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On a minimax theorem of Terkelsen's

By

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1. Mean functions. In the following let D be an infinite convex subset of the set $\bar{\mathbb{R}}$ of extended reals. A function $\xi\colon D \times D \to D$ is called *mean function* (or *mean*, for short) if

(1) $\xi(\cdot, \beta)$, $\beta \in D$ and $\xi(\alpha, \cdot)$, $\alpha \in D$ are nondecreasing functions, and

(2) $\xi(\alpha, \alpha) = \alpha$, $\alpha \in D$.

For $\alpha \in D$, $\beta \in D$ we set

$$m(\alpha, \beta) = \min\{\alpha, \beta\} \quad \text{and}$$
$$M(\alpha, \beta) = \max\{\alpha, \beta\}.$$

Remark 1. The functions m and M are means, and for every mean ξ we have

(3) $m \leqq \xi \leqq M$.

We consider the following continuity properties:

(4) For $\alpha, \beta \in D \cap \mathbb{R}$ with $\alpha > \beta$ we have
$$\xi(\cdot, \beta)^n(\alpha) \to \beta \quad \text{and} \quad \xi(\beta, \cdot)^n(\alpha) \to \beta \ (n \to \infty).$$

(5) For $\alpha, \beta \in D \cap \mathbb{R}$ with $\alpha < \beta$ we have
$$\xi(\cdot, \beta)^n(\alpha) \to \beta \quad \text{and} \quad \xi(\beta, \cdot)^n(\alpha) \to \beta \ (n \to \infty).$$

Let $M^+(D)$ and $M^-(D)$ denote the set of all means $\xi\colon D \times D \to D$ which satisfy Condition (4) or (5), respectively.

Example 1. For a fixed $\tau \in \bar{\mathbb{R}}$ define the mean $\xi_\tau\colon D \times D \to D$ according to $\xi_\tau(\alpha, \beta) = \operatorname{med}\{\alpha, \beta, \tau\}$, the middle of the three values α, β and τ. Then

$$\xi_\tau \in M^+(D) \Leftrightarrow \tau \leqq \inf D \Leftrightarrow \xi_\tau = m \quad \text{and}$$
$$\xi_\tau \in M^-(D) \Leftrightarrow \tau \geqq \sup D \Leftrightarrow \xi_\tau = M.$$

Observe that m (resp. M) satisfies Condition (4) (resp. (5)) for *all* $(\alpha, \beta) \in D \times D$.

Example 2 (cf. [15; Lemma 2.2]). Let $\xi\colon D \times D \to D$ be a mean.

a) If $\xi(\cdot, \beta)$, $\beta \in D$ and $\xi(\alpha, \cdot)$, $\alpha \in D$ are continuous from the right, and if $\xi(\alpha, \beta) < M(\alpha, \beta)$ for $\alpha, \beta \in D \cap \mathbb{R}$ with $\alpha \neq \beta$ holds, then $\xi \in M^+(D)$.

b) If $\xi(\cdot, \beta)$, $\beta \in D$ and $\xi(\alpha, \cdot)$, $\alpha \in D$ are continuous from the left, and if $\xi(\alpha, \beta) > m(\alpha, \beta)$ for $\alpha, \beta \in D \cap \mathbb{R}$ with $\alpha \neq \beta$ holds, then $\xi \in M^-(D)$.

Proof. a) We follow [15; p. 232]: Let $\alpha, \beta \in D \cap \mathbb{R}$ with $\alpha > \beta$. Let $\alpha_0 = \alpha$ and $\alpha_n = \xi(\alpha_{n-1}, \beta)$, $n \in \mathbb{N}$. Then we have $\alpha_n \searrow \alpha^*$ for some $\alpha^* \geqq \beta$. But $\alpha^* > \beta$ implies $\alpha^* = M(\alpha^*, \beta) > \xi(\alpha^*, \beta) = \lim_{n\to\infty} \xi(\alpha_n, \beta) = \alpha^*$, a contradiction. Hence, $\xi(\cdot, \beta)^n(\alpha) = \alpha_n \to \beta$. Similarly, $\xi(\beta, \cdot)^n(\alpha) \to \beta$.

b) Compare the proof of a).

Example 3. a) Let $\lambda \in (0, 1)$, let $f: D \to D$ be a strictly monotone continuous function with inverse f^{-1}, and let $\xi: D \times D \to D$ with

$$\xi(\alpha, \beta) = f^{-1}(\lambda f(\alpha) + (1-\lambda) f(\beta)).$$

Then, by Example 2, $\xi \in M^+(D) \cap M^-(D)$.

For $\lambda = \frac{1}{2}$ an axiomatic characterization of these means has been given by Kolmogoroff [24] (Compare also [33], [28], [29].)

The special case $f(x) = x^p$ leads to the *weighted Minkowski means*

$$\xi(\alpha, \beta) = (\lambda \alpha^p + (1-\lambda)\beta^p)^{\frac{1}{p}}.$$

Especially, for $p = 1$ and $D = (-\infty, \infty]$, say, we obtain the *weighted arithmetic mean*

$$\mu_\lambda(\alpha, \beta) := \lambda\alpha + (1-\lambda)\beta,$$

and for $p = -1$ or $p \to 0$, respectively, and $D = (0, \infty)$ we get the *weighted harmonic mean*

$$\kappa_\lambda(\alpha, \beta) := \left(\frac{\lambda}{\alpha} + \frac{1-\lambda}{\beta}\right)^{-1}$$

and the *weighted geometric mean*

$$\gamma_\lambda(\alpha, \beta) := \alpha^\lambda \beta^{1-\lambda}.$$

Finally, for $p \to \pm\infty$ we obtain the means M and m.

b) New means can also be constructed by composition. If ξ_1, ξ_2, and ξ_3 are means, then by $\xi(\alpha, \beta) = \xi_3(\xi_1(\alpha, \beta), \xi_2(\alpha, \beta))$ another mean is defined [15]. Similarly, two means can be "compounded" to a new mean by an appropriate limiting process [26]. Examples are the famous Gaussian arithmetic-geometric mean and the arithmetic-harmonic mean (cf. [1], [2], [7], [8], [26], [39]).

The following limiting process is to some extend related to the compounding procedure:

Lemma 1. *Let $D^* = D \cap \mathbb{R}$.*

a) *Let $\psi \in M^+(D)$, and let $(\gamma_n, \delta_n) \in D^* \times D^*$ with $m(\gamma_n, \delta_n) \geqq \alpha \in D^*$ such that for every $n \in \mathbb{N}$ either*

(i) $\gamma_{n+1} = \gamma_n$ *and* $\delta_{n+1} \leqq \psi(\alpha, \delta_n)$, *or*
(ii) $\delta_{n+1} = \delta_n$ *and* $\gamma_{n+1} \leqq \psi(\gamma_n, \alpha)$

holds. Then $m\left(\lim_{n\to\infty} \gamma_n, \lim_{n\to\infty} \delta_n\right) = \alpha$.

b) *Let $\varphi \in M^-(D)$, and let $(\gamma_n, \delta_n) \in D^* \times D^*$ with $M(\gamma_n, \delta_n) \leqq \alpha \in D^*$ such that for every $n \in \mathbb{N}$ either*

(i) $\gamma_{n+1} = \gamma_n$ *and* $\delta_{n+1} \geqq \varphi(\alpha, \delta_n)$, *or*
(ii) $\delta_{n+1} = \delta_n$ *and* $\gamma_{n+1} \geqq \varphi(\gamma_n, \delta)$

holds. Then $M\left(\lim_{n\to\infty} \gamma_n, \lim_{n\to\infty} \delta_n\right) = \alpha$.

Part b) of this lemma is stated implicitely in [37; p. 408] for the special case $\varphi(\alpha, \beta) = \mu_{\frac{1}{2}}(\alpha, \beta) = \frac{1}{2}\alpha + \frac{1}{2}\beta$.

Proof. a) Let k_m resp. l_m be the number of all $n \leqq m$ such that condition (i) resp. (ii) holds. Then we have $k_m + l_m \geqq m$, and from (4) we infer

$$\alpha \leqq \lim_{n\to\infty} \delta_n \leqq \lim_{m\to\infty} \psi(\alpha, \cdot)^{k_m}(\delta_1) = \alpha$$

in case $k_m \to \infty$. Otherwise we have $l_m \to \infty$ and $\lim_{n\to\infty} \gamma_n = \alpha$.

The proof for b) is similar.

2. Preliminaries. In the following let an infinite convex subset $D \subset \overline{\mathbb{R}}$ and a triplet $\Gamma = (X, Y, a)$ be given. Here X and Y are nonvoid sets and a is a function $a: X \times Y \to D$. The situation may be interpreted as a *game.* Player 1 and player 2 independently choose *strategies* $x \in X$ and $y \in Y$, respectively. Afterwards player 1 receives the (possibly negative) amount $a(x, y)$ from player 2.

$$a_* = a_*(X, Y) := \sup_{x\in X} \inf_{y\in Y} a(x, y), \quad \text{and}$$

$$a^* = a^*(X, Y) := \inf_{y\in Y} \sup_{x\in X} a(x, y)$$

are called the *lower* and *upper value* of the game. The game is called *strictly determined* if $a_* = a^*$, i.e.

$$(6) \qquad \sup_{x\in X} \inf_{y\in Y} a(x, y) = \inf_{y\in Y} \sup_{x\in X} a(x, y)$$

holds. We want to present sufficient conditions which ensure the validity of (6). A standard method for proving such *minimax theorems* proceeds as follows. Suppose that player 1 has to announce in advance a set $A \in \mathscr{E}(X) := \{C \subset X : C \text{ finite}\}$. Afterwards the game (A, Y, a) is played. In this case the guarantee value a^* of player 2 (he can avoid to lose more than a^*) improves to

$$\tilde{a}^* = \tilde{a}^*(X, Y) := \sup_{A\in\mathscr{E}(X)} a^*(A, Y).$$

So, as usual, in the proof of our minimax theorem we proceed in two steps: first $a^* = \tilde{a}^*$ is shown by a compactness argument, and in the proof of $\tilde{a}^* = a_*$ some convexity and connectedness properties are exploited.

We shall make use of the following level sets:

$$Y_\alpha(x) = \{y \in Y : a(x, y) \leqq \alpha\}, \quad \alpha \in \mathbb{R}, \quad x \in X$$

and

$$Y_\alpha(A) = \bigcap_{x\in A} Y_\alpha(x), \quad \alpha\in\mathbb{R}, \quad A\in\mathscr{E}(X).$$

Here we set $Y_\alpha(\emptyset) = Y$.

Γ will be called *subcompact* if for all $\alpha\in\mathbb{R}$ with $Y_\alpha(A) \neq \emptyset$ for all $A\in\mathscr{E}(X)$ we have $Y_\beta(X) \neq \emptyset$ for all $\beta > \alpha$.

Lemma 2 ([18; Satz 5]). *The following conditions ar equivalent*:

(i) *Γ is subcompact.*
(ii) $\tilde{a}^*(X, Y) = a^*(X, Y)$.

Example 4. Let Y be a topological space such that the functions $a(x,\cdot)$, $x\in X$ are lower semicontinuous. If Y is compact or, more generally, if at least one set $Y_\beta(x)$, $x\in X$, $\beta > \tilde{a}^*(X, Y)$ is compact, then Γ is subcompact.

This result is well-known (cf. [17, Lemma 2], [14; (2.6)], or [13; Theorem 1]). Compare also [19; (3.3)].

Example 5. T. Tjoe-Tie [38] called Y (S)-*conditionally compact* if for every $\varepsilon > 0$ there exists a $B\in\mathscr{E}(Y)$ such that

$$Y = \bigcup_{z\in B}\bigcap_{x\in X}\{y\in Y: a(x, y) \geqq a(x, z) - \varepsilon\}.$$

Similarly, X is called (S)-*conditionally compact* if for every $\varepsilon > 0$ there exists an $A\in\mathscr{E}(X)$ with

$$X = \bigcup_{s\in A}\bigcap_{y\in Y}\{x\in X: a(x, y) \leqq a(s, y) + \varepsilon\}.$$

It is not difficult to show (compare [30], [18]) that in both cases Γ is subcompact.

If $\varphi: D\times D\to D$ is a mean, then we set

$$\varphi_B(x_1, x_2) := \bigcap_{y\in B}\{x\in X: a(x, y) \geqq \varphi(a(x_1, y), a(x_2, y))\}$$

for $\emptyset \neq B\subset Y$, $(x_1, x_2)\in X\times X$. Γ will be called (*finitely*) *φ-concave* iff for all $(x_1, x_2)\in X\times X$ $\varphi_Y(x_1, x_2) \neq \emptyset$ (resp. $\varphi_B(x_1, x_2) \neq \emptyset$ for all $B\in\mathscr{E}(Y)$).

Similarly, for a mean $\psi: D\times D\to D$ let

$$\psi_A(y_1, y_2) := \bigcap_{x\in A}\{y\in Y: a(x, y) \leqq \psi(a(x, y_1), a(x, y_2))\}$$

for $\emptyset \neq A\subset X$, $(y_1, y_2)\in Y\times Y$. Then Γ will be called (*finitely*) *ψ-convex* iff for all $(y_1, y_2)\in Y\times Y$ $\psi_X(y_1, y_2) \neq \emptyset$ (resp. $\psi_A(y_1, y_2) \neq \emptyset$ for all $A\in\mathscr{E}(X)$).

Remark 2. Γ is always m-concave and M-convex.

In our further investigations the following concept turns out to be useful. Let $\varphi: D\times D\to D$ be a mean. For $(x_1, x_2, \alpha, A)\in X\times X\times D\times\mathscr{E}(X)$ consider the conditions

(7) $A\cap\varphi_Y(x_1, x_2) \neq \emptyset$

(8) $\infty > \alpha > a_*(X, Y)$, $Y_\alpha(A\cup\{x_i\}) \neq \emptyset$, $i\in\{1, 2\}$, and $Y_\alpha(A) \subset Y_\alpha(\{x_1\})\cup Y_\alpha(\{x_2\})$

(9) $Y_\beta(A\cup\{x_1, x_2\}) \neq \emptyset$ for all $\beta > \alpha$.

Then Y will be called *Γ-connected* (resp. *φ-connected*) if Condition (8) (resp. Conditions (7) and (8) together) imply (9).

Remark 3. a) Y is always M-connected.
b) If Y is Γ-connected, then Y is φ-connected for every mean $\varphi: D\times D\to D$.
c) If Γ is finitely m-convex, then Y is m-connected.

Example 6. Let Y be a topological space such that all nonvoid sets $Y_\alpha(A)$, $a_*(X, Y) < \alpha < \infty$, $A\in\mathscr{E}(X)$ are connected (as subspaces). If either (i) or (ii) is satisfied:

(i) Every function $a(x,\cdot)$, $x\in X$ is upper semicontinuous.
(ii) Every function $a(x,\cdot)$, $x\in X$ is lower semicontinuous,

then Y is Γ-connected.

Proof. Let (x_1, x_2, α, A) satisfy Condition (8).

If (i) holds, then $Y_\beta^*(A) := \bigcap_{x\in A}\{y\in Y: a(x, y) < \beta\}$, $\beta > \alpha$, is open, and we have $Y_\alpha(A)\cap Y_\beta^*(x_i) \neq \emptyset$, $i\in\{1, 2\}$ and $Y_\alpha(A) = (Y_\alpha(A)\cap Y_\beta^*(x_1))\cup(Y_\alpha(A)\cap Y_\beta^*(x_2))$.

Now, the connectedness of $Y_\alpha(A)$ implies

$$\emptyset \neq Y_\alpha(A)\cap Y_\beta^*(x_1)\cap Y_\beta^*(x_2) \subset Y_\beta(A\cup\{x_1, x_2\}).$$

In the proof of case (ii) replace Y_β^* by Y_β.

The following example will be fundamental for our Theorem 2 below:

Example 7. Let Γ be finitely ψ-convex w.r.t. a $\psi\in M^+(D)$. Then

(i) Y is Γ-connected if $\infty\notin D$.
(ii) Y is φ-connected for every mean $\varphi: D\times D\to D$ with

(10) $(\alpha, \infty)\in D\times D \Rightarrow \varphi(\alpha, \infty) = \varphi(\infty, \alpha) = \infty$.

Proof (cf. [6; p. 44f] and [15; p. 235f]). For $(x_1, x_2, \alpha, A)\in X\times X\times(D\cap\mathbb{R})\times\mathscr{E}(X)$ – with $A\cap\varphi_Y(x_1\ x_2) \neq \emptyset$ in case (ii) – let $S = Y_\alpha(x_1)$, $T = Y_\alpha(x_2)$, and $R = Y_\alpha(A)$ such that $S\cap R \neq \emptyset$, $T\cap R \neq \emptyset$ and $R\subset S\cup T$. Choose $v_1\in S\cap R$ and $w_1\in T\cap R$. Then we have $\gamma_1 := a(x_2, v_1) < \infty$ and $\delta_1 := a(x_1, w_1) < \infty$. Under Assumption (10) this is true because for $y\in R$ and $\bar{x}\in A\cap\varphi_Y(x_1, x_2)$ we have $\infty > \alpha \geqq a(\bar{x}, y) \geqq \varphi(a(x_1, y), a(x_2, y))$ which implies $a(x_i, y) < \infty$, $i\in\{1, 2\}$. If $v_n\in S\cap R$ and $w_n\in T\cap R$ with $\gamma_n := a(x_2, v_n) < \infty$ and $\delta_n := a(x_1, w_n) < \infty$ are choosen, then choose any $y_n\in\psi_{A\cup\{x_1, x_2\}}(v_n, w_n)$. From

$$a(x, y_n) \leqq \psi(a(x, v_n), a(x, w_n)) \leqq \psi(\alpha, \alpha) = \alpha, \quad x\in A$$

we infer $y_n\in R\subset S\cup T$. In case $y_n\in S$ we set $(v_{n+1}, w_{n+1}) = (y_n, w_n)$, otherwise let $(v_{n+1}, w_{n+1}) = (v_n, y_n)$. In the first case we infer from $w_n\in T$

$$\gamma_{n+1} := a(x_2, v_{n+1}) \leqq \psi(a(x_2, v_n), a(x_2, w_n)) \leqq \psi(\gamma_n, \alpha) < \infty,$$

and in the second case $v_n\in S$ implies

$$\delta_{n+1} := a(x_1, w_{n+1}) \leqq \psi(\alpha, \delta_n) < \infty.$$

If $R\cap S\cap T = \emptyset$, then $m(\gamma_n, \delta_n) > \alpha$, $n\in\mathbb{N}$. By Lemma 1 a) there exists for every $\beta > \alpha$ a $k\in\mathbb{N}$ with $m(\gamma_k, \delta_k) < \beta$, hence

$$Y_\beta(A\cup\{x_1, x_2\})\cap\{v_k, w_k\} \neq \emptyset.$$

3. The main theorem. We are now in the position to prove our main result. It has been observed by Wu [42] that the only property of convex sets which is actually needed in the proof of minimax theorems is connectedness. Compare also the papers [40], [41], [3], [9], [10], [11], [19], [25], [36], [37]. The proof of the following result was mainly inspired by Terkelsen's paper [37].

Theorem 1. *Let Γ be φ-concave and Y φ-connected with respect to a mean $\varphi \in M^-(D)$ such that $-\infty < \inf_{y\in Y} a(x, y) \in D$ for all $x \in X$. Then*

(11) $a^*(A, Y) \leqq a_*(X, Y)$ *for all* $A \in \mathscr{E}(X)$.

Moreover, Condition (6) *holds iff Γ is subcompact.*

Proof. Suppose that, in contrast to (11), there exists an $A \in \mathscr{E}(X)$ and an $\alpha \in \mathbb{R}$ such that $a^*(A, Y) > \alpha > a_* := a_*(X, Y)$ and

(12) $a^*(C, Y) \leqq a_*$ for all $C \in \mathscr{E}(X)$ with $|C| < |A|$.

We choose $s_1, t_1 \in A$ with $s_1 \neq t_1$ and set $E = A - \{s_1, t_1\}$. If s_n, t_n are chosen with $a^*(A_n, Y) > \alpha$ for $A_n = E \cup \{s_n, t_n\}$, then we construct s_{n+1} and t_{n+1} as follows. We choose an $x_n \in \varphi_Y(s_n, t_n)$ and set

$$S_\lambda = Y_\lambda(s_n), \quad T_\lambda = Y_\lambda(t_n) \quad \text{and} \quad R_\lambda = Y_\lambda(E \cup \{x_n\}), \quad \lambda \in \mathbb{R}.$$

We choose β and γ such that $a^*(A_n, Y) > \beta > \gamma > \alpha$. Then from

$$a(x_n, y) \geqq \varphi(a(s_n, y), a(t_n, y)) > \gamma \quad \text{for} \quad y \notin S_\gamma \cup T_\gamma$$

we infer $R_\gamma \subset S_\gamma \cup T_\gamma$. Hence, $R_\beta \cap S_\beta \cap T_\beta = \emptyset$ implies either $T_\gamma \cap R_\gamma = \emptyset$ or $S_\gamma \cap R_\gamma = \emptyset$, as Y is φ-connected. We set $(s_{n+1}, t_{n+1}) = (x_n, t_n)$ in the first case and $= (s_n, x_n)$ otherwise.

Now we set $W = Y_\alpha(E)\,(\neq \emptyset$ by (12)), $\gamma_n = \inf_{y\in W} a(s_n, y)$, and $\delta_n = \inf_{y \in W} a(t_n, y)$. Then we have

(13) $T_\gamma \cap R_\gamma = \emptyset \Rightarrow a^*(A_{n+1}, Y) > \alpha \geqq \gamma_{n+1} \geqq \varphi(\gamma_n, \alpha)$, and

(14) $T_\gamma \cap R_\gamma \neq \emptyset \Rightarrow a^*(A_{n+1}, Y) > \alpha \geqq \delta_{n+1} \geqq \varphi(\alpha, \delta_n)$.

To prove (13), say, observe that (12) implies $Y_\alpha(E \cup \{s_n\}) \neq \emptyset$, hence $\gamma_n \leqq \alpha$, $n \in \mathbb{N}$. Moreover, $R_\gamma \cap T_\gamma = \emptyset$ implies $a^*(A_{n+1}, Y) \geqq \gamma > \alpha$.

For $y \in R_\gamma \cap W$ we have $y \notin T_\gamma$, hence

$$a(s_{n+1}, y) = a(x_n, y) \geqq \varphi(a(s_n, y), a(t_n, y)) \geqq \varphi(\gamma_n, \alpha)$$

and for $y \in W - R_\gamma$ we obtain

$$a(s_{n+1}, y) = a(x_n, y) > \gamma > \alpha \geqq \varphi(\gamma_n, \alpha),$$

which implies $\gamma_{n+1} \geqq \varphi(\gamma_n, \alpha)$.

Now from Lemma 1b) we infer $M\left(\lim_{n\to\infty} \gamma_n, \lim_{n\to\infty} \delta_n\right) = \alpha > a_*$, and thus $a^*(E \cup \{x^*\}, Y) > a_*$ for some $x^* \in \bigcup_{n=1}^{\infty} \{s_n, t_n\}$, in contradiction to (12).

Hence, we have shown that (11) holds which is equivalent to the equality $\tilde{a}^* = a_*$. By Lemma 2 the last assertion follows.

Remark 4. If the mean φ in Theorem 1 satisfies Condition (5) for *all* $(\alpha, \beta) \in D \times D$, then it is not necessary to assume $\inf_{y\in Y} a(x, y) > -\infty$, $x \in X$.

4. Some minimax theorems. The above theorem can be used to derive several old and new minimax theorems. We present some examples.

Corollary 1 (Dini Theorem [20], [37]). *Let $D = [-\infty, \infty]$.*

a) *Let Γ be M-concave, and let Y be a compact topological space such that every function $a(x, \cdot)$, $x \in X$ is lower semicontinuous. Then* (6) *holds.*
b) *Let Γ be m-convex, and let X be a compact topological space such that every function $a(\cdot, y)$, $y \in Y$ is upper semicontinuous. Then* (6) *holds.*

Proof. a) By Examples 1 and 4 and Remarks 3a) and 4 we can apply Theorem 1.
b) Apply part a) to (Y, X, b) with $b(y, x) = -a(x, y)$.

The following version of Dini's Theorem seems to be less known:

Corollary 2 ("Dax Theorem" [21]). *Let $D = [-\infty, \infty]$.*

a) *Let Γ be finitely m-convex, and let Y be a compact topological space such that the functions $a(x, \cdot)$, $x \in X$ are lower semicontinuous. Then* (6) *holds.*
b) *Let Γ be finitely M-concave and let X be a compact topological space such that the functions $a(\cdot, y)$, $y \in Y$ are upper semicontinuous. Then* (6) *holds.*

Proof. a) Let $A \in \mathscr{E}(X)$ be endowed with the discrete topology. Then from Corollary 1b) we get $a^*(A, Y) = a_*(A, Y) \leqq a_*(X, Y)$, hence $\tilde{a}^* = a_*$. Now from Lemma 2 and Example 4 the assertion follows.

b) Apply part a) to (Y, X, b) with $b(y, x) = -a(x, y)$.

Remark 5. a) Corollary 2a) cannot be derived directly from Theorem 1 by Remarks 2, 3c), and 4, because the mean $\varphi = m$ does not satisfy continuity property (5).

b) Corollary 1a), say, turns wrong if Γ is only supposed to be finitely M-concave. For a counterexample, take $X = Y = \mathbb{N}$ endowed with the cofinite topology and set $a(x, y) = 1$ (0) for $x \neq y$ ($x = y$).

Corollary 3. *Let $-\infty \notin D$ and let Γ be φ-concave w.r.t. a $\varphi \in M^-(D)$. Suppose that Y is a compact topological space such that every function $a(x, \cdot)$, $x \in X$ is lower semicontinuous and every nonvoid set $Y_\alpha(A)$, $A \in \mathscr{E}(X)$, $\alpha \in \mathbb{R}$ is connected. Then* (6) *holds.*

The special case $\varphi = \mu_{\frac{1}{2}}$ (cf. Example 3a)) and $D = \mathbb{R}$ is due to Terkelsen [37; Theorem 2].

Proof. Apply Lemma 2, Examples 4 and 6, Remark 3b) and Theorem 1. Observe that for every $x \in X$ there is a $z \in Y$ such that $-\infty < a(x, z) = \inf_{y\in Y} a(x, y)$.

Now let Y be a topological space, and let $\langle\cdot,\cdot\rangle: Y \times Y \to 2^Y$ be a mapping such that every "interval" $\langle y_1, y_2\rangle$, $(y_1, y_2) \in Y \times Y$ is connected and contains y_1 and y_2. In this case, Y is called an *interval space* [36], [19]. A subset $Z \subset Y$ is called *convex* if $\{y_1, y_2\} \subset Z$ implies $\langle y_1, y_2\rangle \subset Z$, and a function $a(x,\cdot)$, $x \in X$ is called *quasiconvex* if every set $Y_\alpha(x)$, $\alpha \in \mathbb{R}$ is convex.

Corollary 4. *Let $-\infty \notin D$, and let Γ be φ-concave w.r.t. a $\varphi \in M^-(D)$. Suppose that Y is a compact interval space such that the functions $a(x,\cdot)$, $x \in X$ are lower semicontinuous and quasiconvex. Then* (6) *holds.*

A special case of this result is due to Terkelsen [37; Corollary 1].

P r o o f. In an interval space, the intersection of convex sets is convex, and every convex set is connected [19; Remark 2.1]. Hence, Corollary 3 can be applied.

In his lecture on mathematical economics in Karlsruhe (cf. [21]), which culminated in the book [23], König formulated the following problem:

K ö n i g ' s P r o b l e m. Let $D = (-\infty, \infty]$. Characterize those pairs of functions $\varphi, \psi: D \times D \to D$ with Property (3) and the following property:

(P) For every pair of nonvoid sets X, Y and for every function $a: X \times Y \to D$ such that $\Gamma = (X, Y, a)$ is φ-concave and ψ-convex Condition (11) is satisfied.

A partial solution of this problem has been presented by Irle [15], [16] for a special class of continuous means φ and ψ which he called *averaging functions* (and which play also an important role in the theory of fuzzy sets [5]). The following theorem is closely related to Irle's main theorem in [15]:

Theorem 2. *Let $-\infty < \inf_{y \in Y} a(x, y) \in D$ for all $x \in X$. Let $\varphi \in M^-(D)$ satisfy* (10) *and let $\psi \in M^+(D)$. Suppose that Γ is φ-concave and finitely ψ-convex. Then* (11) *holds, and* (6) *is true iff Γ is subcompact.*

P r o o f. This follows from Lemma 2, Example 7 and Theorem 1.

R e m a r k 6. The above theorem has a long history. By combining it with Example 4 we obtain Ky Fan's classical minimax theorem [6] as well as – up to some epsilontics – the generalized versions of König [20], [22] and Irle [15]. (I abstained from presenting Theorems 1 and 2 in the greatest possible generality in order to keep the proofs as short and lucid as possible.) In connection with Example 5 we get versions of Teh Tjoe Tie's minimax theorem [38], [30], [31], [18]. Finally, we obtain a generalization of a minimax theorem of De Wilde [4] and the author [18]:

Corollary 5. *Let $D \subset \mathbb{R}$ be a compact interval. Let Γ be φ-concave and ψ-convex w.r.t. means $\varphi \in M^-(D)$ and $\psi \in M^+(D)$. If*

$$\liminf_{m\to\infty} \limsup_{n\to\infty} a(x_m, y_n) \geqq \limsup_{n\to\infty} \liminf_{m\to\infty} a(x_m, y_n)$$

is true for all sequences (x_m) in X and (y_n) in Y, then (6) *holds.*

P r o o f. From Theorem 2 we infer $\tilde{a}^* = a_*$. By symmetry we have $\tilde{a}_* := \inf_{B \in \mathscr{E}(Y)} a_*(X, B) = a^*$. But the "double limit condition" implies $\tilde{a}^* \geqq \tilde{a}_*$ [4], [18].

The following example shows that the continuity properties (4) and (5) cannot be dispensed with:

E x a m p l e 8. Let $X = Y = D = [0, 1]$ and $a(x, y) = (x - y)^2$. Then $\Gamma = (X, Y, a)$ is subcompact, and for $\varphi = m$ and $\psi = \mu_{\frac{1}{2}}$ all assumptions of Theorem 2 and Corollary 5 are fulfilled with the only exception that φ does not satisfy Condition (5). Of course, $a_* = 0 < \frac{1}{4} = a^*$, i.e. (6) is violated.

Addendum. After the present paper had been submitted Lin and Quan published the following result:

Theorem A (Lin-Quan [27]). *Let Y be a compact topological space and let every function $a(x,\cdot)$, $x \in X$ be real valued and lower semicontinuous. If there exist s, t in* (0, 1) *such that X is s-concave and Y is t-convex then* (6) *holds.*

Here X (Y) is called *λ-concave* (*λ-convex*) iff – in our terminology – Γ is ξ_λ-concave (resp. ξ_λ-convex) w.r.t. the composed mean $\xi_\lambda := \mu_\lambda(M, m)$ (cf. Example 3).
By Example 2, $\xi_\lambda \in M^+(\mathbb{R}) \cap M^-(\mathbb{R})$. Hence the above theorem is a special case of our Theorem 2. Similarly, several other related results of Geraghty and Lin [9], [11], [12] are easy consequences of the present results.
Theorem A has recently been generalized by Simons. He calls a function $a: X \times Y \to \mathbb{R}$ *upward on Y* if

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall y_1, y_2 \in Y \, \exists y_0 \in M_X(y_1, y_2) \, \forall x \in X:$$
$$|a(x, y_1) - a(x, y_2)| \geqq \varepsilon \Rightarrow a(x, y_0) \leqq M(a(x, y_1), a(x, y_2)) - \delta$$

Similarly, a is *downward on X* if $b: Y \times X \to \mathbb{R}$ with $b(y, x) = -a(x, y)$ is upward on Y.

Theorem B (Simons [34]). *Let $a: X \times Y \to \mathbb{R}$ be upward on Y, downward on X, and let $\inf_{y \in Y} a(x, y) > -\infty$, for all $x \in X$. Then* (11) *holds.*

This theorem is similar to our Theorem 2. By a slight modification of our proofs, one gets the following versions of Example 7 and Theorem 1, which demonstrate again the usefulness of our concept of connectedness.

E x a m p l e 7*. Let $D = \mathbb{R}$, and let a be upward on Y. Then Y is Γ-connected.

Theorem 1*. *Let $D = \mathbb{R}$, let a be downward on X, and let Y be Γ-connected. If $\inf_{y \in Y} a(x, y) > -\infty$ for all $x \in X$, then* (11) *holds.*

By combining both results we obtain Theorem B. Finally, Theorem 1* together with Example 6 imply Simons' version of Terkelsen's minimax theorem [35].

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