f^\beta(x) \subset L_f^\beta(x_0) \cup L_g^\beta(x_1)$.*

(ii) *The sets $O_y := \{ x \in C : g(x,y) < \beta \}$ are open in C for any y in Z;*

(iii) *For any x in C, there exists $y \in Z$ such that $g(x,y) < \beta$.*

Then there exists $x \in X$ such that $L_f^\beta(x_0)$ and $L_g^\beta(x_1)$ are joined by $L_f^\beta(x) \cap Z$.

PROOF. Proceed as in Lemma 4 up to (2), where C_0 and C_1 are both nonempty, $C_0 \cap C_1 = \emptyset$, and $C = C_0 \cup C_1$.

For $x \in C$, we have, by (iii),

$$(1) \qquad x \in C_0 \Leftrightarrow \quad \text{there exists} \quad y \in L_g^\beta(x_1) \cap Z \quad \text{and} \quad g(x,y) < \beta.$$

Let $x \in C_0$. By (1), there exists an element $y \in L_g^\beta(x_1) \cap Z$ such that $g(x,y) < \beta$. Hence $x \in O_y$. If there exists $x' \in O_y \setminus C_0$, then $L_f^\beta(x') \cap L_g^\beta(x_1) \cap Z = \emptyset$. But $x' \in O_y$, hence $y \in L_f^\beta(x')$. This implies that $L_f^\beta(x') \cap L_g^\beta(x_1) \cap Z \neq \emptyset$, a contradiction. Hence, $x \in O_y \subset C_0$. Therefore, C_0 is an open set in C. Similarly, C_1 is also an open set in C by the following condition:

$$x \in C_1 \Leftrightarrow \quad \text{there exists} \quad y \in L_f^\beta(x_0) \cap Z \quad \text{and} \quad f(x,y) < \beta.$$

However, C is a connected set, we obtain a contradiction. □

THEOREM 6. *Let X and Y be topological spaces with Y compact. Let $f, g : X \times Y \to R$. Suppose that for any $\beta > g_*$, the conditions* (i) *and* (v) *of Theorem* 1, (i) *of Lemma* 5, *as well as the following conditions are satisfied:*

(i) *For any finite set $A \subset X$, $\cap_{x \in A} \{ y \in Y : f(x,y) \leqq \beta \}$ is connected;*

(ii) *The sets $\{ (x,y) \in C \times Y : f(x,y) \leqq \beta \}$ and $\{ (x,y) \in C \times Y : g(x,y) \leqq \beta \}$ are closed in $C \times Y$.*

Then $(*)$ *holds.*

PROOF. Let $\beta > g_*$ and let W be a good set for (g_*, g). For $x \in C$, since $L_g^\beta(x) \cap L_f^\beta(W) \supset L_g^\beta(x) \cap L_g^\beta(W) \supset L_g^{g_*}(x) \cap L_g^{g_*}(W) \neq \emptyset$, hence condition (ii) of Lemma 4 is satisfied with $Z = L_f^\beta(W)$. Consequently, by Lemma 4, there exists an element x in X such that $L_f^\beta(x_0)$ and $L_g^\beta(x_1)$ are joined by $L_f^\beta(x) \cap L_f^\beta(W)$. By Theorem 1, the minimax result follows. □

REMARK 2. Condition (i) of Lemma 5 is satisfied if for any $x_0, x_1 \in X$, there exists a connected set C in X such that $\{x_0, x_1\} \subset C$ and for all x in C

$$f(x,y) \geqq \min\left\{ f(x_0,y), g(x_1,y)\right\};$$

or if $f(\cdot, y)$ is quasi-concave on X for any y in Y. Hence, we obtain the following

COROLLARY 7. *Let X and Y be topological spaces with Y compact. Let $f, g : X \times Y \to R$. Suppose that for any $\beta > g_*$, condition* (i) *of Theorem* 1, (i), (ii) *of Theorem* 6 *are satisfied and*

(iii) *For any x_0, $x_1 \in X$, there exists a connected set C in X such that $\{x_0, x_1\} \subset C$ and for all x in C,*

$$f(x,y) \geqq \min\left\{ f(x_0,y), g(x_1,y)\right\},$$

or,

(iii′) *$f(\cdot, y)$ is quasi-concave on X for any y in Y.*

Then $(*)$ *holds.*

REMARK 3. Corollary 7 gives a two-function generalization of Theorem 2 in [12] and hence Ha's minimax theorem [8]. It is worth to note that Ha's minimax theorem is different from Sion's minimax theorem in the continuity conditions of f. See [8] and [21].

THEOREM 8. *Let X and Y be topological spaces with Y compact. Let $f, g : X \times Y \to R$. Suppose that for any $\beta > g_*$, the conditions* (i) *and* (v) *of Theorem* 1, (ii) *of Theroem* 6, (i) *of Lemma* 5, *as well as the following condition is satisfied.*

(i) *The sets $\left\{x \in C : g(x,y) < \beta\right\}$ are open in C for any y in Y.*

Then $(*)$ *holds.*

PROOF. Let $\beta > g_*$ and W a good set for (g_*, g). Then there exists an element y in $L_g^{g_*}(W) \cap L_g^{g_*}(x)$ for any x in X. Since $y \in L_g^{g_*}(W) \subset L_g^\beta(W)$ and $g(x,y) \leqq g_* < \beta$, so Lemma 5 is satisfied with $Z = L_f^\beta(W)$. Therefore, by Theorem 1 and Lemma 5, the minimax result follows.

By using Theorem 1 and Lemma 5, we can also obtain a non-compact version of Theorem 8 as shown in [2].

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(Received September 15, 1997; revised February 2, 1998)

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