



The Existence of Nash Equilibrium in n -Person Games with C -Concavity

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Abstract—The purpose of this paper is to introduce the C -concavity condition, and next prove a new existence theorem of Nash equilibrium in n -person games with C -concavity. And, as an application, we shall prove a minimax theorem. Finally, we shall give some examples of a two-person game where the C -concavity can be applied, but the previous general concavity conditions cannot be applied. Our results generalize the corresponding results due to Nash and Forgö in several ways. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In mathematical economics, showing the existence of equilibrium is the main problem of investigating various kind of economic models, and till now, a number of equilibrium existence results in general economic models have been investigated by several authors, e.g., Debreu [1,2], Nash [3], Friedman [4], and others.

In 1951, Nash established the following well-known theorem.

THEOREM A. (See [3].) Let I be a finite set of players. Assume that for all $i \in I$,

- (a) the set $X_i \subset \mathbb{R}^{k_i}$ is nonempty compact and convex;
- (b) the function $f_i : X := \prod_{i \in I} X_i \rightarrow \mathbb{R}$ is continuous on X ;
- (c) the function $y_i \mapsto f_i(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ is concave on X_i .

Then there exists an $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that for every $i \in I$,

$$f_i(\bar{x}_1, \dots, \bar{x}_i, \dots, \bar{x}_n) \geq f_i(\bar{x}_1, \dots, x_i, \dots, \bar{x}_n), \quad \text{for all } x_i \in X_i.$$

Next, in 1977, Friedman [4] established a generalization of Theorem A using the quasiconcavity assumption on every payoff function. Since then, the classical results of Nash [3], Debreu [1],

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Nikaido and Isoda [5], and Friedman [4] have served as basic references for the existence of Nash equilibrium for noncooperative generalized games. In all of them, convexity of strategy spaces, continuity, and concavity/quasiconcavity of the payoff functions were assumed. Till now, there have been a number of generalizations, and also many applications of those theorems have been found in several areas; e.g., see [4,6] and the references therein.

Two important concepts for removing the concavity/quasiconcavity assumptions of payoff functions are marked by the seminal papers of Fan [7,8] for two-person zero-sum games, and the complete abandonment of concavity in [9]. In fact, the concept of concavelike payoffs due to Fan [8] does not require any linear structure on the strategy space. However, in [10], Joó gave a general sum two-person game where the payoff functions are continuous and concavelike, but the game has no Nash equilibrium. Horváth and Joó [11] also show that higher smoothness of the payoff functions does not change the situation. On the other hand, Fan's existence results have been extended to two-person games defined over certain convexity structures by Joó and Stachó [12], Horváth and Sövegjártó [13], and Borgmér *et al.* [14]. Joó [10] also proved that in a certain sense, partial concavity of the payoff functions is necessary for a two-person game to have an equilibrium. And there have been a number of generalized concepts of concavity by several authors, and using those concepts, there have been also many applications; e.g., see [15] and the references therein.

In this paper, we first introduce the \mathcal{C} -concavity which generalizes both concavity and CF-concavity without assuming the linear structure. Using the \mathcal{C} -concavity and the partition of unity argument, we shall prove the Nash equilibrium for noncooperative n -person games. And, as an application, we shall prove a minimax theorem. Finally, we shall give some examples of two-person games where the \mathcal{C} -concavity can be applied, but the quasiconcavity cannot be applied.

2. PRELIMINARIES

We begin with some notations and definitions. Let X_i be a nonempty topological space for each $i \in I$, and denote $X_{\bar{i}} := \prod_{j \in I \setminus \{i\}} X_j$. If $x = (x_1, \dots, x_n) \in X$, we shall write $x_{\bar{i}} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X_{\bar{i}}$. If $x_i \in X_i$ and $x_{\bar{i}} \in X_{\bar{i}}$, we shall use the notation $(x_i, x_{\bar{i}}) := (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = x \in X$. Denote by $[0, 1]^n$ the Cartesian product space of unit intervals $\underbrace{[0, 1] \times \dots \times [0, 1]}_{n \text{ times}}$.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

Let $I = \{1, \dots, n\}$ be a set of players. A *noncooperative n -person game* of normal form is an ordered $2n$ -tuple $(X_1, \dots, X_n; f_1, \dots, f_n)$, where for each player $i \in I$, the nonempty set X_i is the player's pure strategy space, and $f_i : X = \prod_{i=1}^n X_i \rightarrow \mathbb{R}$ is the player's payoff function. The set X , *joint strategy space*, is the Cartesian product of the individual strategy sets, and an element of X_i is called a *strategy*. A strategy n -tuple $(\bar{x}_1, \dots, \bar{x}_n) \in X$ is called a *Nash equilibrium* for the game if the following system of inequalities holds:

$$f_i(\bar{x}_i, \bar{x}_{\bar{i}}) \geq f_i(x_i, \bar{x}_{\bar{i}}), \quad \text{for all } x_i \in X_i \quad \text{and} \quad i = 1, \dots, n.$$

Here we note that the model of a game in this paper is a noncooperative game, i.e., there is no replay communicating between players, and so players act as free agents, and each player is trying to maximize his/her own payoff according to his/her strategy.

Next, we recall some concepts which generalize the convexity/concavity as follows: let X be a nonempty convex subset of a vector space E and let $f : X \rightarrow \mathbb{R}$. We say that f is *quasiconcave* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \geq t\}$ is convex, and that f is *quasiconvex* if $-f$ is quasiconcave. It is easy to see that if f is quasiconcave, then

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{f(x_1), f(x_2)\},$$

for every $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$. It should be noted that if f, g are quasiconcave, then $f + g$ is not quasiconcave in general.

When X and Y are any arbitrary sets, recall that $f : X \times Y \rightarrow \mathbb{R}$ is *concavelike* [8] on X if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, there exists an $x_0 \in X$ such that

$$f(x_0, y) \geq \lambda f(x_1, y) + (1 - \lambda)f(x_2, y), \quad \text{for every } y \in Y.$$

We now prove the following, which is equivalent to the concavelike condition.

LEMMA 1. *Let X be a nonempty topological space, Y an arbitrary set. Then $f : X \times Y \rightarrow \mathbb{R}$ is concavelike on X , if and only if, for every $n \geq 2$, whenever $x_1, \dots, x_n \in X$ are given and for any $\lambda_i \in [0, 1]$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$, there exists a point $x_0 \in X$ such that*

$$f(x_0, y) \geq \lambda_1 f(x_1, y) + \dots + \lambda_n f(x_n, y), \quad \text{for all } y \in Y. \tag{*}$$

PROOF. The sufficiency is clear. For the necessity, we shall use the induction argument on n . When $n = 2$, condition (*) is exactly the same as the definition of a concavelike condition. Assume that condition (*) holds for all $k \leq n - 1$ ($n \geq 3$). Let $\{x_1, \dots, x_n\} \subset X$ be given, and $\lambda_i \in [0, 1]$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$ be arbitrarily given. Without loss of generality, we may assume $\sum_{i=1}^{n-1} \lambda_i > 0$ by reindexing i . Then, for a given set $\{x_1, \dots, x_{n-1}\}$, the induction assumption assures that there exists a point $\bar{x} \in X$ such that

$$f(\bar{x}, y) \geq \frac{\lambda_1}{\sum_{i=1}^{n-1} \lambda_i} f(x_1, y) + \dots + \frac{\lambda_{n-1}}{\sum_{i=1}^{n-1} \lambda_i} f(x_{n-1}, y), \quad \text{for all } y \in Y.$$

Then, by the induction assumption on two points \bar{x}, x_n , there exists a point $x_0 \in X$ such that for all $y \in Y$,

$$f(x_0, y) \geq \left(\sum_{i=1}^{n-1} \lambda_i \right) f(\bar{x}, y) + \lambda_n f(x_n, y).$$

Therefore, we finally have

$$\begin{aligned} f(x_0, y) &\geq \left(\sum_{i=1}^{n-1} \lambda_i \right) f(\bar{x}, y) + \lambda_n f(x_n, y) \\ &\geq \sum_{i=1}^{n-1} \lambda_i \left(\frac{\lambda_1}{\sum_{i=1}^{n-1} \lambda_i} f(x_1, y) + \dots + \frac{\lambda_{n-1}}{\sum_{i=1}^{n-1} \lambda_i} f(x_{n-1}, y) \right) + \lambda_n f(x_n, y) \\ &= \lambda_1 f(x_1, y) + \dots + \lambda_n f(x_n, y), \end{aligned}$$

for all $y \in Y$. Therefore, by the induction, for every $n \geq 2$, we can obtain the desired conclusion. ■

Lemma 1 is a convenient tool in proving the existence of equilibrium for generalized games with the concavelike condition.

In a recent paper [16], adding the continuity to concavelike functions, Forgö introduced the CF-concavity as follows: let X be a nonempty topological space, Y an arbitrary set. Then $f : X \times Y \rightarrow \mathbb{R}$ is said to be *CF-concave on X* with respect to Y if there exists a continuous function $\Psi : X \times X \times [0, 1] \rightarrow X$ such that for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$f(\Psi(x_1, x_2, \lambda), y) \geq \lambda f(x_1, y) + (1 - \lambda)f(x_2, y), \quad \text{for all } y \in Y.$$

Moreover, Forgö [16, Lemma 1] obtained the following, which is equivalent to the CF-concavity: let X be a nonempty topological space, Y an arbitrary set. Then $f : X \times Y \rightarrow \mathbb{R}$ is CF-concave on X with respect to Y if and only if for each $n \geq 2$, there exists a continuous function $\Psi_n : \underbrace{X \times \cdots \times X}_n \times [0, 1]^n \rightarrow X$ such that for any $x_1, \dots, x_n \in X$, and for all $\lambda_i \in [0, 1]$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$,

$$f(\Psi_n(x_1, \dots, x_n; \lambda_1, \dots, \lambda_n), y) \geq \lambda_1 f(x_1, y) + \cdots + \lambda_n f(x_n, y),$$

for all $y \in Y$.

Next, we will introduce the following, which generalizes the concavity condition.

DEFINITION. Let X be a nonempty topological space, Y an arbitrary set. Then $f : X \times Y \rightarrow \mathbb{R}$ is called \mathcal{C} -concave on X if for every $n \geq 2$, whenever n points $x_1, \dots, x_n \in X$ are arbitrarily given, there exists a continuous function $\phi_n : [0, 1]^n \rightarrow X$ such that

$$f(\phi_n(\lambda_1, \dots, \lambda_n), y) \geq \lambda_1 f(x_1, y) + \cdots + \lambda_n f(x_n, y), \tag{1}$$

for all $\lambda_i \in [0, 1]$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$, and for all $y \in Y$.

REMARK. As remarked in [16], CF-concavity is closely related to a concavelike condition. The concavity clearly implies the \mathcal{C} -concavity by letting $\phi_n(\lambda_1, \dots, \lambda_n) := \lambda_1 x_1 + \cdots + \lambda_n x_n$, whenever $x_1, \dots, x_n \in X$ are given. Note that the continuous function ϕ_n need not be globally defined on $X_1 \times \cdots \times X_n \times [0, 1]^n$, but defined only on $[0, 1]^n$ for each $n \geq 2$ in the definition. In fact, by defining $\phi_n(\lambda_1, \dots, \lambda_n) := \Psi_n(x_1, \dots, x_n; \lambda_1, \dots, \lambda_n)$, for any given n points $x_1, \dots, x_n \in X$, we can see that the CF-concavity implies the \mathcal{C} -concavity. However, we do not know the implications between the quasiconcavity and the \mathcal{C} -concavity. Therefore, the following implication diagram holds:

$$\text{concave} \implies \text{CF-concave} \implies \mathcal{C}\text{-concave.}$$

Finally, recall that a topological space X is said to have the *fixed-point property* (or is a *fixed-point space*) [17] if every continuous mapping $f : X \rightarrow X$ has a fixed point in X .

Clearly this property is topologically invariant, and note that the product of two fixed-point spaces need not be a fixed-point space. In contrast with finite products, an infinite product of nonempty compact fixed-point spaces will be a fixed-point space whenever every finite product of those spaces is a fixed-point space; e.g., see [17, p. 174].

3. EXISTENCE OF NASH EQUILIBRIUM

By following the method in [5], let us define the total sum of payoff functions $H : X \times X \rightarrow \mathbb{R}$ associated with the noncooperative game Γ as follows:

$$H(x, y) := \sum_{i=1}^n f_i(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n), \tag{2}$$

for every $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X = \prod_{i=1}^n X_i$.

Then we shall need the following.

LEMMA 2. (See [5].) Let Γ be a noncooperative n -person game of normal form. If there exists a point $\bar{x} \in X$ for which

$$H(\bar{x}, \bar{x}) \geq H(x, \bar{x}), \quad \text{for any } x \in X,$$

then \bar{x} is a Nash equilibrium for Γ .

PROOF. For any $x = (\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \in X$, $x_i \in X_i$, by substitution, we can see that \bar{x} is a Nash equilibrium. ■

Here we note that if for each $i \in I$, the function $y_i \mapsto f_i(y_i, x_i)$ is concave as in Theorem A, then the function $x \mapsto H(x, y)$ is concave, and so it is \mathcal{C} -concave.

Using the \mathcal{C} -concavity and the partition of unity argument, we now prove the following new existence theorem of Nash equilibrium.

THEOREM 1. *Let I be a finite set of players, and let Γ be a noncooperative game satisfying the following:*

- (i) *the strategy space $X := \prod_{i=1}^n X_i$ is nonempty compact and has the fixed-point property;*
- (ii) *the function $H(x, y)$ is continuous on $X \times X$;*
- (iii) *the function $x \mapsto H(x, y)$ is \mathcal{C} -concave on X ;*

then Γ has at least one Nash equilibrium.

PROOF. Suppose the contrary. Then, by Lemma 2, for all $x \in X$, there exists an $y \in X$ such that $H(x, x) < H(y, x)$.

For any $z \in X$, we let

$$U(z) := \{x \in X \mid H(x, x) < H(z, x)\}.$$

Then, by Assumption (ii), H is continuous, so that each $U(z)$ is open in X , and also $\bigcup_{z \in X} U(z) = X$. Since X is compact, there exists a finite number of nonempty open sets $U(z_1), \dots, U(z_n)$ such that $\bigcup_{i=1}^n U(z_i) = X$. Let $\{\alpha_i \mid i = 1, \dots, n\}$ be the partition of unity subordinate to the open covering $\{U(z_i) \mid i = 1, \dots, n\}$ of X ; i.e.,

$$0 \leq \alpha_i(x) \leq 1, \quad \sum_{i=1}^n \alpha_i(x) = 1 \text{ for all } x \in X, \quad i = 1, \dots, n;$$

and if $x \notin U(z_j)$, for some j , then $\alpha_j(x) = 0$.

For such $\{z_1, \dots, z_n\} \subset X$, since H is \mathcal{C} -concave, there exists a continuous mapping $\phi_n : [0, 1]^n \rightarrow X$ satisfying the condition

$$H(\phi_n(\lambda_1, \dots, \lambda_n), y) \geq \lambda_1 H(z_1, y) + \dots + \lambda_n H(z_n, y),$$

for all $\lambda_i \in [0, 1]$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$, and for all $y \in X$.

Now consider a continuous mapping $\Psi : X \rightarrow X$, defined by

$$\Psi(x) := \phi_n(\alpha_1(x), \dots, \alpha_n(x)), \quad \text{for all } x \in X.$$

Since ϕ_n and every α_i are continuous, Ψ is continuous on X . Moreover, Ψ maps X , which is a fixed-point space, into itself. Therefore, there exists a fixed point $\bar{x} \in X$ such that $\Psi(\bar{x}) = \bar{x}$.

Next, by the \mathcal{C} -concavity of H , we have

$$H(\Psi(z), x) \geq \alpha_1(z)H(z_1, x) + \dots + \alpha_n(z)H(z_n, x), \quad \text{for all } x \in X,$$

and so we have

$$H(\bar{x}, \bar{x}) \geq \sum_{i=1}^n \alpha_i(\bar{x}) H(z_i, \bar{x}). \tag{3}$$

However, if $\bar{x} \in U(z_j)$ for some $1 \leq j \leq n$, then we have $H(\bar{x}, \bar{x}) < H(z_j, \bar{x})$ and $\alpha_j(\bar{x}) > 0$; and if $\bar{x} \notin U(z_k)$ for some $1 \leq k \leq n$, then $\alpha_k(\bar{x}) = 0$. Thus, we have

$$\sum_{i=1}^n \alpha_i(\bar{x}) H(z_i, \bar{x}) > \sum_{i=1}^n \alpha_i(\bar{x}) H(\bar{x}, \bar{x}) = H(\bar{x}, \bar{x}),$$

which contradicts the fact (3). This completes the proof. ■

REMARKS.

- (1) Theorem 1 generalizes the previous equilibrium existence theorems due to Nash [3] and Forgö [16] in the following aspects:

- (a) the strategy sets X_1, \dots, X_n need not be convex, but $\prod_{i=1}^n X_i$ has the fixed-point property (in fact, if X_i is homeomorphic to a compact convex subset of finite-dimensional Euclidean space as in [16], then $\prod_{i=1}^n X_i$ is clearly a fixed-point space);
 - (b) all payoff functions f_1, \dots, f_n need not be continuous nor concave, and also H need not be CF-concave on X .
- (2) It is well known that the Nash-equilibrium problem has only one family of players and the payoff functions are single-valued functions. Note that by following the method in [18], the payoff functions might be families of multimaps. So we can improve the equilibrium existence theorems in more general settings.
- (3) Our proof is different from those of Nikaido and Isoda [5] and Forgö [16] where they used a kind of symmetrization procedure.

Therefore, we can obtain the following immediate consequences of Theorem 1.

COROLLARY 1. (See [3].) *Let I be a finite index set, and Γ be a noncooperative game. Assume that*

- (a) for all $i \in I$, the set $X_i \subset \mathbb{R}^{k_i}$ is nonempty compact and convex;
- (b) for all $i \in I$, the function f_i is continuous;
- (c) the function $y_i \mapsto f_i(y_i, x_i)$ is concave.

Then there exists at least one Nash equilibrium for Γ .

COROLLARY 2. (See [16].) *Let I be a finite index set, and Γ be a noncooperative game. Assume that*

- (a) for all $i \in I$, the set $X_i \subset \mathbb{R}^{k_i}$ is nonempty compact and convex;
- (b) for all $i \in I$, the function f_i is continuous;
- (c) for all $y \in X$, the function $H(x, y)$ is CF-concave on X .

Then there exists at least one Nash equilibrium for Γ .

As an application of Theorem 1, we shall prove the following minimax theorem.

THEOREM 2. *Let X and Y be nonempty compact sets and $X \times Y$ be a fixed-point space. Assume that*

- (a) the function $f : X \times Y \rightarrow \mathbb{R}$ is continuous on $X \times Y$;
- (b) for each $y \in Y$, the function $-f(\cdot, y)$ is \mathcal{C} -concave on X ;
- (c) for each $x \in X$, the function $f(x, \cdot)$ is \mathcal{C} -concave on Y .

Then we have

$$\max_{y \in Y} \min_{x \in X} f(x, y) = \min_{x \in X} \max_{y \in Y} f(x, y).$$

PROOF. Let $f_1(x, y) := -f(x, y)$ and $f_2(x, y) := f(x, y)$. In order to apply Theorem 1, we first define the mapping $H : (X \times Y) \times (X \times Y) \rightarrow \mathbb{R}$ by

$$H((x_1, y_1), (x_2, y_2)) := f_1(x_1, y_2) + f_2(x_2, y_1), \quad \text{for every } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Then H is clearly continuous, so it suffices to show that Assumption (iii) of Theorem 1 is satisfied. Let two points $(x_1, y_1), (x_2, y_2) \in X \times Y$ be given arbitrarily. Then for $\{x_1, x_2\}$, by Assumption (b), there exists a continuous function $\phi_1 : [0, 1]^2 \rightarrow X$ such that

$$f_1(\phi_1(\lambda, 1 - \lambda), v) \geq \lambda f_1(x_1, v) + (1 - \lambda) f_1(x_2, v),$$

for every $\lambda \in [0, 1]$ and every $v \in Y$. Also, for $\{y_1, y_2\}$, by Assumption (c), there exists a continuous function $\phi_2 : [0, 1]^2 \rightarrow Y$ such that

$$f_2(u, \phi_2(\lambda, 1 - \lambda)) \geq \lambda f_2(u, y_1) + (1 - \lambda) f_2(u, y_2),$$

for every $\lambda \in [0, 1]$ and every $u \in X$.

Now we define a continuous function $\Phi_2 : [0, 1]^2 \rightarrow X \times Y$ by

$$\Phi_2(\lambda, 1 - \lambda) := (\phi_1(\lambda, 1 - \lambda), \phi_2(\lambda, 1 - \lambda)), \quad \text{for every } \lambda \in [0, 1].$$

Then it is easy to see that Φ_2 is a continuous function on $[0, 1]^2$. Also, for every $\lambda \in [0, 1]$, we have

$$\begin{aligned} &\lambda H((x_1, y_1), (u, v)) + (1 - \lambda) H((x_2, y_2), (u, v)) \\ &= \lambda(f_1(x_1, v) + f_2(u, y_1)) + (1 - \lambda)(f_1(x_2, v) + f_2(u, y_2)) \\ &= [\lambda f_1(x_1, v) + (1 - \lambda)f_1(x_2, v)] + [\lambda f_2(u, y_1) + (1 - \lambda)f_2(u, y_2)] \\ &\leq f_1(\phi_1(\lambda, 1 - \lambda), v) + f_2(u, \phi_2(\lambda, 1 - \lambda)) \\ &= H(\Phi_2(\lambda, 1 - \lambda), (u, v)), \quad \text{for all } (u, v) \in X \times Y. \end{aligned}$$

For arbitrarily given n points $(x_1, y_1), \dots, (x_n, y_n) \in X \times Y$, we can similarly define a continuous function $\Phi_n : [0, 1]^n \rightarrow X \times Y$ by

$$\Phi_n(\lambda_1, \dots, \lambda_n) := (\psi_1(\lambda_1, \dots, \lambda_n), \psi_2(\lambda_1, \dots, \lambda_n)),$$

for every $\lambda_i \in [0, 1], i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$, where $\psi_1 : [0, 1]^n \rightarrow X$ is a continuous function suitable for f_1 with respect to $\{x_1, \dots, x_n\}$, and $\psi_2 : [0, 1]^n \rightarrow Y$ is a continuous function suitable for f_2 with respect to $\{y_1, \dots, y_n\}$ in the \mathcal{C} -concavity condition (1). Thus, we can also show condition (1), and hence, H is \mathcal{C} -concave on $X \times Y$.

Therefore, by Theorem 1, there exists a Nash equilibrium $(x_0, y_0) \in X \times Y$ such that

$$f_1(x_0, y_0) = \max_{x \in X} f_1(x, y_0) \quad \text{and} \quad f_2(x_0, y_0) = \max_{y \in Y} f_2(x_0, y).$$

Therefore, we have

$$-f(x_0, y_0) = f_1(x_0, y_0) \geq f_1(x, y_0) = -f(x, y_0), \quad \text{for all } x \in X,$$

and

$$f(x_0, y_0) = f_2(x_0, y_0) \geq f_2(x_0, y) = f(x_0, y), \quad \text{for all } y \in Y.$$

Hence,

$$\max_{y \in Y} f(x_0, y) \leq f(x_0, y_0) \leq \min_{x \in X} f(x, y_0),$$

which implies

$$\min_{x \in X} \max_{y \in Y} f(x, y) \leq f(x_0, y_0) \leq \max_{y \in Y} \min_{x \in X} f(x, y).$$

And the reverse inequality

$$\max_{y \in Y} f(x, y) \geq \max_{y \in Y} \min_{x \in X} f(x, y)$$

is trivial, and so we have the conclusion. ■

Using any (possibly uncountable) set of players and typical strategy spaces, Theorem 1 can be reformulated as follows.

THEOREM 3. *Let I be a (possibly uncountable) set of players, and let Γ be a noncooperative generalized game satisfying the following:*

- (i) *the strategy space $X := \prod_{i \in I} X_i$ is nonempty and homeomorphic to a compact convex subset of a locally convex Hausdorff topological vector space;*
- (ii) *the total sum of all payoff functions $H(x, y) := \sum_{i \in I} f_i(x_i, y_i)$ is continuous on $X \times X$;*
- (iii) *the function $H(x, y)$ is \mathcal{C} -concave on X ;*

then Γ has at least one Nash equilibrium.

PROOF. Since X is compact, we can repeat the same proof of Theorem 1 except for finding a fixed point of Ψ . But X is homeomorphic to a compact convex subset of a locally convex Hausdorff topological vector space, and we can apply the Tychonoff fixed-point theorem in this case, so that we can obtain the conclusion. ■

As we mentioned before, the game described previously has an equilibrium if the payoff function f_i satisfies either CF-concavity or quasiconcavity. Indeed, many of the assumptions made in the preceding theorems have been weakened and the existence of equilibrium has been proved; however, it is hard to improve the equilibrium theorem by relaxing quasiconcavity assumption of the payoff functions and the convexity assumption on the strategy space. In fact, the Nash equilibrium is applied in many areas of mathematical economics including oligopoly theory, general equilibrium, and social choice theory; hence, the \mathcal{C} -concavity should be helpful in developing the theory of Nash equilibrium.

Finally, note that Theorem 3 can be improved to more general spaces by using general fixed-point theorems, e.g., Eilenberg-Montgomery’s fixed-point theorem or Himmelberg’s fixed-point theorem without assuming finite-dimensional Euclidean spaces or compact strategy spaces. Also, by using Berge’s maximum theorem or its generalizations, it is possible to improve the existence theorem of social equilibrium for generalized games with the \mathcal{C} -concavity.

4. EXAMPLES OF TWO-PERSON GAMES WITHOUT QUASICONCAVITY

Next, we shall give two examples of a two-person game where Theorem 1 can be applied but the previous theorems due to Nash [3] and Friedman [4] cannot be applied. They also show that Theorem 1 generalizes the corresponding results due to Nash [3] and Forgö [16] in several ways.

EXAMPLE 1. Let $\Gamma = \{X_1, X_2; f_1, f_2\}$ be a two-person game where $X_1 = [-1, 1]$, $X_2 = [0, 1]$, and

$$\begin{aligned} f_1(x_1, x_2) &:= x_1^2, & \text{for every } (x_1, x_2) \in X = X_1 \times X_2; \\ f_2(y_1, y_2) &:= \sqrt{y_2}, & \text{for every } (y_1, y_2) \in X = X_1 \times X_2. \end{aligned}$$

Clearly $f_1(\cdot, x_2)$ is not quasiconcave for any $x_2 \in [0, 1]$, and thus, theorems of Nash [3] and Friedman [4] cannot be applied. For this game, the related total sum of payoff functions $H : X \times X \rightarrow \mathbb{R}$ is given by

$$H((x_1, x_2), (y_1, y_2)) = f_1(x_1, y_2) + f_2(y_1, x_2) = x_1^2 + \sqrt{x_2},$$

for every $((x_1, x_2), (y_1, y_2)) \in X \times X$. For arbitrarily given two points $(x_1, x_2), (x_3, x_4) \in X$, we now define a continuous function $\phi_2 : [0, 1]^2 \rightarrow X$ by

$$\phi_2(\lambda, 1 - \lambda) := \left(\sqrt{\lambda x_1^2 + (1 - \lambda)x_3^2}, [\lambda\sqrt{x_2} + (1 - \lambda)\sqrt{x_4}]^2 \right), \quad \text{for all } \lambda \in [0, 1].$$

Then it is easy to see that ϕ_2 is a continuous function on $[0, 1]^2$. Also, for every $\lambda \in [0, 1]$ and $(y_1, y_2) \in X$, we have

$$\begin{aligned} &H(\phi_2(\lambda, 1 - \lambda), (y_1, y_2)) \\ &= H\left(\left(\sqrt{\lambda x_1^2 + (1 - \lambda)x_3^2}, [\lambda\sqrt{x_2} + (1 - \lambda)\sqrt{x_4}]^2\right), (y_1, y_2)\right) \\ &= (\lambda x_1^2 + (1 - \lambda)x_3^2) + (\lambda\sqrt{x_2} + (1 - \lambda)\sqrt{x_4}) \\ &\geq (\lambda x_1^2 + \lambda\sqrt{x_2}) + ((1 - \lambda)x_3^2 + (1 - \lambda)\sqrt{x_4}) \\ &= \lambda H((x_1, x_2), (y_1, y_2)) + (1 - \lambda) H((x_3, x_4), (y_1, y_2)). \end{aligned}$$

For arbitrarily given n points $(x_1, x_2), \dots, (z_1, z_2) \in X$, we can similarly define a continuous function ϕ_n by

$$\phi_n(\lambda_1, \dots, \lambda_n) := \left(\sqrt{\lambda_1 x_1^2 + \dots + \lambda_n z_1^2}, [\lambda_1 \sqrt{x_2} + \dots + \lambda_n \sqrt{z_2}]^2 \right),$$

for all $\lambda_i \in [0, 1]$, $i = 1, \dots, n$, with $\sum_{i=1}^n \lambda_i = 1$. Thus, we can show that H is \mathcal{C} -concave on X . Therefore, we can apply Theorem 1 to the game Γ ; and clearly, $(1, 1)$ is a Nash equilibrium for Γ . In fact, we have

$$\begin{aligned} 1 &= f_1(1, 1) \geq f_1(x_1, 1) = x_1^2, & \text{for every } x_1 \in X_1; \\ 1 &= f_2(1, 1) \geq f_2(1, y_2) = \sqrt{y_2}, & \text{for every } y_2 \in X_2. \end{aligned}$$

The next example demonstrates that the strategy sets X_1, X_2 need not be convex but homeomorphic to compact convex sets, so that $X_1 \times X_2$ is a fixed-point space.

EXAMPLE 2. Let $\Gamma = \{X_1, X_2; f_1, f_2\}$ be a two-person game where $X_1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, -1 \leq x \leq 1, 0 \leq y \leq 1\}$, $X_2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, 0 \leq x, y \leq 1\}$, and let

$$\begin{aligned} f_1((x, y), (u, v)) &:= x^2 u + v^2, & \text{for every } ((x, y), (u, v)) \in X = X_1 \times X_2; \\ f_2((x, y), (u, v)) &:= -xv^2 + y, & \text{for every } ((x, y), (u, v)) \in X = X_1 \times X_2. \end{aligned}$$

Then clearly $f_1((x, y), (u, v))$ is not quasiconcave for any fixed $(u, v) \in X_2$, but $f_2((x, y), (u, v))$ is clearly quasiconcave for any fixed $(x, y) \in X_1$; thus, theorems due to Nash [3] and Friedman [4] cannot be applied. Also note that the strategy sets X_1 and X_2 are not convex, but homeomorphic to a compact convex set $\{(x, 0) \mid 0 \leq x \leq 1\}$ using the projection mapping.

For this game, the related total sum of payoff function $H : X \times X \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} H(((x, y), (u, v)), ((t_1, t_2), (t'_1, t'_2))) &= f_1((x, y), (t'_1, t'_2)) + f_2((t_1, t_2), (u, v)) \\ &= x^2 t'_1 + t'_2{}^2 - t_1 v^2 + t_2, & \text{for every } (((x, y), (u, v)), ((t_1, t_2), (t'_1, t'_2))) \in X \times X. \end{aligned}$$

For arbitrarily given two points $((x_1, y_1), (u_1, v_1)), ((x_2, y_2), (u_2, v_2)) \in X$, we now define a continuous function $\phi_2 : [0, 1]^2 \rightarrow X$ by

$$\phi_2(\lambda, \mu) := \left(\left(\sqrt{\lambda x_1^2 + \mu x_2^2}, \sqrt{1 - \lambda x_1^2 - \mu x_2^2} \right), \left(\sqrt{1 - \lambda v_1^2 - \mu v_2^2}, \sqrt{\lambda v_1^2 + \mu v_2^2} \right) \right),$$

for all $(\lambda, \mu) \in [0, 1]^2$, where $\lambda + \mu = 1$.

Then it is easy to see that ϕ_2 is a continuous function which depends on the given two points $((x_1, y_1), (u_1, v_1)), ((x_2, y_2), (u_2, v_2)) \in X$. Also, for every $((t_1, t_2), (t'_1, t'_2)) \in X$ and every $(\lambda, \mu) \in [0, 1]^2$ with $\lambda + \mu = 1$, we have

$$\begin{aligned} H(\phi_2(\lambda, \mu), ((t_1, t_2), (t'_1, t'_2))) &= H \left(\left(\left(\sqrt{\lambda x_1^2 + \mu x_2^2}, \sqrt{1 - \lambda x_1^2 - \mu x_2^2} \right), \left(\sqrt{1 - \lambda v_1^2 - \mu v_2^2}, \sqrt{\lambda v_1^2 + \mu v_2^2} \right) \right), \right. \\ &\quad \left. ((t_1, t_2), (t'_1, t'_2)) \right) \\ &\geq (\lambda x_1^2 + \mu x_2^2) t'_1 + t'_2{}^2 - t_1 (\lambda v_1^2 + \mu v_2^2) + t_2 \\ &= \lambda (x_1^2 t'_1 + t'_2{}^2 - t_1 v_1^2 + t_2) + \mu (x_2^2 t'_1 + t'_2{}^2 - t_1 v_2^2 + t_2) \\ &= \lambda H(((x_1, y_1), (u_1, v_1)), ((t_1, t_2), (t'_1, t'_2))) + \mu H(((x_2, y_2), (u_2, v_2)), ((t_1, t_2), (t'_1, t'_2))). \end{aligned}$$

For arbitrarily given n points $((x_1, y_1), (u_1, v_1)), \dots, ((x_n, y_n), (u_n, v_n)) \in X$, as in Example 1, we can similarly define a continuous function ϕ_n , and hence, H is \mathcal{C} -concave on X . Therefore, we can apply Theorem 1 to the game Γ ; and clearly, $((1, 0), (1, 0))$ is a Nash equilibrium for Γ . In fact, we have

$$\begin{aligned} 1 &= f_1((1, 0), (1, 0)) \geq f_1((x, y), (1, 0)) = x^2, & \text{for every } (x, y) \in X_1, \\ 0 &= f_2((1, 0), (1, 0)) \geq f_2((1, 0), (u, v)) = -v^2, & \text{for every } (u, v) \in X_2. \end{aligned}$$

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