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Intersecting sets in midset spaces. I

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1. Introduction. Many mathematical existence problems can be reduced to the following

Intersection Problem. *Let Y be a nonvoid set, X an index set and $\{\Phi(x): x \in X\}$ a system of subsets of Y. When does the family intersect, i.e., when is $\bigcap_{x \in X} \Phi(x)$ nonvoid?*

It is convenient to formulate the problem in terms of correspondences. Recall that for nonvoid sets X and Y a set-valued mapping $\Phi: X \to 2^Y$ is called a *correspondence* from X to Y iff every *value* $\Phi(x), x \in X$, is nonvoid. Of course, w.l.g. we may assume Φ to be a correspondence in our Intersection Problem. Hence, in the following let two nonvoid sets X and Y and a correspondence Φ from X to Y be given.

In the present paper work begun in [11], [12], [13], [14] is continued. We generalize the concept of an interval space introduced by Stachó [27] and study situations where Y (or X) is endowed with a generalized interval structure ("midset function") such that the values of Φ (or of Φ^*, the dual of Φ) are convex. Just as in our former papers all proofs are entirely elementary.

In a subsequent paper, appearing in the same journal, several applications of the present results will be presented.

2. Preliminaries. For a correspondence Φ from X to Y we set

$$\Phi^\cap(E) := \bigcap_{x \in E} \Phi(x) \text{ for nonvoid } E \subset X \text{ and } \Phi^\cap(\emptyset) = Y.$$

Furthermore, $\operatorname{val} \Phi := \{\Phi(x): x \in X\}$ is the *value set* of Φ, $\Phi^*: Y \to 2^X$ with $\Phi^*(y) = \{x \in X: y \notin \Phi(x)\}$ is the *dual* of Φ, and $\operatorname{Gr} \Phi := \{(x, y) \in X \times Y: y \in \Phi(x)\}$ is the *graph* of Φ. We also use $\hat{\Phi}: X \to 2^X$ with $\hat{\Phi}(x) := \{t \in X: \Phi(x) \cap \Phi(t) = \emptyset\}$, $x \in X$, and we set $\mathscr{E}(X) = \{A \subset X: A \text{ finite nonvoid}\}$ and $\mathscr{E}_\Phi(X) = \{A \in \mathscr{E}(X) \cup \{\emptyset\}: \Phi^\cap(A \cup \{x\}) \neq \emptyset \text{ for all } x \in X\}$.

If $\mathscr{F}$ is a nonvoid system of subsets of a set S then we say that

$\mathscr{F}$ has the *pairwise intersection property* iff $F_1 \cap F_2 \neq \emptyset$ for all $F_i \in \mathscr{F}$, $i \in \{1, 2\}$,

$\mathscr{F}$ has the *finite intersection property* iff $\bigcap_{i=1}^{n} F_i \neq \emptyset$ for all $F_i \in \mathscr{F}$, $i \leq n \in \mathbb{N}$.

Let S_0, S_1, and S_2 be subsets of a set S. We say that
S_0 *meets* S_1 *and* S_2 iff $S_0 \cap S_i \neq \emptyset$, $i \in \{1,2\}$,
S_0 *joins* S_1 *and* S_2 iff $S_0 \subset S_1 \cup S_2$ and S_0 meets S_1 and S_2.
Let $\emptyset \neq \mathcal{M} \subset 2^S$. We say that
S_0 *is pseudoconnected for* $\mathcal{M}$ iff $S_1 \cap S_2 \neq \emptyset$ for every pair $(S_1, S_2) \in \mathcal{M} \times \mathcal{M}$ which is joined by S_0,
S_0 *is connected for* $\mathcal{M}$ iff $S_0 \cap S_1 \cap S_2 \neq \emptyset$ for every pair $(S_1, S_2) \in \mathcal{M} \times \mathcal{M}$ which is joined by S_0.
Finally, $\mathcal{M}$ will be called *(pseudo)-connected* iff every $M \in \mathcal{M}$ is (pseudo-)connected for $\mathcal{M}$.

Abstract connectivity conditions have been introduced into the theory of minimax theorems by Kindler [11] ("Γ-connectedness"), Simons [25], [26] (pseudoconnectedness), König [18] and König-Zartmann [19] (connectedness). Of course, every connected subset of a topological space is connected for any family of open (resp. closed) subsets.

For a nonvoid set S a set-valued mapping $Z: S \times S \to 2^S$ will be called *midset function* for S. The pair (S, Z) will be called *mitset space.* In case $Z \supset \{\cdot,\cdot\}$ (*i.e.*, $Z(s,t) \supset \{s,t\}$, $(s,t) \in S \times S$) Z is called *segment function* for S and (S,Z) is a *segment space.* A subset $T \subset S$ is *(Z-)convex* iff $\{s,t\} \subset T$ implies $Z(s,t) \subset T$.

If, in addition, S is endowed with a topology $\mathcal{T}$ such that every midset/segment $Z(s,t)$ is connected, then Z will be called *topological midset/segment function* and $(S, \mathcal{T}, Z)$, or (S,Z) for short, is a *topological midset/segment space.*

Midset functions with nonvoid midsets have been studied by Calder [6] under the name *interval convexities* (compare also [2], [8]) whereas topological midset spaces with symmetric midsets have been introduced by Stachó [27] under the name *interval spaces* (compare also [17], [14]). Every midset function Z for S gives rise to a *ternary relation* $R \subset S \times S \times S$ according to $R = \{(s,u,t) \in S \times S \times S : u \in Z(s,t)\}$. Here $s\,u\,t :\Leftrightarrow (s,u,t) \in R$ is to be interpreted as "u lies between s and t". Conversely, to every ternary ("betweenness") relation there corresponds a midset function $Z(s,t) = \{u \in S : s\,u\,t\}$.

The notion of betweenness is ubiquitous in mathematics:

An axiomatic treatment of *geometric betweenness* was initiated by Pasch (compare [22] for references). Similarly the *join geometries* of Prenowitz and Jantosciak [23] lead in a natural way to a midset structure. If (S,d) is a metric space then *metric betweenness* in the sense of Menger [5], is defined by the relation $s\,u\,t \Leftrightarrow s \neq t$, $u \notin \{s.t\}$, and $d(s,u) + d(u,t) = d(s,t)$.

A *ternary algebra* $(S, (\cdot,\cdot,\cdot))$, where $(\cdot,\cdot,\cdot): S \times S \times S \to S$ is a ternary operation on S, defines a ternary relation $s\,u\,t :\Leftrightarrow (s,u,t) = u$ with the midset function $Z(s,t) = \{u \in S : (s,u,t) = u\}$. A particular case is the *median operator* $(s,u,t) = (s \wedge u) \vee (s \wedge t) \vee (u \wedge t)$ in a distributive lattice. Compare [1] for further examples.

In a *partially ordered set* $(S, \leqq)$ the midsets $Z(s,t) = \{u \in S : s \leqq u \leqq t\}$ resp. $Z(s,t) = \{u \in S : s \leqq u \leqq t$ or $t \leqq u \leqq s\}$ are called *order intervals* and Z-convex subsets are called *order convex.* On $(S, \leqq)$ *order topologies* (or *interval topologies*) can be defined [3], [21]. If the order intervals are connected (the classical result on linear orders is in [10], p. 55 f.) then we obtain a topological midset space. Compare also [24] and the references therein.

In *lattices* S the betweenness relation $s\,u\,t \Leftrightarrow s \wedge t \leqq u \leqq s \vee t$ resp. $s\,u\,t \Leftrightarrow (s \wedge u) \vee (u \wedge t) = u = (s \vee u) \wedge (u \vee t)$ is also frequently used [22], [28]. If the lattice is endowed with a norm (positive valuation) $|\cdot|$, then $d(s,t) = |s \vee t| - |s \wedge t|$ defines a metric on S. For interrelations between metric betweenness and order betweenness compare [4], [5], [20], [22]. For lattice topologies we refer to [9].

3. Pairwise intersection – abstract version. For a given midset function Z on X we consider the following conditions:

(A) $\Phi(x_1) \cap \Phi(x_2) = \emptyset$.
(B) $\Phi(x) \subset \Phi(x_1) \cup \Phi(x_2)$.
(C) $\Phi(x)$ meets $\Phi(x_1)$ and $\Phi(x_2)$.
(D) $x \in Z(x_1,x_2)$ implies $\Phi(x) \subset \Phi(x_1)$ or $\Phi(x) \subset \Phi(x_2)$.
(E) $Z(x_1,x_2)$ meets $\hat{\Phi}(x_1)$ and $\hat{\Phi}(x_2)$.
(F) $Z(x_1,x_2)$ is connected for $\{\hat{\Phi}(x_1), \hat{\Phi}(x_2)\}$.

We say that

Φ is *Z-concave* iff for all $((x_1,x_2),x) \in \operatorname{Gr} Z$ (A) implies (B),
Φ is *Z-connected* iff for all $((x_1,x_2),x) \in \operatorname{Gr} Z$ (A), (B) and (C) are inconsistent,
$\hat{\Phi}$ is *Z-linked* iff for all $(x_1,x_2) \in X \times X$ (A) and (D) imply (E),
Z is *$\hat{\Phi}$-connected* iff for all $(x_1,x_2) \in X \times X$ (A) and (D) imply (F).

Finally we say that Z is *Φ-shrinking* iff for every pair $(x_1,x_2) \in X \times X$ there exists a pair $(t_1,t_2) \in Z(x_1,x_2) \times Z(x_1,x_2)$ such that $\Phi(t_1) \subset \Phi(x_1)$ and $\Phi(t_2) \subset \Phi(x_2)$.

We want to demonstrate that the following simple observation can be very useful in the solution of the Intersection Problem. The idea is to split the problem into four subproblems which can be treated separately.

Proposition 1. val Φ *has the pairwise intersection property iff X admits a midset function Z such that* (a) *Φ is Z-concave,* (b) *Φ is Z-connected,* (c) *$\hat{\Phi}$ is Z-linked, and* (d) *Z is $\hat{\Phi}$-connected.*

P r o o f. Let (a)–(d) be satisfied. Suppose that for some pair $(x_1,x_2) \in X \times X$ condition (A) holds. Then (a) and (b) lead to (D) and to $Z(x_1,x_2) \subset \hat{\Phi}(x_1) \cup \hat{\Phi}(x_2)$. Now, by (c), $Z(x_1,x_2)$ joins $\hat{\Phi}(x_1)$ and $\hat{\Phi}(x_2)$. So by (d) there is an $\bar{x} \in Z(x_1,x_2) \cap \hat{\Phi}(x_1) \cap \hat{\Phi}(x_2)$. Hence, by (a), $\emptyset \neq \Phi(\bar{x}) = (\Phi(\bar{x}) \cap \Phi(x_1)) \cup (\Phi(\bar{x}) \cap \Phi(x_2)) = \emptyset$, a contradiction.

To prove the converse, take $Z = \{\cdot,\cdot\}$, say.

R e m a r k 1. For every midset function Z on X the following implications hold:

a) $\Phi(x) \subset \Phi(x_1) \cup \Phi(x_2)$ $\forall ((x_1,x_2),x) \in \operatorname{Gr} Z \Leftrightarrow \Phi^*$ is Z-convex-valued $\Rightarrow \Phi$ is Z-concave.
b) Φ is Z-connected $\Leftrightarrow \Phi(x)$ is pseudoconnected for $\{\Phi(x_1), \Phi(x_2)\}$ $\forall ((x_1,x_2),x) \in \operatorname{Gr} Z$.
c) Z is a segment function $\Rightarrow Z$ is Φ-shrinking $\Rightarrow \hat{\Phi}$ is Z-linked.

Proposition 2. *The following are equivalent:*

(1) val Φ *has the pairwise intersection property.*
(2) *X admids a Φ-shrinking midset function Z such that Φ^* is Z-convex-valued and for every pair $(x_1,x_2) \in X \times X$*
 (i) *every $\Phi(x), x \in Z(x_1,x_2)$, is pseudoconnected for $\{\Phi(x_1), \Phi(x_2)\}$, and*
 (ii) *$Z(x_1,x_2)$ is connected for $\{\hat{\Phi}(x_1), \hat{\Phi}(x_2)\}$.*

Proof. (1) $\Rightarrow$ (2): Take $Z = \{\cdot, \cdot\}$. (2) $\Rightarrow$ (1): Apply Proposition 1 and Remark 1.

4. Pairwise intersection – topological version. In the sequel we shall present some topological versions of Propositions 1 and 2 where the crucial abstract connectedness assumptions will be replaced by "ordinary" topological ones. The following observation is the key for our further investigations:

Remark 2. Let Z be a midset function for X. Suppose that for every pair $(x_1, x_2) \in X \times X$ which satisfies conditions (A) and (D) there exists a topology on X such that $Z(x_1, x_2)$ is connected or empty and the sets $Z(x_1, x_2) \cap \hat{\Phi}(x_k)$, $k \in \{1, 2\}$, are either both closed or both open in $Z(x_1, x_2)$. Then Z is $\hat{\Phi}$-connected.

As an immediate consequence we obtain:

Proposition 3. *The following are equivalent:*

(1) *val Φ has the pairwise intersection property.*

(2) *X admits a midset function Z such that $\hat{\Phi}$ is Z-linked, Φ^* is Z-convex-valued, and for every pair $(x_1, x_2) \in X \times X$ there exist topologies on X and Y such that*
  - (i) *$Z(x_1, x_2)$ is connected or empty, and the values $\Phi(x)$, $x \in Z(x_1, x_2)$, are connected,*
  - (ii) *$\Phi(x_1)$ and $\Phi(x_2)$ are both closed, and*
  - (iii) *the sets $Z(x_1, x_2) \cap \hat{\Phi}(x_k)$, $k \in \{1, 2\}$, are both open in $Z(x_1, x_2)$.*

(3) *As* (2) *with* (ii) *replaced by*
  - (ii)* *$\Phi(x_1)$ and $\Phi(x_2)$ are both open.*

Proof. (1) $\Rightarrow$ (2), (3): Take $Z = \{\cdot, \cdot\}$ and the topologies $\{\emptyset, X\}$ and $\{G \subset Y : \hat{y} \notin G\} \cup \{Y\}$ in case (2) resp. $\{G \subset Y : \hat{y} \in G\} \cup \{\emptyset\}$ in case (3), with arbitrary $\hat{y} \in \Phi(x_1) \cap \Phi(x_2)$.

(2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1): From Remarks 1 and 2 we infer that the assumptions of Proposition 1 are satisfied.

In applications the verification of condition (iii) in Proposition 3 may cause difficulties. Therefore, we shall present a modified version where (iii) is replaced by another condition (iii)* fitting better into the frame of correspondences. We shall make use of the following continuity conditions:

Let X and Y be topological spaces. Then a correspondence Φ from X to Y is called

upper semicontinuous at a point $x \in X$ iff for every open set G with $G \supset \Phi(x)$ there is a neighborhood U of x such that $\Phi(u) \subset G$, $u \in U$,

lower semicontinuous at a point $x \in X$ iff for every open set G with $G \cap \Phi(x) \neq \emptyset$ there is a neighborhood U of x such that $\Phi(u) \cap G \neq \emptyset$, $u \in U$,

quartercontinuous at a point $x \in X$ iff for every open set G with $G \supset \Phi(x)$ there is a neighborhood U of x such that $\Phi(u) \cap G \neq \emptyset$, $u \in U$.

Φ is called upper semicontinuous, ... iff Φ is upper semicontinuous, ... at every point $x \in X$.

The notion of upper and lower semicontinuity is classical [15]. In our framework, however, the weaker *quartercontinuity*, which has recently been introduced by Komiya [16], and, under the name of semicontinuity, by Correa et al. [7], is *essential*:

Remark 3. The following are equivalent:

(1) val Φ has the pairwise intersection property.
(2) $\hat{\Phi}$ is open-valued w.r.t. every topology on X.
(3) Φ is quartercontinuous w.r.t. all topologies on X and Y.

The following lemma summarizes some interrelations, most of which ought to be well-kown.

Lemma 1. *Let X and Y be topological spaces. For a correspondence Φ from X to Y consider the following properties:*

(a) *Φ is upper semicontinuous.*
(a)* *$\bigcap \{\Phi^*(y) : y \in F\}$ is open for every closed $F \subset Y$.*
(b) *Φ is lower semicontinuous.*
(b)* *$\bigcap \{\Phi^*(y) : y \in G\}$ is closed for every open $G \subset Y$.*
(c) *Φ is quartercontinuous.*
(d) *Φ is open-valued.*
(e) *Φ is closed-valued.*
(f) *$\hat{\Phi}$ is open-valued.*
(g) *$\hat{\Phi}$ is closed-valued.*
(h) *Φ^* is closed-valued.*
(i) *$\mathrm{Gr}\,\Phi$ is closed.*
(k) *Y is compact.*

Then the following implications hold: 1. (a) $\Leftrightarrow$ (a)* $\Rightarrow$ (c); 2. (b) $\Leftrightarrow$ (b)* $\Rightarrow$ (c); 3. (a) $\wedge$ (e) $\Rightarrow$ (f); 4. (b) $\wedge$ (d) $\Rightarrow$ (g); 5. (h) $\Rightarrow$ (g); 6. (g) $\Rightarrow$ (c); 7. (i) $\wedge$ (k) $\Rightarrow$ (a) $\wedge$ (e).

Proof. 1. and 2. are obvious, and 3., 4., and 5. follow from $\hat{\Phi}(x) = \bigcap \{\Phi^*(y) : y \dot{\in} \Phi(x)\}$ together with 1. and 2. To see 6. observe that for $U = X - \hat{\Phi}(x)$ and $G \supset \Phi(x)$ we have $\Phi(u) \cap G \neq \emptyset$, $u \in U$. Finally 7. is well-known [15].

Remark 2*. Let Z be a midset function for X. Suppose that for every pair $(x_1, x_2) \in X \times X$ which satisfies conditions (A) and (D) there exist topologies on X and Y such that the values $\Phi(x_1)$ and $\Phi(x_2)$ are either both open or both closed, Φ is quartercontinuous at every $x \in Z(x_1, x_2)$, and $Z(x_1, x_2)$ is connected or empty. Then Z is $\hat{\Phi}$-connected.

Proof. Let (x_1, x_2) satisfy (A) and (D). For $k \in \{1, 2\}$ let $M_k = Z(x_1, x_2) \cap \hat{\Phi}(x_k)$. By Remark 2 it is sufficient to show that M_1 and M_2 are both open in $Z(x_1, x_2)$. To see this, let $\bar{x} \in M_1$, say. We set $G = Y - \Phi(x_1)$ if $\Phi(x_1)$ and $\Phi(x_2)$ are both closed and $G = \Phi(x_2)$ otherwise. Then G is open and contains $\Phi(\bar{x})$. Hence, there is a neighborhood U of $\bar{x}$ such that $\Phi(u) \cap G \neq \emptyset$, $u \in U$. Now (A) and (D) imply $U \cap Z(x_1, x_2) \subset M_1$ in both cases.

Now, as above, we obtain:

Proposition 3*. *Proposition* 3 *remains true if condition* (iii) *is replaced by*
(iii)* *Φ is quartercontinuous at every $x \in Z(x_1, x_2)$.*

5. Correspondences with constant selector – solution of the intersection problem. We shall now attack the problem mentioned in the introduction. In order to be able to apply our results from Sections 3 and 4 we reduce it to a pairwise intersection problem:

For $E \in \mathscr{E}(X) \cup \{\emptyset\}$ let $\Phi_E(x) := \Phi^\cap(E \cup \{x\})$, $x \in X$. If P is any property then we say that Φ *has* P *hereditarily* iff every Φ_E, $E \in \mathscr{E}_\Phi(X)$, has P.

Remark 4. a) The correspondence Φ possesses a constant selector iff val Φ has the finite intersection property and Y admits a topology such that Φ is closed-valued and at least one value of Φ is compact.

b) val Φ has the finite intersection property iff it has the pairwise intersection property hereditarily.

Proof. a) For $\wp \in \Phi^\cap(X)$ the topology $\{G \subset Y : \wp \notin G\} \cup \{Y\}$ is compact and Φ is closed-valued. The converse is obvious.

b) Suppose that val Φ does not possess the finite intersection property, i.e., $\Phi^\cap(A) = \emptyset$ for some $A \in \mathscr{E}(X)$. Then we have $\Phi_E(x_1) \cap \Phi_E(x_2) = \emptyset$ for $\{x_1, x_2\} \subset A, x_1 \neq x_2$, and $E = A - \{x_1, x_2\}$. If such an A is chosen with minimal cardinality, then $E \in \mathscr{E}_\Phi(X)$. So Φ fails to have the pairwise intersection property hereditarily. The converse is obvious.

Remark 5. For a midset function Z on X the conditions "Φ^* is Z-convex-valued" and "Z is Φ-shrinking" are hereditary.

The following "hereditary version" of Lemma 1 will enable us to apply Remark 4:

Lemma 2 (Compare also [16].) *Let X and Y be topological spaces and Φ a correspondence from X to Y. Consider the same conditions* (a), (b), ... *as in Lemma* 1. *We write* (aH), (bH), ... *iff condition* (a), (b), ... *holds hereditarily.*

The following implications hold: 1. (a) $\wedge$ (e) $\Leftrightarrow$ (aH) $\wedge$ (eH) $\Rightarrow$ (cH) $\wedge$ (fH); 2. (a) $\wedge$ (d) $\Rightarrow$ (aH) $\wedge$ (dH) $\Rightarrow$ (cH); 3. (b) $\wedge$ (d) $\Leftrightarrow$ (bH) $\wedge$ (dH) $\Rightarrow$ (cH) $\wedge$ (gH); 4. (h) $\Leftrightarrow$ (hH) $\Rightarrow$ (gH) $\Rightarrow$ (cH); 5. (i) $\wedge$ (k) $\Leftrightarrow$ (iH) $\wedge$ (kH) $\Rightarrow$ (cH) $\wedge$ (fH).

Proof. (a) $\wedge$ (e) $\Rightarrow$ (aH): Let $E \in \mathscr{E}_\Phi(X)$, $x \in X$, and G an open neighborhood of $\Phi_E(x)$. Then $\Phi(x) \subset G_0 := G \cup (Y - \Phi^\cap(E))$. Hence there is a neighborhood U of x such that $\Phi(u) \subset G_0$ and therefore $\Phi_E(u) \subset G$, $u \in U$.

(a) $\wedge$ (d) $\Rightarrow$ (aH): Conclude as above with $G_0 = \Phi(x)$.

(b) $\wedge$ (d) $\Rightarrow$ (bH): Let $E \in \mathscr{E}_\Phi(X)$, $x \in X$, and G an open set intersecting $\Phi_E(x)$. Then for $G_0 = G \cap \Phi^\cap(E)$ there is a neighborhood U of x with $\Phi_E(u) \cap G = \Phi(u) \cap G_0 \neq \emptyset$, $u \in U$. The rest of the proof follows with Lemma 1.

Theorem 1. *For a correspondence Φ from X to Y the following are equivalent*:

(1) val Φ *has the finite intersection property.*

(2) *There exist topologies on X and Y and a Φ-shrinking topological midset function on X such that*
 (i) Φ^* *is convex-valued,*
 (ii) *every* $\Phi^\cap(A), A \in \mathscr{E}(X)$, *is connected or empty,*
 (iii) Φ *is hereditarily quartercontinuous, and*
 (iv) Φ *is open-valued.*

(3) *There exist topologies on X and Y and a topological segment function on X such that*
 (v) Φ *is lower semicontinuous,*

and conditions (i), (ii), *and* (iv) *as in* (2) *hold.*

Proof. (1) $\Rightarrow$ (3): (Compare [12]). Take the topologies $\mathscr{T}_X = \{\emptyset, X\}$ and $\mathscr{T}_Y = \{G \subset Y : \exists A \in \mathscr{E}(X)$ such that $G \supset \Phi^\cap(A)\} \cup \{\emptyset\}$ and the segment function $Z = \{\cdot,\cdot\}$.

(3) $\Rightarrow$ (2): This follows with the second implication in Lemma 2.

(2) $\Rightarrow$ (1): By Remark 4 it is sufficient to show that every $\Phi_E, E \in \mathscr{E}_\Phi(X)$, has the pairwise intersection property. But this follows with Proposition 3* together with Remarks 1 c) and 5.

Remark 6. The implication "(2) $\Rightarrow$ (1)" in Theorem 1 remains true if condition (iv) is replaced by

(iv)* Φ is closed-valued.

But "(1) $\Rightarrow$ (2)" fails in this case as the example $X = Y = \mathbb{N}$ and $\Phi(x) = \mathbb{N} - \{x\}$, $x \in \mathbb{N}$, shows. (Here (iv)* is only satisfied for $\mathscr{T}_Y = 2^Y$ which violates (ii).)

Theorem 2. *For a correspondence Φ from X to Y the following are equivalent*:

(1) Φ *possesses a constant selector* $\left(i.e., \bigcap_{x \in X} \Phi(x) \text{ is nonvoid}\right)$.

(2) *There exist topologies on X and Y and a Φ-shrinking topological midset function on X such that*
 (0) *at least one value of Φ is compact,*
 (i) Φ^* *is convex-valued,*
 (ii) *every* $\Phi^\cap(A), A \in \mathscr{E}(X)$, *is connected or empty,*
 (iii) Φ *is hereditarily quartercontinuous, and*
 (iv) Φ *is closed-valued.*

(3) *There exist topologies and topological segment functions on X and Y such that*
 (0) Y *is compact,*
 (i) Φ^* *is convex-valued,*
 (ii) Φ *is convex-valued,*
 (iii) Φ *is upper semicontinuous, and*
 (iv) Φ *is closed-valued.*

Proof. (1) $\Rightarrow$ (3): (Compare [14], [12]). Choose $\wp \in \Phi^\cap(X)$. Take $\mathscr{T}_X = \{\emptyset, X\}$, $Z_X = \{\cdot,\cdot\}$, $\mathscr{T}_Y = \{G \subset Y : \wp \notin G\} \cup \{Y\}$ and $Z_Y = \{\cdot,\cdot\} \cup \{\wp\}$.

(3) $\Rightarrow$ (2): This follows with the first implication in Lemma 2. (Observe that every $\Phi^\cap(A)$, $A \in \mathscr{E}(X)$, is convex and therefore connected.)

(2) $\Rightarrow$ (1): This follows with Remarks 4 a) and 6.

Remark 7. Conditions (2) or (3) in Theorem 2 without the compactness assumption (0) imply the *finite* intersection property of val Φ.

Theorem 3. *For a correspondence Φ from X to Y the following are equivalent:*

(1) *Φ has a constant selector.*

(2) *There exist topologies on X and Y and a topological midset function Z_X for X such that $\hat{\Phi}$ is hereditarily Z_X-linked and*

 (0) *Φ is compact-valued,*

 (i) *Φ^* is Z_X-convex-valued,*

 (ii) *every $\Phi^{\cap}(A)$, $A \in \mathscr{E}(X)$, is connected or empty,*

 (iii) *$\operatorname{Gr}\Phi \cap (Z_X(x_1, x_2) \times Y)$ is closed in $Z_X(x_1, x_2) \times Y$ for every $(x_1, x_2) \in X \times X$, and*

 (iv) *Φ is closed-valued.*

(3) *There exist topologies on X and Y, a topological midset function Z_X for X such that $\hat{\Phi}$ is hereditarily Z_X-linked, and a topological segment function Z_Y for Y with the properties*

 (0) *Y is compact,*

 (i) *Φ^* is Z_X-convex-valued,*

 (ii) *Φ is Z_Y-convex-valued,*

 (iii) *$\operatorname{Gr}\Phi \cap (Z_X(x_1, x_2) \times Y)$ is closed in $Z_X(x_1, x_2) \times Y$ for every $(x_1, x_2) \in X \times X$, and*

 (iv) *Φ is closed-valued.*

Proof. (1) $\Rightarrow$ (3): Take $Z_X \equiv \emptyset$ and $\mathscr{T}_X, \mathscr{T}_Y$ and Z_Y as in the proof of Theorem 2.

(3) $\Rightarrow$ (2): Compare the proof of Theorem 2.

(2) $\Rightarrow$ (1): By Remark 4 it is sufficient to show that val Φ has the pairwise intersection property, because the assumptions are hereditary for Φ. So by Proposition 3 it remains to show that for every pair $(x_1, x_2) \in X \times X$ the sets $M_k = Z_X(x_1, x_2) \cap \hat{\Phi}(x_k)$, $k \in \{1, 2\}$, are both open in $Z_X(x_1, x_2)$. To this end, consider a net (z_n) in $Z_X(x_1, x_2) - M_1$, say, which converges to some $z \in Z_X(x_1, x_2)$. For every n choose $y_n \in \Phi(x_1) \cap \Phi(z_n)$. As $\Phi(x_1)$ is compact there exists a subnet (y_m) converging to some $y \in \Phi(x_1)$. Condition (iii) implies $y \in \Phi(z)$, hence $z \notin M_1$.

Remark 8. Theorem 3 "(2) $\Rightarrow$ (1)" contains Stachó's Proposition 2 in [27] and, in essence, Simons' Theorem 8 in [25]. In these results Z_X is assumed to be a topological *segment* function. However, under this stronger assumption the implication "(1) $\Rightarrow$ (2)" fails to hold any more: Consider $X = \{1, 2\}$, $Y = \{1, 2, 3\}$, $\Phi(1) = \{1, 2\}$, and $\Phi(2) = \{2, 3\}$ as in [12], Example 3. It is easily seen that there exist no topologies on X and Y and no Φ-shrinking topological midset function (in particular, no topological segment function) Z_X for X such that condition (iii) in Theorem 3 is satisfied.

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