Acta Math. Acad. Sci. Hungar.
40 (1—2), (1982), 159—163.

MINIMAX THEOREMS FOR UPPER SEMICONTINUOUS FUNCTIONS

By
V. KOMORNIK (Budapest)

1. The various generalizations of von Neumann's classical minimax theorem [1] constitute an important chapter of the modern analysis. In the economic applications it might have some interest to prove minimax theorems for vector-valued functions, e.g. for functions mapping into $\mathbf{R}^n$, endowed with the lexicographic order. As the first theorem of this paper shows, without any further conditions Neumann's result does not remain true for such functions.

In a recent publication [5], C.-W. HA generalized Neumann's minimax theorem (see also in [3]) for upper semicontinuous functions. Our second theorem establishes a slightly more general form of this result, which contains also Theorem 1 in [4]. Our proof is based on the considerations, developed by I. Joó in [2] and [3]; thus we can eliminate the application of Brouwer's fixed point theorem, essentially used in [5]. Theorem 2 is formulated for functions mapping into a linearly ordered space. Thus we obtain a positive answer for the minimax problem of vector-valued functions.

The third theorem of this paper asserts that in case if one of the underlying spaces is a convex subset of some topological vector space, the continuity conditions of Theorem 2 can be weakened.

The author is grateful to I. Joó for proposing the minimax problem of vector-valued functions.

2. THEOREM 1. *There exists a continuous function* $f\colon [0,1]\times[-1,1]\to[-1,1]\times[-1,1]$ *such that*

(1^x) *the subfunctions* $f(\cdot, y)$ *are concave for any fixed* $y\in[-1,1]$,

(1^y) *the subfunctions* $f(x,\cdot)$ *are convex for any fixed* $x\in[0,1]$;

nevertheless

$$\max_x \min_y f(x,y) = (0,-1) \neq (0,0) = \min_y \max_x f(x,y) \tag{2}$$

($[-1,1]\times[-1,1]$ *is equipped with the lexicographic order*).

PROOF. Consider the continuous function

$$f\colon [0,1]\times[-1,1]\to[-1,1]\times[-1,1], \quad f(x,y)=(xy,-y).$$

It is easy to see that

$$\min_y f(x,y) = \begin{cases} (0,-1) & \text{if } x=0, \\ (-x,1) & \text{if } 0<x\leqq 1, \end{cases}$$

$$\max_x f(x,y) = \begin{cases} (0,-y) & \text{if } -1\leqq y\leqq 0, \\ (y,-y) & \text{if } 0<y\leqq 1. \end{cases}$$

From these relations we obtain (2) at once.

To prove (1^x) and (1^y), we have to show that

$$f(tx_1+(1-t)x_2, y) \geqq tf(x_1, y)+(1-t)f(x_2, y)$$

for any $x_1, x_2 \in [0, 1]$, $y \in [-1, 1]$, $t \in [0, 1]$, and

$$f(x, ty_1+(1-t)y_2) \leqq tf(x, y_1)+(1-t)f(x, y_2)$$

for any $x \in [0, 1]$, $y_1, y_2 \in [-1, 1]$, $t \in [0, 1]$. But these conditions are obviously satisfied; moreover, we have not only inequality but also equality in both cases:

$$(tx_1y+(1-t)x_2y, -y) = t(x_1y, -y)+(1-t)(x_2y, -y)$$

and

$$(xty_1+x(1-t)y_2, -ty_1-(1-t)y_2) = t(xy_1, -y_1)+(1-t)(xy_2, -y_2).$$

The theorem is proved.

3. We recall that by an *interval space* (see [4]) we mean a topological space X endowed with a mapping $[\cdot, \cdot]: X \times X \to \{$connected subsets of $X\}$ such that $x_1, x_2 \in [x_1, x_2] = [x_2, x_1]$ for all $x_1, x_2 \in X$. A subset K of an interval space is *convex* if for every $x_1, x_2 \in K$ we have $[x_1, x_2] \subset K$. Any convex subset of a real topological vector space is an interval space with its natural interval structure.

A *linearly ordered space* (see [6]) is called *complete* if every subset has a least upper bound. Such spaces are the extended real line $\overline{\mathbf{R}}$, the extended euclidean n-space $\overline{\mathbf{R}}^n$ or any compact (in the euclidean topology) subset of $\mathbf{R}^n$ with respect to the lexicographic order.

Let X be an interval space and Z a complete linearly ordered space. A function $f: X \to Z$ is called *quasiconvex* (resp. *quasiconcave*) if the sets

$$\{x \in X: f(x) \leqq z\} \quad (\text{resp. } \{x \in X: f(x) \geqq z\})$$

are convex for all $z \in Z$. Furthermore, f is called *upper semicontinuous* if all the sets

$$\{x \in X: f(x) \geqq z\}, \quad z \in Z,$$

are closed in X.

If X is compact and $f: X \to Z$ is upper semicontinuous, then there exists an $x_0 \in X$ such that $f(x_0) = \sup_{x \in X} f(x)$. Given a family $(f_i)_{i \in I}$ of upper semicontinuous functions from X into Z, the map $\inf_{i \in I} f_i$ is also upper semicontinuous. These statements are proved in the same way as in case $Z = \overline{\mathbf{R}}$.

THEOREM 2. *Let X be a compact interval space, Y an arbitrary interval space, Z a complete linearly ordered space and $f: X \times Y \to Z$ an upper semicontinuous function such that*

(3^x) *the subfunctions $f(\cdot, y)$ are quasiconcave on X for any fixed $y \in Y$,*

(3^y) *the subfunctions $f(x, \cdot)$ are quasiconvex on Y for any fixed $x \in X$.*

Then

$$\max_x \inf_y f(x, y) = \inf_y \max_x f(x, y). \tag{4}$$

PROOF. The expressions in (4) make sense by the two statements mentioned just before this theorem. Being the relation $\max_x \inf_y f(x, y) \leqq \inf_y \max_x f(x, y)$ obvious, it is enough to show that the family of sets

$$\{K(y) \equiv \{x\in X : f(x, y) \geqq \inf_y \max_x f(x, y)\} : y\in Y\}$$

has a non-empty intersection.

For any $y\in Y$, the set $K(y)$ is convex by (3^x) and non-empty by the definition of $\inf_y \max_x f(x, y) \equiv z^*$. Moreover, $K(y)$ is compact because X is compact and f is upper semicontinuous.

It follows from (3^y) that for any $y_1, y_2\in Y$ and $y\in[y_1, y_2]$, $K(y)\subset K(y_1)\cup K(y_2)$. Finally, if $\lim_{i\in I} x_i = x$, $\lim_{i\in I} y_i = y$ and $x_i\in K(y_i)$ for all $i\in I$, then $x\in K(y)$. Indeed, we have $f(x_i, y_i)\geqq z^*$ for all $i\in I$ and $\lim_{i\in I}(x_i, y_i) = (x, y)$. Hence, by the upper semicontinuity of f, $f(x, y)\geqq z^*$, i.e. $x\in K(y)$.

On the basis of these properties, our theorem follows from the fixed point theorem of I. Joó [2], which can be proved by simple tools (the present formulation is due to L. L. Stachó [4]):

Let X, Y be interval spaces and $K(\cdot)$ a mapping of Y into the family of compact convex subsets of X, such that

(i) $K(y)\neq\emptyset$ for all $y\in Y$,

(ii) $K(y)\subset K(y_1)\cup K(y_2)$ whenever $y\in[y_1, y_2]$ and $y_1, y_2\in Y$;

(iii) $x\in K(y)$ whenever $y = \lim_{i\in I} y_i$; $x = \lim_{i\in I} x_i$ and $x_i\in K(y_i)$ for all $i\in I$.

Then we have

$$\bigcap_{y\in Y} K(y) \neq \emptyset.$$

4. Theorem 3. *Let X be a compact interval space, Y a convex subset of some real topological vector space, Z a complete linearly ordered space and $f: X\times Y\to Z$ a function, having the properties*

(5^x) *the subfunctions $f(\cdot, y)$ are quasiconcave on X and upper semicontinuous on X for all fixed $y\in Y$,*

(5^y) *the subfunctions $f(x, \cdot)$ are quasiconvex on Y and upper semicontinuous on any interval of Y for all fixed $x\in X$.*

Then

$$\max_x \inf_y f(x, y) = \inf_y \max_x f(x, y).$$

Remark. As Theorem 2 in [4] shows, this assertion is true if we require in (5^y) lower semicontinuity instead of upper semicontinuity.

Proof. It suffices again to prove that the family of sets

$$\mathscr{F} \equiv \{K(y) \equiv \{x\in X : f(x, y) \geqq \inf_y \max_x f(x, y)\} : y\in Y\}$$

has a non-empty intersection. Being the elements of $\mathscr{F}$ compact (because of (5^x) and the compactness of X), it suffices to show that $\mathscr{F}$ has the finite intersection property. The definition of $\inf_y \max_x f(x, y)\equiv z^*$ ensures that $K(y)\neq\emptyset$ for all $y\in Y$. Assume now that $\bigcap_{i=1}^{n} K(y_i)\neq\emptyset$ for every choice of $y_1, \dots, y_n\in Y$, but

$\bigcap_{i=1}^{n+1} K(y_i^*) = \emptyset$ for some $y_1^*, \ldots, y_{n+1}^* \in Y$. To complete the proof, we show that this is impossible. Set $K^*(y) \equiv \bigcap_{i=3}^{n+1} K(y_i^*) \cap K(y)$ for all $y \in Y$, then

(6) $$K^*(y_1^*) \cap K^*(y_2^*) = \emptyset.$$

It follows from the inductive hypothesis and (5^x) that

(7) $K^*(y)$ is non-empty, convex and compact for all $y \in Y$.

(5^y) implies that

(8) $K^*(y) \subset K^*(y_1) \cup K^*(y_2)$ whenever $y_1, y_2 \in Y$ and $y \in [y_1, y_2]$.

Furthermore,

(9) either $K^*(y) \subset K^*(y_1^*)$ or $K^*(y) \subset K^*(y_2^*)$ for any $y \in [y_1^*, y_2^*]$.

Indeed, if there were points x_1, x_2 such that $x_i \in K^*(y) \cap K^*(y_i^*)$ $(i=1, 2)$ for some $y \in [y_1^*, y_2^*]$, then — using (6), (7) and (8) — the connected set $[x_1, x_2]$ could be represented as the union of two closed, non-empty and disjoint subsets:

$$[x_1, x_2] = \bigcup_{i=1}^{2} [x_1, x_2] \cap K^*(y) \cap K^*(y_i^*),$$

which is impossible.

For brevity, we write henceforth $[y_1, y_2) = (y_2, y_1]$ instead of $[y_1, y_2] \setminus \{y_2\}$. It follows from (6)—(9) that the sets

$$\{y \in [y_1^*, y_2^*] : K^*(y) \subset K^*(y_i^*)\}, \quad i = 1, 2$$

are disjoint convex sets and their union is $[y_1^*, y_2^*]$. Therefore there exists a point $y_0 \in [y_1^*, y_2^*]$ such that

(10) $$K^*(y) \subset K^*(y_i^*) \quad \text{for all} \quad y \in [y_i^*, y_0), \quad i = 1, 2.$$

Suppose

(11) $$K^*(y_0) \subset K^*(y_1^*)$$

(the case $K^*(y_0) \subset K^*(y_2^*)$ is similar). Then $\bigcap_{y \in [y_2^*, y_0)} K^*(y) \neq \emptyset$. Indeed, being the sets $K^*(y)$ compact, it is enough to show that for any $y_1 \in (y_0, y_2]$, $y_2 \in (y_0, y_2^*]$: $K^*(y_1) \subset K^*(y_2)$. But this is true: the application of (8), (11), (10) and (6) gives

$$K^*(y_1) \subset (K^*(y_0) \cup K^*(y_2)) \cap K^*(y_2^*) \subset (K^*(y_1^*) \cup K^*(y_2)) \cap K^*(y_2^*) =$$

$$= (K^*(y_1^*) \cap K^*(y_2^*)) \cup (K^*(y_2) \cap K^*(y_2^*)) = \emptyset \cup K^*(y_2) = K^*(y_2).$$

Choosing an arbitrary $x_0 \in \bigcap_{y \in [y_2^*, y_0)} K^*(y)$, we have by definition $f(x_0, y) \geqq z^*$ for all $y\in[y_2^*, y_0)$; and taking the limit $y\to y_0$, we obtain by (5^y)

$$f(x_0, y_0) \geqq z^*. \tag{12}$$

On the other hand; $x_0\in K^*(y_2^*)$, (6) and (11) imply $x_0\neq K^*(y_0)$ i.e. $f(x_0; y_0)<z^*$, contradicting (12). This contradiction proves the theorem.

References

[1] J. von Neumann, Zur Theorie der Gesellschaftsspiele, *Math. Ann.*, **100** (1928), 295—320.
[2] I. Joó and A. P. Sövegjártó, A fixed point theorem, *Ann. Univ. Sci. Budapest, Sect. Math.* (to appear).
[3] I. Joó, A simple proof for von Neumann's minimax theorem, *Acta Sci. Math. Szeged*, **42** (1980), 91—94.
[4] L. L. Stachó, Minimax theorems beyond topological vector spaces, *Acta Sci. Math. Szeged*, **42** (1980), 157—164.
[5] C.-W. Ha, Minimax and fixed point theorems, *Math. Ann.*, **248** (1980), 73—77.
[6] R. Engelking, *General topology* (Warszawa, 1977).

(Received May 14, 1981)

DEPARTMENT II OF ANALYSIS
LORÁND EÖTVÖS UNIVERSITY
H-1445 BUDAPEST 8, PF. 323.