JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **225,** 312–325 (1998)
ARTICLE NO. AY986042

Two-Function Topological Minimax Theorems

Jürgen Kindler

Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstr. 7, D–64289 Darmstadt, Germany

Submitted by Ying-ming Liu

Received June 25, 1997

Three recent minimax theorems of Lin and Quan, Cheng and Lin, and Chang, Cao, Wu, and Wang are generalized, where two functions are involved and where the classical convexity assumptions are replaced by connectedness properties of certain level sets. © 1998 Academic Press

1. INTRODUCTION

As in [8, 9] a topological space X will be called a *topological midset space* if it is endowed with a set-valued map $Z: X \times X \to 2^X$, a *topological midset function*, such that every *midset* $Z(x_1, x_2)$, $(x_1, x_2) \in X \times X$, is connected. A subset A of X is *convex* iff $Z(x_1, x_2) \subset A$ for all $\{x_1, x_2\} \subset A$, and A is called *midset-closed* iff $Z(x_1, x_2) \cap A$ is closed in $Z(x_1, x_2)$ for every midset $Z(x_1, x_2)$ in X.

Following Stachó [20] a topological midset space X will be called an *interval space* iff $\{x_1, x_2\} \subset Z(x_1, x_2)$, $(x_1, x_2) \in X \times X$. In this case we write $Z = \langle \cdot, \cdot \rangle$ for the *interval function*, and the midsets $\langle x_1, x_2 \rangle$ are called *intervals*. Many examples can be found in [8, 9].

Let $\Phi: X \to 2^Y$ be a set-valued map. Then Φ is called a *correspondence* iff every value $\Phi(x)$, $x \in X$, is nonvoid. We denote by $\mathscr{E}(X)$ the system of all nonvoid finite subsets of X, and for $A \in 2^X$ we set $\Phi^\cap(A) := \bigcap_{x \in A} \Phi(x)$ with $\bigcap_{x \in \emptyset} \Phi(x) = Y$. Furthermore, the *dual* Φ^* of Φ is defined according to $\Phi^*(y) := \{x \in X: y \notin \Phi(x)\}$, $y \in Y$.

If X and Y are topological spaces then, according to Komiya [13], a set-valued map $\Phi: X \to 2^Y$ will be called *quartercontinuous* iff for all $x \in X$ and for any open set $G \supset \Phi(x)$ there is a neighborhood U of x such that $\Phi(u) \cap G \neq \emptyset$ for all $u \in U$.

0022-247X/98 $25.00

As usual, we set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, $\alpha \vee \beta = \max\{\alpha, \beta\}$, and $\alpha \wedge \beta = \min\{\alpha, \beta\}$.

We want to generalize the following three minimax theorems:

THEOREM A (Lin and Quan [15]). *Let X be a Hausdorff interval space, Y be a compact Hausdorff space, and f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$ with the properties*

(0) $f(x, y) \le g(x, y)$ *for all* $(x, y) \in X \times Y$,

(i) *every function* $f(x, \cdot)$, $x \in X$, *is lower semicontinuous,*

(ii) *every function* $g(x, \cdot)$, $x \in X$, *is lower semicontinuous,*

(iii) *for each nonvoid closed subset F of Y and each real number γ with $\sup_{x\in X} \inf_{y\in F} g(x, y) < \gamma$ the correspondence Λ with $\Lambda(x) = \{y \in Y\colon g(x, y) \le \gamma\} \cap F$, $x \in X$, is quartercontinuous,*

(iv) *for any $x_1, \dots, x_n \in X$ and any $\beta_1, \dots, \beta_n \in \mathbb{R}$ the set $\bigcap_{i=1}^{n}\{y \in Y\colon f(x_i, y) \le \beta_i\}$ is either connected or empty, and*

(v) $f(x, y) \ge f(x_1, y) \wedge g(x_2, y)$ *for all* $x \in \langle x_1, x_2\rangle$, $(x_1, x_2) \in X \times X$, *and all* $y \in Y$.

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) \le \sup_{x\in X} \inf_{y\in Y} g(x, y)$.

The special case $f = g$ of Theorem A is owing to Komiya [13].

THEOREM B (Cheng and Lin, [4]). *Let X be a Hausdorff interval space, Y be a Hausdorff space, and f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$ with the properties*

(0) $f(x, y) \le g(x, y)$ *for all* $(x, y) \in X \times Y$,

(i) *every function* $f(x, \cdot)$, $x \in X$, *is lower semicontinuous,*

(ii) *every function* $g(x, \cdot)$, $x \in X$, *is lower semicontinuous,*

(iii) *every function* $g(\cdot, y)$, $y \in Y$, *is upper semicontinuous on every interval of* X,

(iv) *for all $x_1, \dots, x_n \in X$ and $\beta \in \mathbb{R}$ the set $\bigcap_{i=1}^{n}\{y \in Y\colon f(x_i, y) < \beta\}$ is either connected or empty,*

(v) $f(x, y) \ge f(x_1, y) \wedge g(x_2, y)$ *for all* $x \in \langle x_1, x_2\rangle$, $(x_1, x_2) \in X \times X$, *and all* $y \in Y$, *and*

(vi) *there exists an $x_0 \in X$ and a $\beta_0 \in \mathbb{R}$ with $\beta_0 \ge \inf_{y\in Y} \sup_{x\in X} f(x, y)$ such that the set $\{y \in Y\colon f(x_0, y) \le \beta_0\}$ is compact.*

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) \le \sup_{x\in X} \inf_{y\in Y} g(x, y)$.

THEOREM C (Chang *et al.* [3]). *Let X be a topological midset space, Y be a topological space, and f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$ with the properties*

(0) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in X \times Y$,

(i) *every function* $f(x, \cdot)$, $x \in X$, *is lower semicontinuous*,

(ii) *every function* $g(x, \cdot)$, $x \in X$, *is lower semicontinuous*,

(iii) *every function* $g(\cdot, y)$, $y \in Y$, *is upper semicontinuous on every midset of* X,

(iv) *for all* $x_1, \ldots, x_n \in X$ *and* $\beta \in \mathbb{R}$ *the set* $\bigcap_{i=1}^{n}\{y \in Y\colon g(x_i, y) < \beta\}$ *is either connected or empty*,

(v) $g(x, y) \geq g(x_1, y) \wedge g(x_2, y)$ *for all* $x \in Z(x_1, x_2)$, $(x_1, x_2) \in X \times X$, *and all* $y \in Y$,

(vi) $\forall (x_1, x_2) \in X \times X\ \exists \{s_1, s_2\} \subset Z(x_1, x_2)\ \forall y \in Y\ \forall i \in \{1, 2\}$: $g(s_i, y) \geq g(x_i, y)$, *and*

(vii) *there exists an* $x_0 \in X$, *a* $\beta_0 \in \mathbb{R}$ *with* $\beta_0 > \sup_{x\in X} \inf_{y\in Y} g(x, y)$, *and a compact subset L of Y such that* $f(x_0, y) > \beta_0$ *for all* $y \in Y - L$.

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} \inf_{y\in Y} g(x, y)$.

In fact, Theorem C is formulated in [3, Theorem 3.1] in a dual form.

It turns out that assumption (ii) in Theorems A and B and assumption (i) in Theorem C are dispensable, and the other assumptions can also be relaxed. It will be shown that under these weaker assumptions even the "minimax equality" $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$ in Theorems A and B and $\inf_{y\in Y} \sup_{x\in X} g(x, y) = \sup_{x\in X} \inf_{y\in Y} g(x, y)$ in Theorem C hold.

We assume our functions to be $\overline{\mathbb{R}}$-valued. As in [3] and [4] our proofs carry over to Z-valued functions, where Z is an order complete, order dense, linearly ordered space.

2. PRELIMINARIES

Quartercontinuous set-valued maps were introduced, under the name semicontinuity, by Correa *et al.* [5]. They proved the following result:

LEMMA 1 [5]. *Let X and Y be topological spaces. Then every quartercontinuous correspondence* $\Phi\colon X \to 2^Y$ *with connected values is connected, i.e.,* $\bigcup_{x\in C} \Phi(x)$ *is connected for every connected* $C \subset X$.

We shall use this result in the following version:

LEMMA 2. *Let Y and Z be topological spaces with Z connected, let F_1, F_2 be closed subsets of Y, and let $\Lambda: Z \to 2^Y$ be a quartercontinuous correspondence such that*

(i) $\Lambda(x) \subset F_1 \cup F_2$ *for all* $x \in Z$,

(ii) *every value* $\Lambda(x)$, $x \in Z$, *is connected, and*

(iii) *for* $i \in \{1, 2\}$ *there exist* $s_i \in Z$ *with* $\Lambda(s_i) \cap F_i \neq \emptyset$.

Then $C \cap F_1 \cap F_2 \neq \emptyset$ *for* $C = \bigcup_{x \in Z} \Lambda(x)$.

Proof. By Lemma 1, the set C is connected. From $C \subset F_1 \cup F_2$ and $C \cap F_i \supset \Lambda(s_i) \cap F_i \neq \emptyset$, $i \in \{1, 2\}$, the assertion follows. ∎

The quartercontinuity of a certain correspondence can often be established by making use of the following simple facts:

Remark 1. Let X and Y be topological spaces and let $\Phi, \Psi: X \to 2^Y$ be correspondences with $\Psi \subset \Phi$, i.e., $\Psi(x) \subset \Phi(x)$ for all $x \in X$. If Ψ is quartercontinuous, then Φ is quartercontinuous as well.

Remark 2. Let X and Y be topological spaces, $\Phi: X \to 2^Y$ be a correspondence and H be a subset of Y such that $\Lambda(x) := \Phi(x) \cap H \neq \emptyset$, $x \in X$. If every set $\Phi^*(y)$, $y \in H$, is closed, then Λ is quartercontinuous.

Proof. From the identity $\{x \in X: \Lambda(x) \cap G \neq \emptyset\} = X - \bigcap_{y \in G \cap H} \Phi^*(y)$ it follows that Λ is even lower semicontinuous. ∎

Remark 3 [13]. Let X and Y be topological spaces, H be a nonvoid subset of Y, and h be an $\overline{\mathbb{R}}$-valued function on $X \times Y$. For $\gamma > \sup_{x \in X} \inf_{y \in H} h(x, y)$ and $x \in X$ set $\Lambda(x) = \{y \in Y: h(x, y) \leq \gamma\} \cap H$. Suppose that every set $\{x \in X: h(x, y) \geq \gamma\}$, $y \in H$, is closed. Then the correspondence Λ is quartercontinuous.

Proof. By Remark 2 the correspondence Ξ with $\Xi(x) = \{y \in Y: h(x, y) < \gamma\} \cap H$, $x \in X$, is quartercontinuous. By Remark 1, Λ is quartercontinuous as well. ∎

Remark 4 [13]. Let X and Y be topological spaces with Y compact. Then every correspondence $\Phi: X \to 2^Y$ with closed graph is quartercontinuous.

Remark 5 [1, Proposition 11.14]. Let $X \subset \mathbb{R}^n$ be a polytope, Y be a topological space, and $\Phi: X \to 2^Y$ be a correspondence such that Φ has closed values and Φ^* has open convex values. Then the graph of Φ is closed.

3. MAIN RESULTS

In the sequel we shall first derive a generalization of Theorem A. Following Komiya [13] we combine this result with Remark 3 to obtain generalizations of Theorems B and C and with Remarks 4 and 5 to get two-function minimax theorems where one function is (separately) lower semicontinuous on $X \times Y$.

Let X and Y be nonvoid sets and h be an $\overline{\mathbb{R}}$-valued function on $X \times Y$. Then a nonvoid subset B of $\mathbb{R}$ will be called a *border set* for h iff $\beta > \sup_{x\in X} \inf_{y\in Y} h(x,y)$ for all $\beta \in B$ and $\inf B = \sup_{x\in X} \inf_{y\in Y} h(x,y)$.

If f and g are $\overline{\mathbb{R}}$-valued functions on $X \times Y$, then we put $\mathscr{E}(f,g)$ for the set of all $A \in \mathscr{E}(X) \cup \{\emptyset\}$ with

$$\{y \in Y\colon g(x,y) \le \beta\} \cap \bigcap_{t\in A} \{y \in Y\colon f(t,y) \le \beta\} \neq \emptyset$$

$$\text{for all } x \in X \text{ and all } \beta > \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

Note that $\mathscr{E}(f,g)$ always contains the empty set.

PROPOSITION 1. *Let X be a topological midset space, Y be a topological space, f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$, and B be a border set for g with the properties*

(i) *every set $\{y \in Y\colon f(x,y) \le \beta\}$, $x \in X$, $\beta \in B$, is closed,*

(ii) *every set $\{y \in Y\colon g(x,y) \le \beta\}$, $x \in X$, $\beta \in B$, is closed,*

(iii) *for every $\beta \in B$ and every $A \in \mathscr{E}(f,g)$ there exists a $\gamma \in B$ with $\gamma < \beta$ and sets C_t and D_x with*

$$\{y \in Y\colon f(t,y) < \gamma\} \subset C_t \subset \{y \in Y\colon f(t,y) \le \beta\}, \qquad t \in A,$$

and

$$\{y \in Y\colon g(x,y) < \gamma\} \subset D_x \subset \{y \in Y\colon g(x,y) \le \beta\}, \qquad x \in X$$

such that the correspondence Λ with $\Lambda(x) := D_x \cap \bigcap_{t\in A} C_t$, $x \in X$, is quartercontinuous on every midset and the values $\Lambda(x)$, $x \in X$, are connected,

(iv) $\forall (x_1,x_2) \in X \times X\ \forall x \in Z(x_1,x_2)\ \forall y \in Y\colon g(x,y) \ge f(x_1,y) \wedge g(x_2,y)$, *and*

(v) $\forall (x_1,x_2) \in X \times X\ \exists \{s_1,s_2\} \subset Z(x_1,x_2)\ \forall y \in Y\colon g(s_1,y) \ge f(x_1,y)$ *and* $g(s_2,y) \ge g(x_2,y)$.

Then $\mathscr{E}(f,g) = \mathscr{E}(X) \cup \{\emptyset\}$.

Proof. We show by induction that

$$\mathscr{E}_k := \{A \in \mathscr{E}(X) \cup \{\emptyset\}\colon |A| = k\} \subset \mathscr{E}(f,g) \qquad \forall k \in \{0,1,2,\ldots\}.$$

Obviously, $\mathscr{E}_0 = \{\emptyset\} \subset \mathscr{E}(f, g)$ holds. Suppose that $\mathscr{E}_k \subset \mathscr{E}(f, g)$ and let $E \in \mathscr{E}_{k+1}$. Choose $x_1 \in E$ and set $A = E - \{x_1\}$. By assumption we have $A \in \mathscr{E}(f, g)$. For a fixed $\beta \in B$ take γ, C_t, D_x, and Λ according to condition (iii). For arbitrary $x \in X$ let $F_1 = \{y \in Y\colon f(x_1, y) \le \beta\}$ and $F_2 = \{y \in Y\colon g(x, y) \le \beta\}$. Then by conditions (iv) and (v),

$$\Lambda(z) \subset \{y \in Y\colon g(z, y) \le \beta\} \subset F_1 \cup F_2, \qquad z \in Z(x_1, x),$$

and there exist $\{s_1, s_2\} \subset Z(x_1, x)$ with

$$\Lambda(s_i) \subset \{y \in Y\colon g(s_i, y) \le \beta\} \subset F_i, \qquad i \in \{1, 2\}.$$

Hence, by Lemma 2,

$$\begin{aligned} \emptyset \ne F_1 \cap F_2 \cap \bigcup_{z \in Z(x_1, x)} \Lambda(z) &\subset \{y \in Y\colon g(x, y) \le \beta\} \\ &\cap \bigcap_{t \in E} \{y \in Y\colon f(t, y) \le \beta\}, \end{aligned}$$

which yields $E \in \mathscr{E}(f, g)$. ∎

THEOREM 1. *Let the assumptions of Proposition* 1 *be satisfied and assume that*

(vi) *for some* $(s_0, t_0) \in X \times X$ *and some* $\beta_0 > \sup_{x \in X} \inf_{y \in Y} g(x, y)$ *the set* $\{y \in Y\colon f(s_0, y) \le \beta_0\} \cap \{y \in Y\colon g(t_0, y) \le \beta_0\}$ *is compact.*

Then $\inf_{y \in Y} \sup_{x \in X} f(x, y) \le \sup_{x \in X} \inf_{y \in Y} g(x, y)$.

Proof. By Proposition 1 the system of closed compact sets

$$\{y \in Y\colon f(x, y) \le \beta\} \cap \{y \in Y\colon f(s_0, y) \le \beta\} \cap \{y \in Y\colon g(t_0, y) \le \beta\}, \qquad x \in X, \quad \beta \in B \cap (-\infty, \beta_0),$$

has the finite intersection property. Therefore, there exists a $\hat{y} \in Y$ with $\sup_{x \in X} f(x, \hat{y}) \le \inf B = \sup_{x \in X} \inf_{y \in Y} g(x, y)$. ∎

We now combine the above results with Remark 3 in order to replace the somewhat artificial condition of quartercontinuity of a certain level-set correspondence by an easier tractable property.

PROPOSITION 2. *Let* X *be a topological midset space,* Y *be a topological space,* f *and* g *be* $\overline{\mathbb{R}}$*-valued functions on* $X \times Y$*, and* B *be a border set for* g *with the properties*

(i) *every set* $\{y \in Y\colon f(x, y) \le \beta\}$, $x \in X$, $\beta \in B$, *is closed,*

(ii) *every set* $\{y \in Y\colon g(x, y) \le \beta\}$, $x \in X$, $\beta \in B$, *is closed,*

(iii) *every set* $\{x \in X\colon g(x, y) \ge \beta\}$, $y \in Y$, $\beta \in B$, *is midset-closed,*

(iv) *for every* $\beta \in B$ *and every* $A \in \mathscr{E}(f, g)$ *there exists a* $\gamma \in B$ *with* $\gamma < \beta$ *and sets* C_t *and* D_x *with*

$$\{y \in Y: f(t, y) < \gamma\} \subset C_t \subset \{y \in Y: f(t, y) \leq \beta\}, \qquad t \in A,$$

and

$$\{y \in Y: g(x, y) < \gamma\} \subset D_x \subset \{y \in Y: g(x, y) \leq \beta\}, \qquad x \in X,$$

such that the sets $D_x \cap \bigcap_{t \in A} C_t$, $x \in X$, *are connected,*

(v) $\forall (x_1, x_2) \in X \times X\ \forall x \in Z(x_1, x_2)\ \forall y \in Y$: $g(x, y) \geq f(x_1, y) \wedge g(x_2, y)$, *and*

(vi) $\forall (x_1, x_2) \in X \times X\ \exists \{s_1, s_2\} \subset Z(x_1, x_2)\ \forall y \in Y$: $g(s_1, y) \geq f(x_1, y)$ *and* $g(s_2, y) \geq g(x_2, y)$.

Then $\mathscr{E}(f, g) = \mathscr{E}(X) \cup \{\emptyset\}$.

Proof. Let $A \in \mathscr{E}(f, g)$ and $\beta \in B$. Choose γ, C_t, and D_x according to condition (iv). Then for $\delta \in B$ with $\delta < \gamma$ and $H = \bigcap_{t \in A} C_t$ the correspondence Ξ with $\Xi(x) = \{y \in Y: g(x, y) \leq \delta\} \cap H$, $x \in X$, is quartercontinuous on the midsets of X by Remark 3. By Remark 1 the correspondence Λ ($\supset \Xi$) with $\Lambda(x) = D_x \cap H$, $x \in X$, is also quartercontinuous on the midsets of X. Therefore, Proposition 1 applies. ∎

THEOREM 2. *Let the assumptions of Proposition* 2 *be satisfied and assume that*

(vii) *for some* $(s_0, t_0) \in X \times X$ *and some* $\beta_0 > \sup_{x \in X} \inf_{y \in Y} g(x, y)$ *the set* $\{y \in Y: f(s_0, y) \leq \beta_0\} \cap \{y \in Y: g(t_0, y) \leq \beta_0\}$ *is compact.*

Then $\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y)$.

Proof. Compare the proof of Theorem 1. ∎

Further minimax theorems can be obtained by combining Theorem 1 with Remark 4 and 5.

4. GENERALIZATIONS OF THEOREMS A, B, AND C

We shall show that the above results, combined with the following simple observation, yield generalized versions of Theorems A, B, and C.

Remark 6. Let X and Y be nonvoid sets and let f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$ such that

$$f(x_2, y) \geq f(x_1, y) \wedge g(x_2, y) \quad \text{for all } (x_1, x_2) \in X \times X \text{ and all } y \in Y.$$

Then for every nonvoid subset H of Y,

$$\sup_{x \in X} \inf_{y \in H} f(x, y) \geq \Big(\inf_{y \in H} \sup_{x \in X} f(x, y)\Big) \wedge \Big(\sup_{x \in X} \inf_{y \in H} g(x, y)\Big).$$

EXAMPLE 1. Let X be an interval space, Y be a compact topological space, f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$, and B be a border set for f or for g with the properties

(0) $f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$,

(i) every function $f(x, \cdot)$, $x \in X$, is lower semicontinuous,

(ii) for each nonvoid subset $F_A = \bigcap_{t \in A}\{y \in Y\colon f(t, y) \leq \alpha\}$, $A \in \mathscr{E}(X) \cup \{\varnothing\}$, with $\alpha := \sup_{x\in X} \inf_{y \in Y} f(x, y) = \sup_{x \in X} \inf_{y \in F_A} g(x, y)$ and for each real $\gamma > \alpha$ the level-set correspondence Ξ with $\Xi(x) = \{y \in Y\colon g(x, y) \leq \gamma\} \cap F_A$ is quartercontinuous on the intervals of X,

(iii) for every $\beta \in B$ there exist sets C_x with $\{y \in Y\colon f(x, y) < \beta\} \subset C_x \subset \{y \in Y\colon f(x, y) \leq \beta\}$, $x \in X$, such that every set $\bigcap_{t \in A} C_t$, $A \in \mathscr{E}(X)$, is either connected or empty, and

(iv) $f(x, y) \geq f(x_1, y) \wedge g(x_2, y)$ for all $x \in \langle x_1, x_2\rangle$, $(x_1, x_2) \in X \times X$, and all $y \in Y$.

Then $\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y\in Y} f(x, y)$.

Proof. Assume that $\alpha = \sup_{x\in X} \inf_{y \in Y} f(x, y) < \inf_{y\in Y} \sup_{x \in X} f(x, y) =: \lambda$. Then, by Remark 6, $\sup_{x\in X} \inf_{y\in Y} f(x, y) = \sup_{x \in X} \inf_{y \in Y} g(x, y)$ holds, i.e., B is a border set for g iff it is a border set for f. For $A \in \mathscr{E}(f, f)$ we have $\sup_{x\in X} \inf_{y \in F_A} g(x, y) = \alpha$, for otherwise Remark 6 yields $\alpha = \sup_{x \in X} \inf_{y\in F_A} f(x, y) = \inf_{y \in F_A} \sup_{x\in X} f(x, y) \geq \lambda > \alpha$. For $\beta \in B$ and $\gamma \in (\alpha, \beta)$ take Ξ according to condition (ii). Then Ξ is quartercontinuous on the intervals and so is Λ with $\Lambda(x) = \bigcap_{t \in A \cup \{x\}} C_t$, $x \in X$, since $\Lambda \supset \Xi$. Therefore all assumptions of Theorem 1 with g replaced by f are satisfied and we obtain $\alpha = \lambda$. ∎

Remark 7. Example 1 generalizes Theorem A. We see that assumption (ii) in Theorem A is superfluous and that X and Y need not be assumed Hausdorff. Moreover, we obtain the stronger minimax equality for f instead of the minimax inequality for f and g.

COROLLARY 1. *Let X be a topological midset space, Y be a topological space, f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$, and B be a border set for f or for g with the properties*

(0) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in X \times Y$,

(i) *every set* $\{y \in Y\colon f(x, y) \leq \beta\}$, $x \in X$, $\beta \in B$, *is closed*,

(ii) *every set* $\{x \in X\colon g(x, y) \geq \beta\}$, $y \in Y$, $\beta \in B$, *is midset-closed*,

(iii) *for every* $\beta \in B$ *there exists a* $\gamma \in B$ *with* $\gamma < \beta$ *and sets* C_x *with*

$$\{y \in Y\colon f(x, y) < \gamma\} \subset C_x \subset \{y \in Y\colon f(x, y) \leq \beta\}, \qquad x \in X,$$

such that every set $\bigcap_{t\in A} C_t$, $A \in \mathscr{E}(X)$, *is either connected or empty,*

(iv) $\forall (x_1, x_2) \in X \times X\ \forall x \in Z(x_1, x_2)\ \forall y \in Y$: $f(x, y) \geq f(x_1, y) \wedge f(x_2, y)$,

(v) $\forall\ (x_1, x_2) \in X \times X\ \exists \{s_1, s_2\} \subset Z(x_1, x_2)\ \forall y \in Y\ \forall i \in \{1, 2\}$: $f(s_i, y) \geq f(x_i, y)$, *and*

(vi) $f(x_2, y) \geq f(x_1, y) \wedge g(x_2, y)$ *for all* $(x_1, x_2) \in X \times X$ *and all* $y \in Y$.

Then $\mathscr{E}(f, f) = \mathscr{E}(X) \cup \{\emptyset\}$, *i.e.*,

$$\inf_{y\in Y} \max_{x\in A} f(x, y) \leq \sup_{x\in X} \inf_{y\in Y} f(x, y) \quad \textit{for all } A \in \mathscr{E}(X).$$

If moreover

(vii) *for some* $x_0 \in X$ *and some* $\beta_0 > \sup_{x\in X} \inf_{y\in Y} f(x, y)$ *the set* $\{y \in Y\colon f(x_0, y) \leq \beta_0\}$ *is compact,*

then $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$.

Proof. In case $\alpha := \sup_{x\in X} \inf_{y\in Y} f(x, y) = \inf_{y\in Y} \sup_{x\in X} f(x, y) =: \lambda$ we are done. Hence, we may assume $\lambda > \sup B$. In particular, by Remark 6, B is a border set for f iff it is a border set for g.

For $\beta \in B$ choose γ and C_x according to condition (iii) and take a $\delta \in B$ with $\delta < \gamma$. For $A \in \mathscr{E}(f, f)$ set $H = \bigcap_{t\in A} C_t$, $\Lambda(x) = C_x \cap H$, and $\Xi(x) = \{y \in Y\colon g(x, y) \leq \delta\} \cap H$, $x \in X$. Then $A \in \mathscr{E}(f, f)$ implies $\sup_{x\in X} \inf_{y\in H} f(x, y) < \delta < \lambda \leq \inf_{y\in H} \sup_{x\in X} f(x, y)$, and from Remark 6 we infer $\sup_{x\in X} \inf_{y\in H} g(x, y) \leq \sup_{x\in X} \inf_{y\in H} f(x, y) < \delta$. By Remark 3 it follows from condition (ii) that Ξ is quartercontinuous on the midsets of X and so is Λ according to Remark 1 since $\Lambda \supset \Xi$ by condition (0).

Now the assertion follows from the special case $f = g$ of Proposition 1 and Theorem 1. ∎

EXAMPLE 2. Let X be an interval space, Y be a topological space, f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$, and B be a border set for f or for g with the properties

(0) $f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$,

(i) every function $f(x, \cdot)$, $x \in X$, is lower semicontinuous,

(ii) every function $g(\cdot, y)$, $y \in Y$, is upper semicontinuous on the intervals of X,

(iii) for every $\beta \in B$ there exist sets C_x with $\{y \in Y\colon f(x, y) < \beta\} \subset C_x \subset \{y \in Y\colon f(x, y) \leq \beta\}$, $x \in X$, such that every set $\bigcap_{t\in A} C_t$, $A \in \mathscr{E}(X)$, is either connected or empty,

(iv) $\forall (x_1, x_2) \in X \times X\ \forall x \in \langle x_1, x_2\rangle\ \forall y \in Y$: $f(x, y) \geq f(x_1, y) \wedge g(x_2, y)$, and

(v) for some $x_0 \in X$ and some $\beta_0 \geq \inf_{y\in Y} \sup_{x\in X} f(x, y)$ the set $\{y \in Y\colon f(x_0, y) \leq \beta_0\}$ is compact.

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$.

Proof. Suppose that $\alpha := \sup_{x\in X} \inf_{y\in Y} f(x, y) < \inf_{y\in Y} \sup_{x\in X} f(x, y)$. Then we have $\beta_0 > \alpha$ and Corollary 1 leads to a contradiction. [Note that condition (v) of Corollary 1 is always satisfied for interval spaces.] ∎

Remark 8. Example 2 generalizes Theorem B. As in Example 1 the lower semicontinuity of the functions $g(x, \cdot)$ is not needed, and we obtain the stronger minimax equality for f.

EXAMPLE 3. Let X be a topological midset space, Y be a topological space, g be an $\overline{\mathbb{R}}$-valued function on $X \times Y$, and B be a border set for g with the properties

(i) every function $g(x, \cdot)$, $x \in X$, is lower semicontinuous,

(ii) every function $g(\cdot, y)$, $y \in Y$, is upper semicontinuous on every midset of X,

(iii) for every $\beta \in B$ there exist sets C_x with $\{y \in Y\colon g(x, y) < \beta\} \subset C_x \subset \{y \in Y\colon g(x, y) \le \beta\}$, $x \in X$, such that every set $\bigcap_{t\in A} C_t$, $A \in \mathscr{E}(X)$, is either connected or empty,

(iv) $g(x, y) \ge g(x_1, y) \wedge g(x_2, y)$ for all $x \in Z(x_1, x_2)$, $(x_1, x_2) \in X \times X$, and all $y \in Y$,

(v) $\forall (x_1, x_2) \in X \times X\ \exists \{s_1, s_2\} \subset Z(x_1, x_2)\ \forall y \in Y\ \forall i \in \{1, 2\}$: $g(s_i, y) \ge g(x_i, y)$, and

(vi) there exists an $x_0 \in X$ and a $\beta_0 \in \mathbb{R}$ with $\beta_0 > \sup_{x\in X} \inf_{y\in Y} g(x, y)$ such that the set $\{y \in Y\colon g(x_0, y) \le \beta_0\}$ is compact.

Then $\sup_{x\in X} \inf_{y\in Y} g(x, y) = \inf_{y\in Y} \sup_{x\in X} g(x, y)$.

Proof. Take $f = g$ in Theorem 2. ∎

Remark 9. Example 3 generalizes Theorem C. [Note that conditions (0), (ii), and (vii) of Theorem C imply condition (vi) of Example 3.] We see that assumption (i) in Theorem C is dispensable and we obtain the stronger minimax equality for g.

5. FURTHER EXAMPLES

It is perhaps remarkable that Theorems 1 and 2 contain a two-function version of Dini's theorem:

EXAMPLE 4. Let X be a nonvoid set, Y be a compact topological space, and f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$ such that

(i) every function $f(x, \cdot)$, $x \in X$, is lower semicontinuous,

(ii) every function $g(x, \cdot)$, $x \in X$, is lower semicontinuous, and

(iii) $\forall (x_1, x_2) \in X \times X\ \exists x_0 \in X \times X\ \forall y \in Y\colon g(x_0, y) \geq f(x_1, y) \vee g(x_2, y)$.

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} \inf_{y\in Y} g(x, y)$.

Proof. Endow X with the topology $\{\emptyset, X\}$ and with the midset function $Z(x_1, x_2) = \{x_0\}$, where x_0 is chosen according to condition (iii), and apply Theorem 1 or 2. ∎

Recall that for an interval space S with interval function $\langle \cdot, \cdot \rangle$ an $\overline{\mathbb{R}}$-valued function f on S is called *quasiconcave* iff $f(s) \geq f(s_1) \wedge f(s_2)$ for all $s \in \langle s_1, s_2 \rangle$, $(s_1, s_2) \in S \times S$ or, equivalently, iff every set $\{s \in S\colon f(s) \geq \alpha\}$, $\alpha \in \mathbb{R}$, is convex, and f is called *quasiconvex* iff $-f$ is quasiconcave.

Remark 10. Let S be a topological midset space and $\mathscr{C}$ be the system of all convex subsets of S. Then $\mathscr{C}$ is an alignment [i.e., $\{\emptyset, S\} \subset \mathscr{C}$ and $\mathscr{C}$ is closed w.r.t. arbitrary intersections and nested unions]. If S is an interval space, then every $C \in \mathscr{C}$ is connected.

As a consequence we obtain:

Remark 11. Let X be a nonvoid set, Y be an interval space, and $h\colon X \times Y \to \overline{\mathbb{R}}$ such that every function $h(x, \cdot)$, $x \in X$, is quasiconvex. Then for $C_x = \{y \in Y\colon h(x, y) \leq \beta\}$ and $D_x = \{y \in Y\colon h(x, y) < \beta\}$, $x \in X$, $\beta \in \mathbb{R}$, the sets $\bigcap_{x\in A} C_x$ and $\bigcap_{x\in A} D_x$, $A \in 2^X - \{\emptyset\}$, are convex and therefore connected or empty.

The following special case of Corollary 1 is owing to Kindler and Trost [11, Corollary 5.2]. It generalizes Sion's classical minimax theorem [19] as well as Proposition 1 of Brézis, Nirenberg, and Stampacchia [2].

EXAMPLE 5. Let X and Y be interval spaces and let f be an $\overline{\mathbb{R}}$-valued function on $X \times Y$ such that

(i) every function $f(x, \cdot)$, $x \in X$, is quasiconvex and lower semicontinuous,

(ii) every function $f(\cdot, y)$, $y \in Y$, is quasiconcave and upper semicontinuous on the intervals of X, and

(iii) for some $x_0 \in X$ and some $\beta_0 > \sup_{x\in X} \inf_{y\in Y} f(x, y)$ the set $\{y \in Y\colon f(x_0, y) \leq \beta_0\}$ is compact.

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$.

In view of Theorems 1 and 2 and Example 3 one may wonder whether Theorem A or B remains true when condition (v) is replaced by the weaker condition

(v*) $g(x, y) \geq f(x_1, y) \wedge g(x_2, y)$ for all $x \in \langle x_1, x_2 \rangle$ and all $y \in Y$

or, at least, by the condition

(v**) $g(x, y) \geq g(x_1, y) \wedge g(x_2, y)$ for all $x \in \langle x_1, x_2 \rangle$ and all $y \in Y$
i.e., every function $g(\cdot, y)$, $y \in Y$, is quasiconcave.

The following counterexample shows that this is not the case:

EXAMPLE 6. Let $X = Y = [0, 1]$, endowed with the natural interval function $\langle x_1, x_2 \rangle = \{\alpha x_1 + (1-\alpha)x_2 \colon \alpha \in [0, 1]\}$, let $F = \{0\} \times [0, 1) \cup \{1\} \times (0, 1]$, $G = F \cup (0, 1) \times (0, 1)$, $f(x, y) = 1_F(x, y)$, and $g(x, y) = 1_G(x, y)$. Then

every function $f(x, \cdot)$, $x \in X$, is lower semicontinuous and quasiconvex,

every function $g(x, \cdot)$, $x \in X$, is lower semicontinuous,

every function $f(\cdot, y)$, $y \in Y$, is upper semicontinuous, and

every function $g(\cdot, y)$, $y \in Y$, is upper semicontinuous and quasiconcave.

In particular, all assumptions of Theorem A (by Remark 3) and of Theorem B are satisfied with condition (v) replaced by (v*) or by (v**).

Of course, $\sup_{x \in X} \inf_{y \in Y} g(x, y) = 0 < 1 = \inf_{y \in Y} \sup_{x \in X} f(x, y)$.

This counterexample should also be compared with Simons Theorems 1 and 5 in [16] and with Example 8 below.

As mentioned above, our main results can also be combined with Remarks 4 and 5. We content ourselves with two concluding examples.

The special case $f = g$ of the following example generalizes minimax theorems of Ha [6, Theorem 4] and Komornik [14, Theorem 2]; compare also Simons' Theorem 8 in [17]:

EXAMPLE 7. Let X be an interval space, Y be a compact topological space, f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$, and B be a border set for f or for g such that

(0) $f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$,

(i) every function $f(x, \cdot)$, $x \in X$, is lower semicontinuous,

(ii) every set $\{(x, y) \in X \times Y \colon g(x, y) \leq \beta\} \cap \langle x_1, x_2 \rangle \times Y$, $(x_1, x_2) \in X \times X$, $\beta \in B$, is closed in $\langle x_1, x_2 \rangle \times Y$,

(iii) for every $\beta \in B$ there exist sets C_x with $\{y \in Y \colon g(x, y) < \beta\} \subset C_x \subset \{y \in Y \colon g(x, y) \leq \beta\}$, $x \in X$, such that every set $\bigcap_{t \in A} C_t$, $A \in \mathscr{E}(X)$, is either connected or empty, and

(iv) $\forall(x_1, x_2) \in X \times X\ \forall x \in \langle x_1, x_2\rangle\ \forall y \in Y$: $f(x, y) \geq f(x_1, y) \wedge g(x_2, y)$.

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$.

Proof. This follows from Example 1 together with Remark 4. ∎

EXAMPLE 8. Let X be a convex subset of a linear topological space, Y be a compact topological space, f and g be $\overline{\mathbb{R}}$-valued functions on $X \times Y$ and B be a border set for f or for g such that

(0) $f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$,

(i) every function $f(x, \cdot)$, $x \in X$, is lower semicontinuous,

(ii) every function $g(x, \cdot)$, $x \in X$, is lower semicontinuous,

(iii) every function $g(\cdot, y)$, $y \in Y$, is lower semicontinuous on the intervals of X and quasiconcave,

(iv) for every $\beta \in B$ there exist sets C_x with $\{y \in Y: g(x, y) < \beta\} \subset C_x \subset \{y \in Y: g(x, y) \leq \beta\}$, $x \in X$, such that every set $\bigcap_{t\in A} C_t$, $A \in \mathscr{E}(X)$, is either connected or empty, and

(v) $\forall(x_1, x_2) \in X \times X\ \forall \alpha \in [0, 1]\ \forall y \in Y$: $f(\alpha x_1 + (1 - \alpha)x_2, y) \geq f(x_1, y) \wedge g(x_2, y)$.

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$.

Proof. We endow X with the "ordinary" interval function $\langle x_1, x_2\rangle = \{\alpha x_1 + (1 - \alpha)x_2: 0 \leq t \leq 1\}$. Then, by Remark 5, Example 7 applies. ∎

The above methods, combined with results from [10], can also be used to derive two-function versions of the topological minimax theorems of König [12] and Tuy [21, 22] as well as minimax theorems based on "abstract connectedness" (cf. [7, 9, 10, 12, 17, 18]).

ACKNOWLEDGMENT

The author would like to thank Bor-Luh Lin for stimulating discussions and for sending the preprints of his work.

REFERENCES

1. K. C. Border, "Fixed Point Theorems with Applications to Economics and Game Theory," Cambridge Univ. Press, 1985.
2. H. Brézis, L. Nirenberg, and G. Stampacchia, A remark on Ky Fan's minimax principle, *Boll. Un. Mat. Ital.* **6** (1972), 293–300.

3. S.-S. Chang, S.-Y. Cao, X. Wu, and D.-C. Wang, Some nonempty intersection theorems in generalized interval spaces with applications, *J. Math. Anal. Appl.* **199** (1996), 787–803.
4. C.-Z. Cheng and B.-L. Lin, A two functions, noncompact topological minimax theorem, *Acta Math. Hungar.* **73** (1996), 65–69.
5. R. Correa, J. B. Hiriart-Urruty, and J.-P. Penot, A note on connected set-valued mappings, *Boll. Un. Mat. Ital. C (6)* **5** (1986), 357–366.
6. Ch.–W. Ha, Minimax and fixed point theorems, *Math. Ann.* **248** (1980), 73–77.
7. J. Kindler, Intersection theorems and minimax theorems based on connectedness, *J. Math. Anal. Appl.* **178** (1993), 529–546.
8. J. Kindler, Intersecting sets in midset spaces I, *Arch. Math.* **62** (1994), 49–57.
9. J. Kindler, Intersecting sets in midset spaces II, *Arch. Math.* **62** (1994), 168–176.
10. J. Kindler, Intersection theorems, minimax theorems and abstract connectedness, in “Minimax Theory and Applications” (B. Ricceri and S. Simons, Eds.) pp. 105–120, Kluwer, Dordrecht, 1998.
11. J. Kindler and R. Trost, Minimax theorems for interval spaces, *Acta Math. Hungar.* **54** (1989), 39-49.
12. H. König, A general minimax theorem based on connectedness, *Arch. Math.* **59** (1992), 55–64; addendum **64** (1995), 139–143.
13. H. Komiya, On minimax theorems without linear structure, *Hiyoshi Rev. Nat. Sci.* **8** (1990), 74–78.
14. V. Komornik, Minimax theorems for upper semicontinuous functions, *Acta Math. Acad. Sci. Hungar.* **40** (1982), 159–163.
15. B.-L. Lin and X.-C. Quan, A two functions nonlinear minimax theorem, *in* “Fixed Point Theory and Applications” (M. A. Théra and J.-B. Baillon, Eds.), Pitman Research Notes Math. Ser., Vol. 252, pp. 321–325, Longman, London, 1991.
16. S. Simons, Minimax and variational inequalities, are they of fixed-point or of Hahn–Banach type?, *in* “Game Theory and Mathematical Economics” (O. Moeschlin and D. Pallaschke, eds.), pp. 379–388, North-Holland, Amsterdam, 1981,
17. S. Simons, A flexible minimax theorem, *Acta Math. Hungar.* **63** (1994), 119–132; addendum **69** (1995), 359–360.
18. S. Simons, Minimax theorems and their proofs, *in* “Minimax and Applications” (D.-Z. Du and P. M. Pardalos, Eds.) pp. 1–23, Kluwer, Dordrecht, 1995.
19. M. Sion, On general minimax theorems, *Pacific J. Math.* **8** (1958), 171–176.
20. L. L. Stachó, Minimax theorems beyond topological vector spaces, *Acta Sci. Math.* **42** (1980), 157–164.
21. H. Tuy, On a general minimax theorem, *Sov. Math. Dokl.* **15** (1974), 1689–1693.
22. H. Tuy, On the general minimax theorem, *Colloq. Math.* **33** (1975), 145–158.