Bounded symmetric domains and generalized operator algebras

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In this lecture we give a short survey how bounded symmetric domains in complex Banach spaces can be described by *JB*-triples*. These can be considered as generalizations of C*-algebras and also of Jordan C*-algebras, the so-called JB*-algebras.

1. Holomorphy in infinite dimensions

Let E, F be arbitrary complex Banach spaces in the following. With $\mathcal{L}(E,F)$ we denote the space of all bounded linear mappings $E \to F$ endowed with the operator norm. In case $E = F$ we also write $\mathcal{L}(E)$ instead of $\mathcal{L}(E,E)$. A (vector valued) function $f : U \to F$, defined in a domain $U \subset E$, is called *holomorphic* if to every $a \in U$ there exists an operator $df_a \in \mathcal{L}(E,F)$ satisfying

$$\lim_{z \mapsto a} \frac{||f(z) - f(a) - df_a(z-a)||}{||z-a||} \;=\; 0\,,$$

where $z \neq a$ runs in U. It is easily seen that for every holomorphic f the operator df_a, called the *complex (Fréchet) derivative of* f *at* a*, is uniquely determined by* a*.* For holomorphic functions in this sense the usual properties of elementary calculus hold - like the chain rule and the implicit function theorem.

For every holomorphic $f : U \to F$ the derivative $df : U \mapsto \mathcal{L}(E,F)$, $u \mapsto df_u$, again is holomorphic. As a consequence, the *second derivative* $d^2f = d(df)$ exists on U and has values in the space $\mathcal{L}\big(E, \mathcal{L}(E,F)\big)$ which is canonically isomorphic to the Banach space $\mathcal{L}^2(E,F)$ of all bounded complex bilinear mappings $E \times E \to F$. By iteration, for every $n \in \mathbb{N}$ the n^{th} derivative $d^n f$ exists on U and has values in the space $\mathcal{L}^n(E,F)$ of all bounded n-multilinear mappings $E^n \to F$. In addition, f has a local power series expansion at every $a \in U$, more precisely, there exists an open neighbourhood $V \subset U$ of a such that the series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\, p_n(z-a)$$

converges uniformly on V, where the n-homogeneous polynomial $p_n : E \to F$ is defined by $p_n(t) = d^n f_a(t, t, \ldots, t)$ for all $t \in E$.

Now suppose that a further domain $V \subset F$ is given. Then every holomorphic function f on U with $f(U) \subset V$ is called a *holomorphic mapping* $f : U \to V$. A bijective mapping $f : U \to V$ is called *biholomorphic* if f and its inverse f^{-1} both are holomorphic. The domains U, V are called *biholomorphically equivalent* if there exists a biholomorphic mapping between them.

1.1 Cartan's uniqueness theorem. *Suppose that $U \subset E$ is a bounded domain, $V \subset F$ is an arbitrary domain and $f, g : U \to V$ are holomorphic mappings with g biholomorphic. Then f, g coincide if there exists at least one point $a \in U$ with $f(a) = g(a)$ and $df_a = dg_a$.*

The proof, originally given by Cartan [3] in the 2-dimensional situation, is astonishingly simple and immediately extends to arbitrary complex Banach spaces. Usually the theorem is applied for the special case $U = V$ and $g = \mathsf{id}_U$. It then states that the holomorphic mapping $f : U \to U$ is already the identity if there is a fixed point $a \in U$ of f with $df_a = \mathsf{id} \in \mathcal{L}(E)$.

Notice that the notion of holomorphy does not depend on the norms but only on the topologies of the Banach spaces involved. A connection between holomorphy and the isometric structure of Banach spaces is given by the following result in [12]:

1.2 Theorem. *The complex Banach spaces E, F are isometrically equivalent if and only if their open unit balls are biholomorphically equivalent.*

This means that the isometric structure of every complex Banach space is completely encoded in the holomorphic structure of its open unit ball. Our interest will be the class of all those complex Banach spaces E for which the corresponding open unit ball B has the following holomorphic property: *To every pair of points $a, b \in B$ there is a biholomorphic mapping $g : B \to B$ with $b = g(a)$.*

2. Automorphism groups and holomorphic vector fields

For every domain U in the complex Banach space E we denote by $\mathsf{Aut}(U)$ the group of all biholomorphic mappings $g : U \to U$ and call it the *biholomorphic automorphism group of U*. In general, $\mathsf{Aut}(U)$ consists only of the identity transformation, but can also be very large: For instance, the group $\mathsf{Aut}(\mathbb{C}^2)$ contains all transformations $(z, w) \mapsto (z, w + f(z))$ as well as $(z, w) \mapsto (e^{f(w)}z, w)$ with f an arbitrary entire function of one complex variable. On the other hand, in case U is bounded in E, the group $\mathsf{Aut}(U)$ cannot be too big. Indeed, by Cartan's uniqueness theorem $u \mapsto (g(u), dg_u)$ gives an injection $\mathsf{Aut}(U) \hookrightarrow U \times \mathcal{L}(E)$.

Our next aim is to describe $\mathsf{Aut}(U)$ infinitesimally where that is possible. To begin with, we call every holomorphic function $f : U \to E$ also a *holomorphic vector field* on U. Since functions and vector fields are structurally different objects we prefer to write symbolically $f(z)\,{}^{\partial}\!/_{\partial z}$ instead of f (here z denotes the 'local variable' in U). Every holomorphic vector field $X = f(z)\,{}^{\partial}\!/_{\partial z}$ on U can be applied to arbitrary Banach space valued holomorphic functions $h : U \to F$ by putting $Xf(u) := dh_u(f(u)) \in F$ for every $u \in U$. Then the space $\mathfrak{hol}(U)$ of all holomorphic vector fields on U becomes a complex Lie algebra with bracket defined by

$$[X,Y] := \big(Xg(z) - Yf(z)\big)\, {\partial}/{\partial z}$$

for all $X = f(z)\, {\partial}/{\partial z}$ and $Y = g(z)\, {\partial}/{\partial z}$ in $\mathfrak{hol}(U)$.

A word of caution is necessary here: Vector fields can also be applied to smooth functions and many authors would write these in the real context as $X = f(z)\, {\partial}/{\partial z} + \overline{f(z)}\, {\partial}/{\partial \overline{z}}$ rather than $X = f(z)\, {\partial}/{\partial z}$ as we do here.

Every holomorphic vector field $X = f(z)\, {\partial}/{\partial z}$ on U gives rise to a flow of local biholomorphic transformations - more precisely - by the elementary theory of ordinary differential equations to every $a \in U$ there are convex domains $T \subset \mathbb{C}$, $V \subset U$ together with a holomorphic mapping $g : T \times V \to U$ such that $0 \in T$, $a \in V$ and g solves the initial value problem ${\partial g(t,v)}/{\partial t} = f(g(t,v))$, $g(0,a) = a$. In case $T \subset \mathbb{C}$ can be chosen to contain the full real line $\mathbb{R}$ we write $\exp(X)(a) := g(1,a)$. In case $\exp(X)(a)$ is defined for every $a \in U$ the mapping $\exp(X) : U \to U$ is biholomorphic and we then call the vector field X *complete* on U. The subset $\mathfrak{aut}(U) \subset \mathfrak{hol}(U)$ of all complete holomorphic vector fields is not closed under addition nor under the bracket in general. Nevertheless we have the exponential mapping $\exp : \mathfrak{aut}(U) \to \mathsf{Aut}(U)$.

In 1935 H. Cartan [3] showed that for every bounded domain $D \subset \mathbb{C}^n$ the group $\mathsf{Aut}(D)$ is a real Lie group in the compact open topology acting analytically on D, that is, the mapping $\mathsf{Aut}(D) \times D \to D$, $(g,z) \mapsto g(z)$, is real analytic (and clearly holomorphic in z). In addition, then $\mathfrak{aut}(D)$ is a real Lie algebra of finite dimension canonically isomorphic to the Lie algebra of the Lie group $\mathsf{Aut}(D)$. An extension of this result to infinite dimension is due to Upmeier [18]: Suppose that U is a bounded domain in the complex Banach space E. Call a domain $V \subset U$ *admissible* if there exist $a \in U$ and $r > 0$ such that $||z - a|| < r$ for all $z \in V$ and $||z - a|| > 5r$ for all $z \notin U$.

2.1 Theorem. *Let $U \subset E$ be a bounded domain. Then $\mathfrak{g} := \mathfrak{aut}(U)$ is a real Lie algebra and for every admissible domain $V \subset U$*

$$||f(z)\, {\partial}/{\partial z}||_V := \sup_{z \in V} ||f(z)||$$

defines a complete norm on $\mathfrak{g}$ making it to a real Banach Lie algebra (that is, the bracket is continuous). For any two admissible domains in U the corresponding norms are equivalent. The group $G := \mathsf{Aut}(U)$ has a unique structure of a real Banach Lie group such that the exponential map is bianalytic in a suitable neighbourhood of the origin in $\mathfrak{g}$. The canonical mapping $G \times U \to U$, $(g,z) \mapsto g(z)$, is real analytic.

We notice that for every bounded domain $U \subset E$ the real Lie algebra $\mathfrak{g} := \mathfrak{aut}(U)$ is totally real in $\mathfrak{hol}(U)$: Indeed, suppose that X is in $\mathfrak{g} \cap i\mathfrak{g}$. Then for every $a \in U$ the holomorphic function $\mathbb{C} \to U$ defined by $t \mapsto \exp(tX)a$ is constant by Liouville, that is, $X = 0$.

The domain $U \subset E$ is called *homogeneous* if the group $\mathsf{Aut}(U)$ acts transitively on U, that is, if $\{g(a) : g \in \mathsf{Aut}(U)\} = U$ for some and hence every $a \in U$. In case U is bounded there always exists a metric on U that is compatible with the topology and is invariant under the action of $\mathsf{Aut}(U)$, for instance the *Carathéodory metric:* Let B be the Banach space of all bounded $\mathbb{C}$-valued holomorphic functions on U and $B^* := \mathcal{L}(B, \mathbb{C})$ its dual. Then $a \mapsto [f \mapsto f(a)]$ defines a continuous embedding $U \hookrightarrow B^*$ and the norm of B^* provides the invariant metric on U. With this metric it can be seen:

2.2 Proposition. *For every bounded domain $U \subset E$ and every $a \in U$ the following conditions are equivalent:*

(i) *U is homogeneous,*
(ii) $\mathsf{Aut}(U)(a)$ *is open in U,*
(iii) *the evaluation map $\mathfrak{aut}(U) \to E$, $f(z)\,{\partial}/{\partial z} \mapsto f(a)$, is surjective.*

3. Operator algebras and generalizations

As a special example let us start with $E := \mathcal{L}(H)$, where H is an arbitrary complex Hilbert space. Then E is a Banach algebra with involution $z \mapsto z^*$. We are interested in holomorphic properties of the open unit ball of E, that we always denote by D in the following. For every $c \in D$ the square roots $(\mathbb{1} - cc^*)^{1/2}$, $(\mathbb{1} - c^*c)^{1/2}$ are well defined as positive hermitian operators on H and

$$g_c(z) := c + (\mathbb{1} - cc^*)^{1/2}(\mathbb{1} - zc^*)^{-1}z(\mathbb{1} - c^*c)^{1/2} \tag{3.1}$$

defines a holomorphic function $g_c : D \to E$. Actually, it is not difficult to see that g_c takes values in D and satisfies $g_c g_{-c} = \mathsf{id}_D$, that is, $g_c \in G := \mathsf{Aut}(D)$ for every $c \in D$. From $g_c(0) = c$ we see that G acts transitively on D, that is, D is homogeneous.

Now consider the *isotropy subgroup* $K := \{g \in G : g(0) = 0\}$. Then every $g \in K$ is linear. Indeed, for every real t the automorphisms $g(e^{it}z)$ and $e^{it}g(z)$ have the same value and the same derivative at the origin and hence coincide by Cartan's uniqueness theorem. Comparing the power series expansions then gives that g is linear. As a consequence, K is just the linear isometry group of the Banach space E. Furthermore, $G = \{g_c k : c \in D, k \in K\}$.

For an abstract generalization later on it is convenient to write (3.1) in a slightly different form: For all $x, y \in E$ define the *Bergman operator* $B(x, y) \in \mathcal{L}(E)$ by

$$B(x, y)z \;=\; (\mathbb{1} - xy^*)z(\mathbb{1} - y^*x) \;=\; z - (xy^*z + zy^*x) + xy^*zy^*x\,. \tag{3.2}$$

For $x, y \in D$ then $B(x,y)$ has spectrum in the right halfplane and therefore $B(x,y)^{1/2} \in \mathcal{L}(E)$ with $B(0,0)^{1/2} = \mathbb{1}$ exists in the sense of the holomorphic functional calculus. Now (3.1) just reads

$$g_c(z) = c + B(c,c)^{1/2}(z + zc^*z + zc^*zc^*z + \ldots)\,. \tag{3.3}$$

Also the Lie algebra $\mathfrak{g} := \mathfrak{aut}(D)$ can be easily described. The isotropy subalgebra

$$\mathfrak{k} := \left\{ f(z)\, \partial\!/\!\partial z \in \mathfrak{g} : f(0) = 0 \right\} \tag{3.4}$$

at the origin is $\mathfrak{k} = \{ i(uz + zv)\, \partial\!/\!\partial z \ : u = u^*, v = v^* \in \mathcal{L}(H) \}$. For every fixed $a \in E$ consider the real 1-parameter family $(g_{ta})_{|t|<1}$ in G, that has derivative $(a - za^*z)$ at $t = 0$. From this it is easily derived that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad \mathfrak{p} := \left\{ (a - za^*z)\, \partial\!/\!\partial z \ : a \in E \right\}. \tag{3.5}$$

From (3.3) as well as (3.5) it is obvious that the above considerations immediately extend to the case were $E \subset \mathcal{L}(H)$ is an arbitrary C*-algebra, that is a closed complex *-invariant subalgebra of $\mathcal{L}(H)$. Actually, only the following is needed for a given closed linear subspace $E \subset \mathcal{L}(H)$: For every $a, z \in E$ also za^*z should be contained in E, or equivalently, E should be invariant under the *Jordan triple product* $\{xyz\} := (xy^*z + zy^*x)/2$. Subspaces with this property were introduced and studied by Harris [8] under the name *J*-algebra* (the defining property there is that E is invariant under the mapping $a \mapsto aa^*a$, a property that is easily seen to be equivalent to the above). For systematic reasons in our notation we prefer to call them JC*-triples in the following.

Examples for JC*-triples are obtained by the closed linear spans in $\mathcal{L}(H)$ of all subsets ABC, where $B \subset \mathcal{L}(H)$ is a C*-subalgebra and $A, C \subset B$ are arbitrary. Further examples are given by the JC*-algebras - these are closed *-invariant linear subspaces of $\mathcal{L}(H)$ that are invariant under the *Jordan product* $x \circ y := (xy + yx)/2$. Indeed, this follows from $zc^*z = 2z \circ (z \circ c^*) - (z \circ z) \circ c^*$.

C*-algebras, JC*-algebras as well as JC*-triples have their abstract analogs (not referring to a Hilbert space H). By definition, a *B*-algebra* is a complex Banach algebra A with (conjugate linear isometric) involution * satisfying $||aa^*|| = ||a||^2$ for all $a \in A$. Every C*-algebra is a B*-algebra, and on the other hand, by the Gelfand-Naimark theorem every B*-algebra is isometrically *-isomorphic to some C*-algebra, see [17] for details.

Abstract Jordan C*-algebras go back to Kaplansky, see [20]. Let J be a complex Banach Jordan algebra, that is, a complex Banach space with commutative bilinear product $x \circ y$ satisfying $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ as well as $||x \circ y|| \le ||x|| \cdot ||y||$, and suppose that on J is given a (conjugate linear) isometric algebra involution $x \mapsto x^*$. Then J is called a *JB*-algebra* if $||\{xxx\}|| = ||x||^3$ for every $x \in J$, where the 'triple product' $\{xyz\}$ on J is given by

$$\{xyz\} := x \circ (z \circ y^*) + z \circ (x \circ y^*) - (x \circ z) \circ y^* . \tag{3.6}$$

Every JC*-algebra is a JB*-algebra with respect to the Jordan product $x \circ y = (xy+yx)/2$, but not every JB*-algebra can be obtained this way. The most prominent counter-example can be described as follows: Let $\mathbb{O}$ be the standard real Cayley algebra, that is the unique (non-associative) real division algebra of dimension 8. Then $\mathbb{O}$ comes with an algebra involution $x \mapsto \overline{x}$ whose fixed point set is a subfield isomorphic to $\mathbb{R}$. Let $V := \mathcal{H}_3(\mathbb{O})$ be the space of all hermitian $3\times 3-$matrices over $\mathbb{O}$, which obviously is a real vector space of dimension 27. With respect to $x \circ y = (xy+yx)/2$ the space V becomes a real Jordan algebra with the unit matrix $\mathbb{1}$ as identity. Now the formal complexification $J := V^{\mathbb{C}} := V \oplus iV$ is a complex Jordan algebra by extending the Jordan product in a complex bilinear way. Also, $(x+iy)^* := x - iy$ defines an algebra involution on J. As in every unital Jordan algebra, to every $z \in J$ there exists a (commutative) associative subalgebra of J containing z and $\mathbb{1}$. This implies that all powers of z and hence also $\exp(z) \in J$ are defined (usual exponential power series). There exists a unique norm on J such that the corresponding closed unit ball is the convex hull of the 'generalized unit circle' $\exp(iV)$. With respect to this norm the Jordan *-algebra J is a JB*-algebra not isomorphic to any JC*-algebra (J is called an *exceptional* JB*-algebra).

The open unit ball D of every JB*-algebra also is homogeneous. We do not go into details here since there also exists an abstract analog to JC*-triples that encloses the JB*-algebras. These are the JB*-triples that will be discussed in the next section.

4. JB*-Triples

4.1 Definition. A *JB*-triple* is a complex Banach space E together with a sesquilinear mapping

$$L : E \times E \to \mathcal{L}(E)$$

such that for all $a, b, x, y, z \in E$ the following conditions hold:

(i) $\{xyz\} := L(x,y)z$ is symmetric in the outer variables (x, z),

(ii) $[L(a,b), L(x,y)] = L(\{abx\}, y) - L(x, L(\{yab\}))$, where $[\,,\,]$ is the usual commutator of operators,

(iii) $L(a,a) \in \mathcal{L}(E)$ is hermitian, that is, $||\exp(itL(a,a))|| = 1$ for all real t,

(iv) $L(a,a)$ has spectrum ≥ 0,

(v) $||L(a,a)|| = ||a||^2$.

Here *sesquilinear* always means *'complex linear in the first and conjugate linear in the second variable'.*

The first two conditions in 4.1 are purely algebraic. Condition (ii) is called the *Jordan triple identity* and is equivalent to $\exp(iL(a,a)) \in \mathsf{GL}(E)$ being an automorphism of the triple product $\{\ ,\ ,\ ,\}$ for every $a \in E$. Condition (iv) only depends on the linear topology of E while (iii) and (v) depend heavily on the norm structure of E. The JB*-triples form in a natural way a category $\mathfrak{JB}^*$: A linear map $\varphi : E \to F$ between JB*-triples is a morphism if $\varphi(\{xyz\}) = \{(\varphi x)(\varphi y)(\varphi z)\}$ holds for all $x, y, z \in E$. It is known that every triple morphism $\varphi : E \to F$ has closed image in F and induces a linear isometry from $E/\ker(\varphi)$ onto $\varphi(E)$. The notion of a JB*-subtriple is obvious and also the notion of an ideal in E (a linear subspace I with $\{IEE\} \subset I$ and $\{EIE\} \subset I\}$). For every closed ideal $I \subset E$ the quotient E/I in a canonical way is also a JB*-triple. We call a JB*-triple *simple* if it does not contain any non-trivial closed ideal.

It is easy to see that every JC*-triple is a JB*-triple with respect to the triple product $\{xyz\} = (xy^*z + zy^*x)/2$. Also every JB*-algebra is a JB*-triple with respect to (3.6). In case the JB*-triple E contains an element e with $L(e,e) = \mathsf{id}_E$ then E becomes a JB*-algebra with unit e in terms of the product $x \circ y := \{xey\}$ and the involution $x^* := \{exe\}$. On the other hand, every unital JB*-algebra is obtained this way. A simple JB*-triple that is neither a JC*-triple nor (triple-isomorphic to) a JB*-algebra is obtained as follows: Consider in $V = \mathcal{H}_3(\mathbb{O})$ the subspace W of all 3×3-matrices $x = (x_{jk})$ with $x_{11} = x_{22} = x_{33} = x_{23} = 0$. Then $W^{\mathbb{C}} \subset \mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$ is a JB*-subtriple of dimension 16 with the required properties. On the other hand, locally every JB*-triple 'looks like a commutative C*-algebra', more precisely, to every $a \in E$ there exists a JB*-subtriple $E_a \subset E$ that is (triple) isomorphic to a commutative C*-algebra and contains a. A global result states that every JB*-triple can be realized as JB*-subtriple of a JB*-algebra, actually the following Gelfand-Naimark theorem for JB*-triples is much more precise [7]:

4.2 Theorem. *Every JB*-triple is isomorphic to a JB*-subtriple of* $\mathcal{L}(H) \oplus^\infty \mathcal{C}\big(S, \mathcal{H}_3(\mathbb{O})^{\mathbb{C}}\big)$, *where H is a suitable complex Hilbert space, S is a suitable compact topological space, $\mathcal{C}(S,J)$ is the JB*-algebra of all continuous functions on S with values in the exceptional Jordan algebra $J = \mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$ of dimension 27 and $\oplus^\infty$ means the ℓ^∞-sum.*

Notice that in 4.2 the summand $\mathcal{L}(H)$ as an associative structure is algebraically easy but complicated in the functional analytic sense. On the other hand, the second summand is algebraically complicated but is topologically easy since $\mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$ has finite dimension.

As usual, we denote by $F^* := \mathcal{L}(F, \mathbb{C})$ the *dual* of the Banach space F. For every JB*-triple E the triple product extends in a unique way to a separately w^*-continuous triple product on the bidual E^{**} making it to a JB*-triple as well. Clearly, the JB*-triple E^{**} is the dual of another complex Banach space (for instance of E^*). In analogy to the W*-algebras in the category of C*-algebras the JB*-triple E is called a *JBW*-triple* if E is (isometrically isomorphic to) the dual of another complex Banach space, which then is called a *predual* of E. Every predual of the JBW*-triple E can be canonically considered as a subspace of E^* which can be shown to be uniquely determined by E. This space is denoted by E_*.

It is easy to see that for every JB*-triple E the corresponding triple product is continuous. Actually it can be shown that always $||L(x,y)|| \le ||x||\cdot||y||$ holds (for JC*-triples this is obvious contrary to the general situation). Also it is clear that the triple product determines the topology and even the norm on E. Indeed, the closed unit ball of E consists of all $a \in E$ such that for every real $t > 1$ the linear operator $\big(t\mathbb{1} - L(a,a)\big)$ is bijective on E. Conversely, the holomorphic structure of the open unit ball of the JB*-triple E - and hence the norm of E - determines the triple product on E uniquely. This will be discussed in more detail in the next section. As a consequence we may consider the category $\mathfrak{JB}^*$ of all JB*-triples as a subcategory of the category of *all* complex Banach spaces (with linear contractions as morphisms).

In contrast to the triple product on JB*-triples the algebra product of C*- or JB*-algebras is not uniquely determined by the norm of the underlying Banach space. For unital JB*-algebras it is uniquely determined by the norm together with the unit element. On every JBW*-algebra and also on every JB*-algebra coming from a C*-algebra any two unital JB*-algebra structures are at least isomorphic, see Lemma (5.2) in [1]. An example of a JB*-triple carrying several non-isomorphic unital JB*-algebra structures is obtained as follows [1]:
Let $A \subset \mathbb{C}^{2\times 2}$ be the JC*-subalgebra of all symmetric complex 2×2-matrices with Jordan product $x \circ y = (xy + yx)/2$ and let $S := e^{i\mathbb{R}} \subset \mathbb{C}$ be the unit circle. Then also $E := \mathcal{C}(S, A)$ is a JB*-algebra with product $\circ$ and involution * defined pointwise. Define the function $e \in E$ by

$$e(s) := \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all} \quad s \in S \ .$$

Then $L(e,e) = \mathsf{id}_E$ and hence the new product $x \,\square\, y := (xe^*y + ye^*x)/2$ and the new involution $x^\star := ex^*e$ give a new JB*-algebra structure on E not isomorphic to the original one.

The category $\mathfrak{JB}^*$ is closed under taking arbitrary ℓ^∞-sums, ultra-powers, biduals as well as taking images of arbitrary contractive projections. A word of caution is necessary for the last statement: Suppose that $P : E \to E$ is a contractive projection (that is, $P^2 = P$ and $||P|| \le 1$) with image $F := P(E)$. Then F may not be a subtriple of E, but it is a JB*-triple in the triple product

$$\{xyz\}_F := P\{xyz\}\,, \qquad x, y, z \in F \ .$$

A simple example where $F \subset E$ is not a subtriple is as follows: For $S := [0,1] \subset \mathbb{R}$ let $E := \mathcal{C}(S)$ be the commutative C*-algebra of all continuous complex-valued functions on S and P the contractive projection on E defined by $(Pf)(s) := (1-s)f(0) + sf(1)$ for all $f \in E$ and $s \in S$.

The open unit ball D of every JB*-triple is homogeneous with respect to biholomorphic automorphisms. As in (3.3) for every $c \in D$ an automorphism $g_c \in \mathsf{Aut}(D)$ with $g_c(0) = c$ is obtained, where the Bergman operator in the triple product reads

$$B(x,y)z = z - 2\{xyz\} + \{x\{yzy\}x\}\,. \tag{4.3}$$

Also (3.5) holds with $\mathfrak{p} = \big\{(a - \{zaz\})\,{}^{\partial}\!/_{\partial z} \,:\, a \in E\big\}$ for $\mathfrak{g} := \mathfrak{aut}(D)$ and the isotropy subalgebra $\mathfrak{k}$ at the origin, see (3.4).

Summarizing we have that every JB*-triple has a (holomorphically) homogeneous open unit ball. Actually, also the converse is true, that is, a complex Banach space is a JB*-triple if and only its open unit ball is homogeneous. In the next section we discuss this in more detail.

5. Bounded symmetric domains

In the following let F be a complex Banach space and $U \subset F$ a bounded domain. Then U is called *symmetric* if to every $a \in U$ there exists an $s \in \mathsf{Aut}(U)$ such that

(i) $s^2 = \mathsf{id}_U$,

(ii) a is an isolated fixed point of s.

It is not difficult to see that condition (ii) can be replaced by $s(a) = a$ and $ds_a = -\mathsf{id}_E$. By Cartan's uniqueness theorem 1.1 the *symmetry s about a* is uniquely determined by a and will always be denoted by s_a. The mapping $U \to \mathsf{Aut}(U)$, $a \mapsto s_a$, is analytic and the subset $\{s_a s_b : a, b \in U\}$ generates a connected subgroup of $\mathsf{Aut}(U)$ acting transitively on U. Therefore every bounded symmetric domain is homogeneous. As an example, the open unit ball D of every JB*-triple E is symmetric. Indeed, $s_0(z) = -z$ and $s_a = g_a s_0 g_{-a}$ for every $a \in D$.

In finite dimensions all bounded symmetric domains have been classified (up to biholomorphic equivalence) by É. Cartan in 1935. The irreducible among these come in 4 series and 2 exceptional domains (the open unit balls of the two exceptional JB*-triples in dimensions 16 and 27 as considered before). A typical example is the matrix domain

$$D := \{z \in \mathbb{C}^{p\times q} : (\mathbb{1} - zz^*) \text{ positive definite}\}\,,$$

where z^* is the conjugate transpose of the $p\times q$-matrix z.

5.1 Riemann mapping theorem for bounded symmetric domains. *Every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple.*

In the following we sketch how for a given bounded symmetric domain $U \subset F$ the corresponding JB*-triple E can be obtained: First we fix an arbitrary base point $o \in U$. Now E as a Banach space is obtained by renorming F in the following way: For every $v \in F$ let

$$||v||_E := \sup\{|df_o(v)| : f \in \mathcal{O}(U) \ \text{ with } \ f(o) = 0\,,\ |f| \leq 1\}\ ,$$

where $\mathcal{O}(U)$ is the space of all $\mathbb{C}$-valued holomorphic functions on U. Then $||\ ||_E$ is an equivalent norm on F, called the *Carathéodory norm* with respect to $o \in U$. For every g in the isotropy subgroup $K := \{g \in \mathsf{Aut}(U) : g(o) = o\}$ the derivative $dg_o \in \mathsf{GL}(E)$ is an isometry. The symmetry $s = s_o$ about the base point $o \in U$ is in the center of K by Cartan's uniqueness theorem and acts in a canonical way by $\mathsf{Ad}(s)$ on the Lie algebra $\mathfrak{g} := \mathfrak{aut}(U)$. From $s^2 = \mathsf{id}_U$ we get a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into a $(+1)$- and a (-1)-eigenspace. $\mathfrak{k}$ is the Lie subalgebra of $\mathfrak{g}$ corresponding to the Banach Lie subgroup $K \subset \mathsf{Aut}(U)$. Furthermore the inclusions $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ hold. In particular, $\mathfrak{p}$ is invariant under the *Lie triple product* $[xyz] := [[x, y], z]$. The canonical evaluation map $\mathfrak{g} \to E$, $f(z)\,{}^{\partial}\!/\!_{\partial z} \mapsto f(o)$, is surjective by 2.2 and hence gives a canonical $\mathbb{R}$-linear isomorphism $\mathfrak{p} \approx E$. Identifying both spaces this way we have a Lie triple product $[xyz]$ on E that is trilinear over $\mathbb{R}$ and complex linear in z. Now define $\{xyz\}$ as half the part of $[xyz]$ that is complex linear in x, more precisely,

$$\{xyz\} := \frac{1}{4}\big([xyz] - i[(ix)yz]\big)\,.$$

Using the power series expansion of every vector field in $\mathfrak{p} \approx E$, Cartan's uniqueness theorem 1.1 and $\mathfrak{g} \cap i\mathfrak{g} = 0$ it is shown that the conditions (i) and (ii) in 4.1 are satisfied. For every $a \in E$ and $t \in \mathbb{R}$ the operator $itL(a, a)$ is the derivative of the automorphism $\exp(t[a, ia]) \in K$ (notice that after the identification $\mathfrak{p} \approx E$ we have the bracket $[\,,\,] : E \times E \to \mathfrak{k}$). This implies condition (iii). The proof of conditions (iv), (v) in 4.1 and of 5.1 is more involved, see [11] for details.

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