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On weighted parallel volumes

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Abstract. The Wills functional of a convex body was originally defined as the sum of its intrinsic volumes. Meanwhile, various integral representations of the Wills functional have been introduced. In this paper we will introduce and examine the weighted parallel volumes as a class of functionals generalising the integral representations of the Wills functional. We will discuss to which extend the weighted parallel volumes are the linear combinations of intrinsic volumes and vice versa. The weighted parallel volumes can be considered as functionals defined on the set of all compact sets. We will study their properties and characterise the weighted parallel volumes which are continuous, additive resp. submodular. We will obtain most of our results in unsymmetric Minkowski spaces. Finally we apply some of our results to the capacity functional of Boolean models.

1 Introduction

Let $\mathcal{C}$ denote the set of all compact subsets of $\mathbb{R}^d$, $\mathcal{C}'$ the set of all bodies, i.e. non-empty compact subsets of $\mathbb{R}^d$, and $\mathcal{K}$ the set of all convex bodies. The dimension d will always be clear from the context.

Let

$$d(K,x) := \inf\{d(y,x)|y \in K\},\ K \in \mathcal{C}',\, x \in \mathbb{R}^d,$$

where $d(y,x)$ denotes the Euclidean distance between two points $x, y \in \mathbb{R}^d$. Recall that

$$K + L := \{x + y|x \in K,\, y \in L\}, \qquad rK := \{rx|x \in K\}$$

for $K, L \subseteq \mathbb{R}^d$ and r in the set of non-negative real numbers $\mathbb{R}_0^+$. Clearly we have

$$K + rB^d = \{x \in \mathbb{R}^d|d(K,x) \leq r\},\ K \in \mathcal{C}',\, r \in \mathbb{R}_0^+,$$

where B^d denotes the d-dimensional unit ball. This set is called the *parallel body* of K at distance r. Moreover, the *parallel volume* $V_d(K + rB^d)$, where V_d denotes the d-dimensional Lebesgue measure, is known to be a polynomial in r for $K \in \mathcal{K}$. This can be used in order to define the intrinsic volumes $V_0(K), \ldots, V_d(K)$ of $K \in \mathcal{K}$ by

$$V_d(K + rB^d) = \sum_{j=0}^{d} \kappa_{d-j} r^{d-j} V_j(K) \tag{1}$$

for all $r \in \mathbb{R}_0^+$, where κ_j denotes the volume of the j-dimensional unit ball. For further information, see [10].

Wills [16] investigated bounds for the number of lattice points, i.e. points with only integer coordinates, contained in a convex body K and conjectured that an upper bound is given by

$$\sum_{j=0}^{d} V_j(K).$$

Hadwiger [5] proved that this functional equals

$$\int_{\mathbb{R}^d} e^{-\pi d(K,x)^2}\, dx$$

and called it the Wills functional. A probabilistic representation is

$$\mathbb{E}\, V_d(K + \Lambda B^d), \tag{2}$$

where Λ is a random variable with distribution function $1 - e^{-\pi t^2}$, $t \geq 0$. A representation quite similar was already obtained in [5], but Vitale [15] was the first to use probabilistic notions in this context.

Hadwiger [6] showed that Wills' conjecture is wrong. However, the Wills functional turned out to have various applications, e.g. in the geometry of numbers [17] or in deriving exponential moment inequalities for Gaussian random processes [15].

The aim of the present paper is to study generalisation of the Wills functional. In Section 2 we will examine the relation between generalisations of the three representations of the Wills functional. We will show that a functional on $\mathcal{K}$ is a linear combination of intrinsic volumes, iff it allows representations of the form

$$\int_{\mathbb{R}^d} G(d(K,x))\, dx, \tag{3}$$

where $G : \mathbb{R}_0^+ \to \mathbb{R}$ is a function fulfilling some weak regularity conditions, and that this is equivalent to having a representation of the form

$$\int_{\mathbb{R}_0^+} V_d(K + \lambda B^d)\, d\rho(\lambda), \tag{4}$$

where ρ is a signed measure on $\mathbb{R}_0^+$ fulfilling some weak integrability conditions. In fact we will derive these results not only in Euclidean space, but also in unsymmetric Minkowski spaces (i.e. finite-dimensional linear spaces with a not necessarily symmetric norm).

The original definition of the Wills functional does only make sense for convex bodies, whereas in the other two representations and their generalisations we can consider arbitrary bodies. We will show that the generalisations of Hadwiger's and the probabilistic representations are equivalent. Therefore it suffices to consider generalisations of the probabilistic representation, which we shall call *weighted parallel volumes.*

Considering the results mentioned so far, which are obtained in an arbitrary, but fixed dimension, the Wills functional does not play a special role. However, it can be characterized by a dimension invariance property.

Section 3 is devoted to the further investigation of weighted parallel volumes as functionals from $\mathcal{C}$ to $\mathbb{R}$. First we shall examine translation invariance, continuity and additivity. Here we will be able to derive characterisation results even for the more general functionals $f_\mu(K) := \int_{\mathcal{K}} V_d(K + A)\, d\mu(A)$, where μ is a signed measure on $\mathcal{K}$, saying that these functionals are

- always translation invariant,
- continuous, iff we can neglect the contribution of lower-dimensional bodies to μ in a certain sense, and
- additive, iff they are multiples of Lebesgue measure.

After this, we will show that f_μ is submodular, if μ is a measure, and proof that a weighted parallel volume is submodular, iff the restriction of the signed measure to the positive real numbers is a measure. Then we derive some necessary and some sufficient conditions for weighted parallel volumes to be monotone. We will point out a connection to the Kneser-Poulson-Conjecture. Finally we show that a signed measure μ is essentially determined by f_μ.

The results of this paper are of potential interest in stochastic geometry and spatial statistics. As a first application of our results we examine the capacity functional of Boolean models in Section 4. Moreover, Meschenmoser and Spodarev [8] will use these results in order to estimate intrinsic volumes.

2 The three representations

In this section we consider three classes of functionals, namely the linear combinations of intrinsic volumes, and the functionals given by (3) and (4). We examine whether a functional which is contained in one of these classes is also contained in the other two classes. There will be a positive answer in any fixed dimension. However we can characterise the Wills functional among these generalisations by a dimension invariance property.

In the first part of this section, there is no need to restrict to Euclidean space. We will replace the unit ball by a convex body $B \subseteq \mathbb{R}^d$. It is well-known that $V_d(K + rB)$ is a polynomial in r. Thus we can define the *mixed volumes* $V(K[d-j], B[j])$, $j = 0, \dots, d$, $K \in \mathcal{K}$, by

$$V_d(K + rB) = \sum_{j=0}^{d} r^j \binom{d}{j} V(K[d-j], B[j]) \tag{5}$$

for all $r \geq 0$. For further information on mixed volumes, see [10, section 5.1].

Now we introduce the notion of a signed measure. A *signed measure* (of finite total variation) on a measure space $(\Omega, \mathcal{A})$ is a σ-additive function from $\mathcal{A}$ to $\mathbb{R}$. So a signed measure is a measure, iff it asigns a non-negative value to each set of $\mathcal{A}$. It is well-known that a signed measure μ has a unique decomposition $\mu = \mu^+ - \mu^-$ with measures μ^+ and μ^- that are singular with respect to each other, the so-called *Jordan-decomposition.* For a measurable map $f : \Omega \to \mathbb{R}$ the integral with respect to μ is defined by

$$\int f \, d\mu := \int f \, d\mu^+ - \int f \, d\mu^-,$$

whenever both integrals are defined and at least one of them is finite. Moreover

$$|\mu| := \mu^+ + \mu^-$$

is the variation measure of μ.

Remark 1. *If μ is a signed measure on Ω and $\Omega' \subseteq \Omega$ is measurable, then we denote the restriction of μ to Ω' by $\mu_{|\Omega'}$. It is easy to see that $(\mu_{|\Omega'})^+ = (\mu^+)_{|\Omega'}$ and $(\mu_{|\Omega'})^- = (\mu^-)_{|\Omega'}$. Hence $|\mu_{|\Omega'}| = |\mu|_{|\Omega'}$.*

If $\Omega \subseteq \mathbb{R}$, then $\int x^j \, d\mu(x)$ is called the j-th moment of μ. For more information on signed measures see e.g. [3, §4].

Now we consider the relationship between linear combinations of mixed volumes and certain functionals that allow integral representations.

Proposition 2. *Let ρ be a signed measure on $\mathbb{R}_0^+$ such that the j-th moment μ_j is finite for $j = 0, \dots, d$. Then we have*

$$\int_{\mathbb{R}_0^+} V_d(K + \lambda B) \, d\rho(\lambda) = \sum_{j=0}^{d} \binom{d}{j} \mu_j V(K[d-j], B[j]), \;\; K \in \mathcal{K}. \tag{6}$$

On the other hand, for arbitrary constants $\alpha_0, \dots, \alpha_d$ there is a signed measure ρ on $\mathbb{R}_0^+$ such that

$$\int_{\mathbb{R}_0^+} V_d(K + \lambda B) \, d\rho(\lambda) = \sum_{j=0}^{d} \alpha_j V(K[d-j], B[j]), \;\; K \in \mathcal{K}. \tag{7}$$

Proof. From (5) we conclude

$$\int_{\mathbb{R}_0^+} V_d(K+\lambda B) \, d\rho(\lambda) = \sum_{j=0}^{d} \binom{d}{j} V(K[d-j], B[j]) \int_{\mathbb{R}_0^+} \lambda^j \, d\rho(\lambda) = \sum_{j=0}^{d} \binom{d}{j} \mu_j V(K[d-j], B[j]),$$

which shows the first statement.

To prove the converse statement, let

$$G : \mathcal{K} \to \mathbb{R},\ K \mapsto \sum_{j=0}^{d} \alpha_j V(K[d-j], B[j])$$

and choose $r_1, \ldots, r_{d+1} \in \mathbb{R}_0^+$ pairwise different. Then by (5) and the well-known fact that the Vandermonde-matrix (r_i^j), where ^{j} denotes the j-th power, is regular, it is easy to see that there are $c_1, \ldots, c_{d+1}$ such that $G(K) = \sum_{i=1}^{d+1} c_i V(K + r_i B)$ for all $K \in \mathcal{K}$. Now put $\rho := \sum_{i=1}^{d+1} c_i \delta_{r_i}$, where δ_r denotes the Dirac measure in r. Then $G(K) = \int V_d(K + \lambda B)\, d\rho(\lambda)$. $\square$

Whereas the right-hand side of (6) is defined only for convex bodies, its left-hand side is defined for arbitrary compact sets. So for a signed measure ρ on $\mathbb{R}_0^+$, for which the 0-th to d-th moments are finite, we consider the functional

$$\mathcal{C} \to \mathbb{R},\ K \mapsto \int_{\mathbb{R}_0^+} V_d(K + rB)\, d\rho(r)$$

and call it the *ρ-weighted B-parallel volume*. (We will omit the ρ and the B if they are clear from the context or in statements that hold for all ρ resp. B.) In fact the requirement that the 0-th to d-th moments of ρ are finite ensures the integral $\int_{\mathbb{R}_0^+} V_d(K + rB)\, d\rho(r)$ to be finite: Since every body is contained in a ball and the volume is monotone, this requirement implies that both $\int_{\mathbb{R}_0^+} V_d(K + rB)\, d\rho^+(r)$ and $\int_{\mathbb{R}_0^+} V_d(K + rB)\, d\rho^-(r)$ are finite and thus $\int_{\mathbb{R}_0^+} V_d(K + rB)\, d\rho(r)$ exists and is finite.

Now we start to prepare a statement generalising the equivalence of Hadwiger's and the probabilistic representation of the Wills functional.

The total variation of a function $f : \mathbb{R}_0^+ \to \mathbb{R}$ is

$$\sup\{\sum_{i=1}^{N-1} |f(s_{i+1}) - f(s_i)|\, |s_1, \ldots, s_N \in \mathbb{R}_0^+,\ s_1 < s_2 < \cdots < s_N,\ N \in \mathbb{N}\} \in \mathbb{R} \cup \{\infty\}.$$

For a signed measure μ we consider the function G on $\mathbb{R}_0^+$ defined by

$$G(r) := \mu([r, \infty)),\ r \in \mathbb{R}_0^+. \tag{8}$$

It is obviously left-continuous, has finite total variation and fulfills $\lim_{r\to\infty} G(r) = 0$. On the other hand for each function G having these properties there is a unique signed measure μ such that (8) is fulfilled (see e.g. [3, Prop. 4.4.3]). It is easy to see, that the j-th moment of ρ exists and is finite, iff $\int r^{j-1} G(r)\, dr$ exists and is finite.

From now on we assume that the origin 0 lies in the interior of B. Then we can define

$$d_B(K, x) := \inf\{r \geq 0 | x \in K + rB\}$$

for an arbitrary body $K \in \mathcal{C}'$ and a point $x \in \mathbb{R}^d$. So we have

$$x \in K + rB \iff d_B(K, x) \leq r$$

for all $K \in \mathcal{C}'$, $x \in \mathbb{R}^d$ and $r \geq 0$.

Proposition 3. *Let ρ be a signed measure on $\mathbb{R}_0^+$ for which the k-th moment is finite, $k = 0, \dots, d$, and let $G : \mathbb{R}_0^+ \to \mathbb{R}$ be the function from (8) with ρ instead of μ. Then we have*

$$\int_{\mathbb{R}_0^+} V_d(K + \lambda B)\, d\rho(\lambda) = \int_{\mathbb{R}^d} G(d_B(K, x))\, dx, \;\; K \in \mathcal{C}'.$$

Proof. The proof is based on Fubini's theorem. Indeed with help of the Jordan-decomposition one can easily show that Fubini's theorem holds for signed measures, where the integrability conditions have now to be satisfied with respect to the variation measure. (In the present proof they will obviously be fulfilled.)

Let $K \in \mathcal{C}'$. Then

$$\begin{aligned}
\int_{\mathbb{R}_0^+} V_d(K + rB)\, d\rho(r) &= \int_{\mathbb{R}_0^+} \int_{\mathbb{R}^d} \mathbf{1}_{K+rB}(x)\, dx\, d\rho(r) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}_0^+} \mathbf{1}_{\{d_B(K,x) \le r\}}\, d\rho(r)\, dx \\
&= \int_{\mathbb{R}^d} \rho([d_B(K, x), \infty))\, dx \\
&= \int_{\mathbb{R}^d} G(d_B(K, x))\, dx. \qquad \square
\end{aligned}$$

So there is a universal one-to-one relation between the generalisations of Hadwiger's and the probabilistic representation. The functionals connected by this relation coincide for all bodies and the only difference is that the probabilistic representation assigns 0 to the empty set, whereas Hadwiger's representation is not defined for it. So there is no need to consider the generalisations of Hadwiger's representation anymore.

However we still have to distinguish between the generalisations of Wills' representation - the linear combinations of mixed volumes - and the generalisations of the probabilistic representation - which we called weighted parallel volumes. The linear combinations of mixed volumes are only defined for convex bodies, whereas the weighted parallel volumes are defined for all compact sets. Therefore it is not surprising that the relation is not one-to-one. Moreover, the relation involves the dimension d of the surrounding space. However considering the probabilistic representation of the Wills functional itself, we see that Hadwiger managed to find a universal measure ρ such that the ρ-weighted B^d-parallel volume represents the Wills functional in all dimensions. So the question arises whether the lack of dependence on the dimension can be overcome in a more general situation.

We will only treat this problem in the Euclidean case ($B = B^d$), since there is no canonical way of making this problem precise for general gauge bodies. A functional defined on subsets of $\mathbb{R}^d$ for every d is called *dimension invariant*, if for any $n > m$ and any isometrical embedding of $\mathbb{R}^m$ into $\mathbb{R}^n$, the functional on the subsets of $\mathbb{R}^m$ is the restriction of the functional on the subsets of $\mathbb{R}^n$. For example, the intrinsic volumes are dimension invariant.

We let W denote the probabilistic representation (2) of the Wills functional defined on $\mathcal{C}$.

Theorem 4. *(i) The functional W is dimension invariant.*

(ii) Let ρ be a signed measure for which all moments are finite. Suppose that there is an $m \in \mathbb{N}$ with

$$\int_{\mathbb{R}_0^+} V_m(K + \lambda B^m)\, d\rho(\lambda) = \int_{\mathbb{R}_0^+} V_n(K + \lambda B^n)\, d\rho(\lambda) \tag{9}$$

for all convex bodies $K \subseteq \mathbb{R}^m$ and all $n > m$. Then there is a constant c such that

$$\int_{\mathbb{R}_0^+} V_d(L + \lambda B^d)\, d\rho(\lambda) = c \cdot W(L) \tag{10}$$

for all convex bodies $L \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$.

(iii) If the signed measure ρ from (ii) is a measure, then it is up to a constant factor c the measure from the probabilistic representation of the Wills functional. In particular

$$\int_{\mathbb{R}_0^+} V_d(K + \lambda B^d)\, d\rho(\lambda) = c \cdot W(K) \tag{11}$$

for all compact sets $K \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$.

Before we come to the proof, we remark that we do not need to assume equation (9) for all convex bodies $K \subseteq \mathbb{R}^m$, but it suffices to assume it for convex bodies $K_1, \dots, K_N \subseteq \mathbb{R}^m$ such that the vector $(V_m(K_1), \dots, V_m(K_N))$ is not a linear combination of the vectors $(V_j(K_1), \dots, V_j(K_N))$, $j = 0, \dots, m-1$.
We like to show that there are really convex bodies having this property.

Example 5. Choose $t_1, \dots, t_N \in \mathbb{R}_0^+$ pairwise different and put $K_i := t_i B^m$, $i = 1, \dots, N$. Then

$$(V_j(K_1), \dots, V_j(K_N)) = V_j(B^d)(t_1^j, \dots, t_N^j), \quad j = 0, \dots, m.$$

Since a Vandermonde-matrix is regular unless it has two identical rows, these vectors are linearly independent, if $N \geq m+1$.

Proof of Theorem 4. (i) This is equivalent to saying that Hadwiger's representation of the Wills functional is dimension invariant and thus is already proven by Hadwiger [5, (2.1)] using Fubini's theorem. Hadwiger states the dimension invariance only on $\mathcal{K}$, but his proof works on $\mathcal{C}'$.
(ii) Let $K \subseteq \mathbb{R}^m$ be a convex body. Then, using the Steiner formula (1), one can rewrite (9) as

$$\sum_{j=0}^{m} \int_{\mathbb{R}_0^+} \lambda^{m-j}\, d\rho(\lambda) \cdot \kappa_{m-j} V_j(K) = \sum_{j=0}^{n} \int_{\mathbb{R}_0^+} \lambda^{n-j}\, d\rho(\lambda) \cdot \kappa_{n-j} V_j(K).$$

Since this holds for all convex bodies K, by Example 5 we conclude

$$\int_{\mathbb{R}_0^+} \lambda^{m-j}\, d\rho(\lambda) \cdot \kappa_{m-j} = \int_{\mathbb{R}_0^+} \lambda^{n-j}\, d\rho(\lambda) \cdot \kappa_{n-j}, \quad j = 0, \dots, m.$$

For an arbitrary number $k \in \mathbb{N}$ this gives, if we put $j = m$ and $n = k + m = k + j$

$$c := \rho(\mathbb{R}_0^+) = \int_{\mathbb{R}_0^+} \lambda^k \, d\rho(\lambda) \cdot \kappa_k. \tag{12}$$

So for a convex body $L \subseteq \mathbb{R}^d, d \in \mathbb{N}$, we have

$$\int_{\mathbb{R}_0^+} V_d(L + \lambda B^d) \, d\rho(\lambda) = \sum_{j=0}^{d} \int_{\mathbb{R}_0^+} \lambda^{d-j} d\rho(\lambda) \kappa_{d-j} V_j(L) = \sum_{j=0}^{d} c \cdot V_j(L) = c \cdot W(L).$$

(iii) Let ρ_0 denote the probability measure with distribution function $1 - e^{-\pi t^2}$. If $\rho = c \cdot \rho_0$, then equation (12) is fulfilled. So in order to conclude $\rho = c \cdot \rho_0$ from (12), it suffices to show that there is at most one measure ρ fulfilling (12). By [12, p. 20] for a sequence $(\mu_k)_{k \in \mathbb{N}_0}$ with $\limsup_{k \to \infty} \frac{1}{k} \sqrt[2k]{\mu_k} < \infty$, there is at most one measure such that its k-th moment equals μ_k for all $k \in \mathbb{N}_0$. Since

$$\kappa_k = \frac{\pi^{k/2}}{\Gamma(\frac{k+2}{2})} > \frac{1}{k^k}$$

for $k \in \mathbb{N}$, we have

$$\limsup_{k \to \infty} \frac{1}{k} \sqrt[2k]{\frac{c}{\kappa_k}} \leq \limsup_{k \to \infty} \frac{1}{k} \sqrt[2k]{c \cdot k^k} = \limsup_{k \to \infty} \sqrt[2k]{c} \frac{\sqrt{k}}{k} = 0.$$

So the measure fulfilling (12) is unique and hence $\rho = c \cdot \rho_0$.

The second assertion is an immediate consequence. □

3 Properties of the weighted parallel volume

In this section we will characterise the signed measures whose weighted B-parallel volumes are translation invariant, continuous, additive resp. submodular, where the gauge body B is a convex body, sometimes with interior points. After this we will derive some results on signed measures whose weighted Euclidean parallel volumes are monotone. Finally we will show that the ρ-weighted B-parallel volume determines the signed measure ρ uniquely.

We will not just prove the desired characterisation results for signed measure whose weighted parallel volumes are translation invariant, continuous resp. additive, but we will show more general statements and then get the desired results as corollaries.

In the following we always consider $\mathcal{K}$ with the Fell-Matheron-σ-field, which is the Borel-σ-field of the Hausdorff-topology as well as the Borel-σ-field of the topology of closed convergence (see e.g. [11, p. 20 f.]). For a signed measure μ on $\mathcal{K}$ with

$$\int_{\mathcal{K}} V_d(K + A) \, d|\mu|(A) < \infty, \ K \in \mathcal{C}, \tag{13}$$

we define

$$f_\mu : \mathcal{C} \to \mathbb{R},\ K \mapsto \int_{\mathcal{K}} V_d(K + A)\, d\mu(A). \tag{14}$$

A functional $\phi : \mathcal{C} \to \mathbb{R}$ is called *translation invariant*, if

$$\phi(K + x) = \phi(K)$$

for all $K \in \mathcal{C}$, $x \in \mathbb{R}^d$, where $K + x := \{y + x | y \in K\}$.

Proposition 6. *Let μ be a signed measure on $\mathcal{K}$ satisfying (13). Then f_μ is translation invariant.*

Proof. This follows immediately from the translation invariance of the ordinary volume. □

The following corollary is an immediate consequence of this theorem.

Corollary 7. *Let ρ be a signed measure on $\mathbb{R}_0^+$ whose 0-th to d-th moments are finite and $B \subseteq \mathbb{R}^d$ a convex body. Then the ρ-weighted B-parallel volume is translation invariant.*

Now we turn to continuity with respect to the Hausdorff metric (see e.g. [11, section 1.2]). We start with two lemmas concerning the continuity of the parallel volume.

Lemma 8. *Let $K \subseteq \mathbb{R}^d$ be a body and $B \subseteq \mathbb{R}^d$ be a convex body with interior points. Then*

$$\mathbb{R}_0^+ \to \mathbb{R}_0^+,\ r \mapsto V_d(K + rB)$$

is continuous.

This was proven by Baddeley, Gill and Hansen [2, Lemma 3].

Lemma 9. *Let $B \subseteq \mathbb{R}^d$ be a convex body with interior points. Then the functional*

$$\mathcal{C} \to \mathbb{R},\ K \mapsto V_d(K + B)$$

is continuous w.r.t. the Hausdorff topology.

Proof. Let $\epsilon > 0$. Let $K \in \mathcal{C}'$. Then, according to Lemma 8, there is $\delta > 0$ such that

$$V_d(K + B) - \epsilon < V_d(K + (1 - \delta)B) < V_d(K + (1 + \delta)B) < V_d(K + B) + \epsilon.$$

Moreover, B has interior points and therefore contains a ball $B_R(x)$ of radius $R > 0$ with center $x \in \mathbb{R}^d$, say. Now let $\tilde{K} \in \mathcal{C}$ with Hausdorff distance from K less than δR. Then

$$K \subseteq \tilde{K} + \delta R B^d \subseteq \tilde{K} + \delta B - \delta x,$$

hence $K + (1 - \delta)B \subseteq \tilde{K} + B - \delta x$. Just the same way one shows $\tilde{K} + B \subseteq K + (1 + \delta)B - \delta x$. Thus

$$V_d(K + B) - \epsilon < V_d(K + (1 - \delta)B) \leq V_d(\tilde{K} + B) \leq V_d(K + (1 + \delta)B) < V_d(K + B) + \epsilon.$$

Since $\emptyset$ is an isolated point in $\mathcal{C}$ w.r.t. the Hausdorff topology, we have proven the statement. □

Further we let $\mathcal{K}_0$ denote the set of all convex bodies with interior points and $\mathbb{R}^+$ the set of all positive real numbers.

Theorem 10. *Let μ be a signed measure on $\mathcal{K}$ satisfying (13). Then f_μ is continuous (w.r.t. the Hausdorff topology), iff $f_\mu = f_{\tilde{\mu}}$, where $\tilde{\mu} := \mu_{|\mathcal{K}_0}$ is the restriction of μ to $\mathcal{K}_0$.*

Proof. First let $f_\mu = f_{\tilde{\mu}}$. Let $K \in \mathcal{C}$ and $(K_i)_{i\in\mathbb{N}}$ a sequence converging to K. Then the dominated convergence theorem, whose conditions we shall check below, and Lemma 9 give

$$\begin{aligned}\lim_{i\to\infty} f_\mu(K_i) &= \lim_{i\to\infty} f_{\tilde{\mu}}(K_i)\\ &= \int_{\mathcal{K}_0} \lim_{i\to\infty} V_d(K_i + A)\, d\tilde{\mu}(A)\\ &= \int_{\mathcal{K}_0} V_d(K + A)\, d\tilde{\mu}(A)\\ &= f_\mu(K).\end{aligned}$$

We have to justify the application of the dominated convergence theorem. Since the sequence $(K_i)_{i\in\mathbb{N}}$ convergences, there is $S > 0$ such that $K_i \subseteq SB^d$ for all $i \in \mathbb{N}$. Now we have $V_d(K_i + A) \leq V_d(SB^d + A)$ for all $i \in \mathbb{N}$ and $A \in \mathcal{K}_0$ and according to Remark 1 and (13)

$$\int_{\mathcal{K}_0} V_d(SB^d + A)\, d|\tilde{\mu}|(A) \leq \int_{\mathcal{K}} V_d(SB^d + A)\, d|\mu|(A) < \infty.$$

Now assume that f_μ is continuous. Recall $\tilde{\mu} := \mu_{|\mathcal{K}_0}$ and put $\eta := \mu - \tilde{\mu}$. Then

$$f_\eta = f_\mu - f_{\tilde{\mu}}. \tag{15}$$

Since f_μ is assumed to be continuous and $f_{\tilde{\mu}}$ is continuous by the direction already proven, f_η is continuous, too. Let $K \subseteq \mathbb{R}^d$ be finite. Then $V_d(K + A) = 0$ for any lower-dimensional convex body A and therefore $\int_{\mathcal{K}} V_d(K + A)\, d\eta(A) = 0$. Since f_η is continuous and the finite sets lie dense in $\mathcal{C}$, we conclude that $f_\eta = 0$ and therefore by (15) that $f_\mu = f_{\tilde{\mu}}$. □

The condition $f_\mu = f_{\tilde{\mu}}$ is not equivalent to $\mu = \tilde{\mu}$. We will further comment on this at the end of this section.

From Theorem 10 we obtain the following corollary:

Corollary 11. *Let ρ be a signed measure on $\mathbb{R}_0^+$ whose 0-th to d-th moments are finite and $B \subseteq \mathbb{R}^d$ a convex body with interior points. Then the ρ-weighted B-parallel volume is continuous, iff $\rho(\{0\}) = 0$.*

Proof. Put $\tilde{\rho} := \rho_{|\mathbb{R}^+}$. Then

$$\int_{\mathbb{R}_0^+} V_d(K + \lambda B)\, d\rho(\lambda) - \int_{\mathbb{R}^+} V_d(K + \lambda B)\, d\tilde{\rho}(\lambda) = \rho(\{0\}) V_d(K).$$

So the ρ-weighted B-parallel volume coincides with the $\tilde{\rho}$-weighted B-parallel volume, iff $\rho(\{0\}) = 0$ and thus the result follows from Theorem 10. □

The *convex ring* is

$$\mathcal{R} := \{\bigcup_{i=1}^{r} K_i | K_1, \ldots, K_r \in \mathcal{K},\, r \in \mathbb{N}\} \cup \{\emptyset\}.$$

A functional $\phi : \mathcal{R} \to \mathbb{R}$ is called *additive*, if

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

for all $K, L \in \mathcal{R}$ and $\phi(\emptyset) = 0$.

Lemma 12. *Let μ be a signed measure on $\mathcal{K}$ satisfying (13), for which the restriction of f_μ to the convex ring is additive. Then $f_\mu(K) = 0$ for all sets $K \in \mathcal{R}$ which are contained in a finite union of hyperplanes.*

Proof. We start by proving the assertion for an element K of the convex ring that is contained in one hyperplane. We let u denote a unit vector orthogonal to that hyperplane and put

$$K_i = \bigcup_{l=1}^{i} (K + \tfrac{l}{i} u), \ i \in \mathbb{N},$$

and

$$L := \{k + \lambda u | k \in K, \ \lambda \in [0,1]\}.$$

Then $K_i \in \mathcal{R}$, $i \in \mathbb{N}$, and $L \in \mathcal{R}$. Using that f_μ is translation invariant and additive by assumption we get

$$\begin{aligned} i \cdot |f_\mu(K)| &= |f_\mu(K_i)| \\ &\leq \int_{\mathcal{K}} V_d(K_i + A) \, d|\mu|(A) \\ &\leq \int_{\mathcal{K}} V_d(L + A) \, d|\mu|(A). \end{aligned}$$

Since $i \in \mathbb{N}$ was arbitrary and the right-hand side is independent of i, we conclude $f_\mu(K) = 0$.

Since f_μ is additive and the intersection of two elements of the convex ring, which are contained in at most n hyperplanes respectively, $n \in \mathbb{N}$, is again an element of the convex ring lying in at most n hyperplanes, an easy induction shows that $f_\mu(K) = 0$ for any compact set K contained in at most n hyperplanes, $n \in \mathbb{N}$. □

Theorem 13. *Let μ be a signed measure on $\mathcal{K}$ satisfying (13). Then the restriction of f_μ to the convex ring is additive, iff f_μ is a scalar multiple of the volume.*

Before proving this theorem we state the desired corollary:

Corollary 14. *Let ρ be a signed measure on $\mathbb{R}_0^+$ whose 0-th to d-th moments are finite and $B \subseteq \mathbb{R}^d$ a convex body. Then the restriction of the ρ-weighted B-parallel volume to the convex ring is additive, iff the ρ-weighted B-parallel volume is a scalar multiple of Lebesgue measure.*

Proof. This is an immediate consequences of Theorem 13. Besides, this is a corollary to Theorem 16 below. □

Proof of Theorem 13. Since the Lebesgue measure and hence restrictions of its scalar multiples are additive, we only have to proof one direction. Let $K \subseteq \mathbb{R}^d$ be a compact set and $\epsilon > 0$. Then

$$\{\operatorname{int} B_\epsilon(x) | x \in K\}$$

is an open cover of K, where int denotes the interior and $B_\epsilon(x)$ denotes the ball with radius ϵ and midpoint x. Since K is compact, there is a finite set $A^\epsilon \subseteq K$ such that

$$\{\operatorname{int} B_\epsilon(x) | x \in A^\epsilon\},$$

is a cover of K. Now we have obviously $A^\epsilon + \epsilon B^d \in \mathcal{R}$ and

$$K \subseteq A^\epsilon + \epsilon B^d \subseteq K + \epsilon B^d. \tag{16}$$

For $i \in \mathbb{N}$ we put

$$K_i^\epsilon := \{(x_1, \ldots, x_d) \in A^\epsilon + \epsilon B^d | \text{there are } s \in \{1, \ldots, d\},\ j \in \mathbb{Z} \text{ such that } x_s = \tfrac{j}{i}\}.$$

This means that K_i^ϵ is the intersection of the set $A^\epsilon + \epsilon B^d$, which approximates K, with a hyperplane lattice that is the finer the greater i is.

Let $B \in \mathcal{K}$ be a convex body containing more than one point. Then we have

$$A^\epsilon + \epsilon B^d + B = \bigcup_{i=1}^{\infty} (K_i^\epsilon + B) \cup \{x \in \mathbb{R}^d | d(A^\epsilon + B, x) = \epsilon\}. \tag{17}$$

The set on the right-hand side is obviously contained in the set on the left-hand side. In order to prove the converse inclusion, let $x \in A^\epsilon + \epsilon B^d + B$. Then $d(A^\epsilon + B, x) \leq \epsilon$. If $d(A^\epsilon + B, x) = \epsilon$, then x is contained in the set on the right-hand side of (17), so that we can assume $d(A^\epsilon + B, x) < \epsilon$ from now on. Then there is $\epsilon' < \epsilon$ with $x \in A^\epsilon + B + \epsilon' B^d$ and hence we get a decomposition $x = k + b$ with $k = (k_1, \ldots, k_d) \in \operatorname{int}(A^\epsilon + \epsilon B^d)$ and $b = (b_1, \ldots, b_d) \in B$. Now there is $\delta > 0$ such that

$$\{(v_1, \ldots, v_d) \in \mathbb{R}^d || k_s - v_s| \leq \delta \text{ for all } s \in \{1, \ldots, d\}\} \subseteq A^\epsilon + \epsilon B^d. \tag{18}$$

Choose a point $b' \in B \setminus \{b\}$ and $i \in \mathbb{N}$ such that $\frac{1}{i} < \min\{\delta, \frac{1}{\sqrt{d}}\|b - b'\|\}$. Then there are $\lambda \in [0, 1]$ such that there is $S \in \{1, \ldots, d\}$ and $j \in \mathbb{Z}$ with

$$k_S + \lambda(b_S - b'_S) = \tfrac{j}{i}.$$

Choose the smallest such λ. Then

$$|k_s - (k_s + \lambda(b_s - b'_s))| = \lambda \cdot |b_s - b'_s| \leq \tfrac{1}{i}$$

for all $s \in \{1, \ldots, d\}$, since otherwise λ could be decreased. By (18) it follows that $k + \lambda(b - b') \in A^\epsilon + \epsilon B^d$. So $k + \lambda(b - b') \in K_i^\epsilon$. Since $\lambda b' + (1 - \lambda) b \in B$, we get

$$x = k + b = k + \lambda(b - b') + \lambda b' + (1 - \lambda) b \in K_i^\epsilon + B.$$

So (17) is proven.

As a consequence of Lemma 8, the set $\{x \in \mathbb{R}^d | d(A^\epsilon + B, x) = \epsilon\}$ has Lebesgue measure 0. Therefore we can conclude from (17) that

$$\lim_{i\to\infty} V_d(K_i^\epsilon + B) = V_d(A^\epsilon + \epsilon B^d + B). \tag{19}$$

Moreover, Lemma 8 shows that

$$\lim_{\epsilon\to 0} V_d(K + B + \epsilon B^d) = V_d(K + B)$$

and by (16) we conclude

$$\lim_{\epsilon\to 0} V_d(A^\epsilon + B + \epsilon B^d) = V_d(K + B). \tag{20}$$

Now let μ' denote the restriction of μ to the set of all convex bodies containing more than one point. Then $\zeta := \mu - \mu'$ is concentrated on the set of bodies containing only one point and therefore f_ζ is a scalar multiple of Lebesgue measure and thus additive. Since f_μ is assumed to be additive, $f_{\mu'}$ is additive, too. As $K_i^\epsilon,\ i \in \mathbb{N}, \epsilon > 0$, is an element of the convex ring contained in the union of finitely many hyperplanes, Lemma 12 gives

$$f_{\mu'}(K_i^\epsilon) = 0. \tag{21}$$

Using (20) together with the dominated convergence theorem, on which we will comment below, (19) together with the dominated convergence theorem and (21) we get

$$\begin{aligned}
f_{\mu'}(K) &= \int_{\mathcal{K}} V_d(K + A)\, d\mu'(A) \\
&= \lim_{\epsilon\to 0} \int_{\mathcal{K}} V_d(A^\epsilon + \epsilon B^d + A)\, d\mu'(A) \\
&= \lim_{\epsilon\to 0} \lim_{i\to\infty} \int_{\mathcal{K}} V_d(K_i^\epsilon + A)\, d\mu'(A) \\
&= \lim_{\epsilon\to 0} \lim_{i\to\infty} f_{\mu'}(K_i^\epsilon) \\
&= 0.
\end{aligned}$$

We were allowed to apply the dominated convergence theorem, since for $i \in \mathbb{N}$ and $\epsilon \in (0,1)$ we have

$$V_d(A^\epsilon + \epsilon B^d + A) \le V_d(K + B^d + A) \text{ resp. } V_d(K_i^\epsilon + A) \le V_d(K + B^d + A)$$

for all $A \in \mathcal{K}$ and

$$\int_{\mathcal{K}} V_d(K + B^d + A)\, d|\mu'|(A) \le \int_{\mathcal{K}} V_d(K + B^d + A)\, d|\mu|(A) < \infty$$

by Remark 1 and (13).
So we have $f_\mu = f_\zeta$ and therefore f_μ is a scalar multiple of the Lebesgue measure. □

A functional $\phi : \mathcal{C} \mapsto \mathbb{R}$ resp. $\phi : \mathcal{R} \mapsto \mathbb{R}$ is called *submodular*, if

$$\phi(K \cup L) + \phi(K \cap L) \leq \phi(K) + \phi(L)$$

for all $K, L \in \mathcal{C}$ resp. $K, L \in \mathcal{R}$. For properties and applications of submodular functions defined on arbitrary lattices see [14].

Proposition 15. *Let μ be a (non-negative) measure on $\mathcal{K}$ satisfying (13). Then f_μ is submodular.*

Proof. For $K, L \in \mathcal{C}$, $A \in \mathcal{K}$ we have

$$V_d((K \cup L) + A) = V_d((K + A) \cup (L + A)) = V_d(K + A) + V_d(L + A) - V_d((K + A) \cap (L + A)).$$

Hence f_μ is submodular, iff

$$\int_{\mathcal{K}} V_d((K + A) \cap (L + A)) - V_d((K \cap L) + A)\, d\mu(A) \geq 0$$

for all $K, L \in \mathcal{C}$. However, this is always true for measures μ, since

$$(K \cap L) + A \subseteq (K + A) \cap (L + A), \quad K, L \in \mathcal{C},\ A \in \mathcal{K}. \quad \square$$

Theorem 16. *Let ρ be a signed measure with finite k-th moment, $k = 0, \ldots, d$, and B a convex body consisting of more than one point. Then the following are equivalent:*

(i) The ρ-weighted B-parallel volume is submodular.

(ii) The restriction of the ρ-weighted B-parallel volume to $\mathcal{R}$ is submodular.

(iii) The restriction of ρ to $\mathbb{R}^+$ is a measure.

Proof. The implication (i)$\Rightarrow$(ii) is clear.

In order to see (iii)$\Rightarrow$(i), put $\eta := \rho - \rho_{|\mathbb{R}^+}$. Then η is concentrated on $\{0\}$, hence f_η is a scalar multiple of Lebesgue measure. Moreover, by Proposition 15 the $\rho_{|\mathbb{R}^+}$-weighted B-parallel volume is submodular. Hence the ρ-weighted B-parallel volume is submodular, too.

Now assume (ii) is satisfied. In order to verify (iii), we first show that for the functions

$$f_{a,b} : \mathbb{R}^+ \to \mathbb{R},\ r \mapsto \begin{cases} r - a & \text{if } a \leq r < \frac{a+b}{2}, \\ b - r & \text{if } \frac{a+b}{2} \leq r < b, \\ 0 & \text{elsewhere}, \end{cases} \qquad b > a > 0,$$

we have

$$\int_{\mathbb{R}^+} f_{a,b}(\lambda)\, d\rho(\lambda) \geq 0, \quad b > a > 0. \tag{22}$$

The idea of this first step of the proof is to construct two bodies $K, L \in \mathcal{R}$ for which the functional

$$r \mapsto V_d(((K + rB) \cap (L + rB)) \setminus ((K \cap L) + rB))$$

is proportional to $f_{a,b}$, if we neglect boundary effects, and to justify that we can neglect them.

So let $b > a > 0$. Choose an unit vector u in the linear space parallel to the affine hull of B and denote the Euclidean ball of radius R in the hyperplane orthogonal to u by B^u_R for $R \geq 0$. By the translation invariance of the volume we may assume $\max\{\langle x, u\rangle | x \in B\} = \max\{\langle x, -u\rangle | x \in B\}$. Now put

$$h := \max\{\langle x, u\rangle | x \in B\} \quad \text{and} \quad h_\perp := \max\{\langle x, v\rangle | x \in B, v \in S^{d-1}, v \perp u\}.$$

Then $h > 0$ and $h_\perp \geq 0$. Let $r \in \mathbb{R}^+_0$ and $R > h_\perp r$. Now set

$$K := (\{-hb \cdot u\} \cup [ha \cdot u, hb \cdot u]) + B^u_R \quad \text{and} \quad L := ([-hb \cdot u, -ha \cdot u] \cup \{hb \cdot u\}) + B^u_R,$$

where $[x, y]$ denotes the line segment from x to y. Then one has, if $a \leq r < \frac{a+b}{2}$,

$$\begin{aligned}&((K + rB) \cap (L + rB)) \setminus ((K \cap L) + rB) \subseteq \\ &\quad [h(a - r) \cdot u, h(r - a) \cdot u] + (B^u_{R+h_\perp r} \setminus B^u_{R-h_\perp r}) \;\cup\; [h(a - r)u, h(r - a)u] + B^u_{R-h_\perp r}. \quad (23)\end{aligned}$$

Indeed, let $x \in ((K + rB) \cap (L + rB)) \setminus ((K \cap L) + rB)$. Then there are $k \in K$ and $m \in B$ such that $x = k + rm$. Since $x \notin ((K \cap L) + rB)$, we have $k \in K \setminus L$. Hence $\langle x, u\rangle = \langle k, u\rangle - r\langle m, -u\rangle \geq ha - rh$. The same way one shows $\langle x, u\rangle \leq (r - a)h$ and $\langle x, v\rangle \leq R + h_\perp r$ for $v \in S^{d-1}$ with $v \perp u$. Hence $x \in [h(a - r)u, h(r - a)u] + B^u_{R+h_\perp r}$ and thus x is contained in the right-hand side of (23).

On the other hand one has

$$[h(a - r)u, h(r - a)u] + B^u_{R-h_\perp r} \subseteq ((K + rB) \cap (L + rB)) \setminus ((K \cap L) + rB).$$

Indeed, let $x \in [h(a - r)u, h(r - a)u] + B^u_{R-h_\perp r}$. Now

$$-h < \tfrac{1}{r}(h(a - r) + ha) \leq \tfrac{1}{r}(\langle x, u\rangle + ha) \leq \tfrac{1}{r}(h(r - a) + ha) = h.$$

Hence there is $m \in B$ with $\langle m, u\rangle = \frac{1}{r}(\langle x, u\rangle + ha)$. Put $l := x - rm$. Then

$$\langle l, u\rangle = \langle x, u\rangle - r \cdot \tfrac{1}{r}(\langle x, u\rangle + ha) = -ha.$$

For $v \in S^{d-1}$ with $v \perp u$ one has $|\langle m, v\rangle| \leq h_\perp$ by the definition of $h_\perp$ and hence $\langle l, v\rangle \leq R$. Thus $l \in L$ and $x \in L + rB$. The same way one shows $x \in K + rB$. For $k \in K \cap L$ one has either $\langle k, u\rangle = -hb$ or $\langle k, u\rangle = hb$. So for any point $y \in (K \cap L) + rB$ either $\langle y, u\rangle \leq h(r - b)$ or $\langle y, u\rangle \geq h(b - r)$ must hold. Since $b - r > r - a$ and $h(a - r) \leq \langle x, u\rangle \leq h(r - a)$ by the choice of x, this shows $x \notin (K \cap L) + rB$.

Similar for $\frac{a+b}{2} \leq r < b$ one shows

$$\begin{aligned}&((K + rB) \cap (L + rB)) \setminus ((K \cap L) + rB) \subseteq \\ &\qquad [h(a - r) \cdot u, h(r - a) \cdot u] + (B^u_{R+h_\perp r} \setminus B^u_{R-h_\perp r}) \;\cup\; (h(r - b)u, h(b - r)u) + B^u_{R-h_\perp r},\end{aligned}$$

and

$$(h(r-b)u, h(b-r)u) + B^u_{R-h_\perp r} \subseteq ((K+rB)\cap(L+rB))\setminus((K\cap L)+rB),$$

where $(x,y) := [x,y]\setminus\{x,y\}$. For $r<a$

$$((K+rB)\cap(L+rB)) = ((K\cap L)+rB)$$

holds and for $r>b$ we have

$$((K+rB)\cap(L+rB))\setminus((K\cap L)+rB) \subseteq [h(a-r)u, h(r-a)u] + (B^u_{R+h_\perp r}\setminus B^u_{R-h_\perp r}).$$

Hence we have

$$\begin{aligned}&2h\cdot f_{a,b}(r)\kappa_{d-1}(R-h_\perp r)^{d-1} \le V_d\big(((K+rB)\cap(L+rB))\ \setminus\ ((K\cap L)+rB)\big)\\ &\le 2h\cdot\max\{0,r-a\}\kappa_{d-1}((R+h_\perp r)^{d-1}-(R-h_\perp r)^{d-1}) + 2h\cdot f_{a,b}(r)\kappa_{d-1}(R-h_\perp r)^{d-1}.\end{aligned}$$

So

$$\lim_{R\to\infty}\frac{V_d\big(((K+rB)\cap(L+rB))\ \setminus\ ((K\cap L)+rB)\big)}{R^{d-1}} = 2h\cdot f_{a,b}(r)\kappa_{d-1},\quad r\in\mathbb{R}^+.$$

By the majorized convergence theorem, which can be applied, since

$$V_d(K+rB)\le 2h(b+r)\kappa_{d-1}(2R)^{d-1},$$

we get

$$\begin{aligned}2h\kappa_{d-1}\int_{\mathbb{R}^+} f_{a,b}(\lambda)\,d\rho(\lambda) &= \int_{\mathbb{R}^+}\lim_{R\to\infty}\frac{V_d\big(((K+\lambda B)\cap(L+\lambda B))\ \setminus\ ((K\cap L)+\lambda B)\big)}{R^{d-1}}\,d\rho(\lambda)\\ &= \lim_{R\to\infty}\int_{\mathbb{R}^+}\frac{V_d\big(((K+\lambda B)\cap(L+\lambda B))\ \setminus\ ((K\cap L)+\lambda B)\big)}{R^{d-1}}\,d\rho(\lambda)\\ &\ge 0,\end{aligned}$$

where the last inequality can be deduced from the submodality by arguments used in the proof of Proposition 15 in the reverse direction. So (22) is proven.

Next we prove $\rho((x,y))\ge 0$ for $y>x>0$. Define $x(n) := x+2^{-n}(y-x)$ and $y(n) := y-2^{-n}(y-x)$, $n\in\mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. Then another application of the majorized convergence theorem gives

$$\begin{aligned}\rho((x,y)) &= \int_{\mathbb{R}^+}\mathbf{1}_{(x,y)}(\lambda)\,d\rho(\lambda)\\ &= \frac{2}{y-x}\int_{\mathbb{R}^+} h_{x,y}(\lambda)+\sum_{n=1}^\infty h_{x(n+1),x(n)}(\lambda)+\sum_{n=1}^\infty h_{y(n),y(n+1)}(\lambda)\,d\rho(\lambda)\\ &= \frac{2}{y-x}\Big(\int_{\mathbb{R}^+} h_{x,y}(\lambda)\,d\rho(\lambda)+\sum_{n=1}^\infty\int_{\mathbb{R}^+} h_{x(n+1),x(n)}(\lambda)\,d\rho(\lambda)\\ &\qquad\qquad+\sum_{n=1}^\infty\int_{\mathbb{R}^+} h_{y(n),y(n+1)}(\lambda)\,d\rho(\lambda)\Big)\\ &\ge 0\end{aligned}$$

by (22).

In order to prove that $\rho_{|\mathbb{R}^+}$ is a measure, it suffices to prove that $x \mapsto \rho([0,x])$ is monotonically increasing on $\mathbb{R}^+$. So let $y > x > 0$. Then

$$\begin{aligned}
\rho([0,y]) - \rho([0,x]) &= \rho((x,y]) \\
&= \rho(\bigcap_{n=1}^{\infty}(x, y+\tfrac{1}{n})) \\
&= \lim_{n\to\infty} \rho((x, y+\tfrac{1}{n})) \\
&\geq 0. \qquad \square
\end{aligned}$$

Another interesting question is, for which signed measures the weighted parallel volumes are monotone, where a functional $\phi : \mathcal{C} \to \mathbb{R}$ is called *monotone*, if for $K, L \in \mathcal{C}$ one has

$$L \subseteq K \;\Rightarrow\; \phi(L) \leq \phi(K).$$

Even in the Euclidean case we are not able to answer this question. However, the following proposition answers a related question and gives a necessary condition for the weighted parallel volume to be monotone.

Proposition 17. *Let ρ be a signed measure on $\mathbb{R}_0^+$ with finite 0-th to d-th moments. Then the restriction of the ρ-weighted Euclidean parallel volume to $\mathcal{K}$ is monotone, iff the 0-th to $d-1$-th moments of ρ are non-negative.*

Proof. By Proposition 2 and the fact $\kappa_j V_{d-j}(K) = \binom{d}{j} V(K[d-j], B^d[j])$ (see [10, (5.3.8)]) we have after changing the summation variable from j to $d-j$

$$\int_{\mathbb{R}_0^+} V_d(K + \lambda B^d)\, d\rho(\lambda) = \sum_{j=0}^{d} \kappa_{d-j}\mu_{d-j} V_j(K)$$

for $K \in \mathcal{K}$, where μ_j denotes the j-th moment of ρ.

So if $\mu_0, \ldots, \mu_{d-1}$ are non-negative, then the monotonicity of the restriction of the ρ-weighted parallel volume follows just from the facts that the intrinsic volumes are monotone and $V_0(K) = 1$ for all $K \in \mathcal{K}$.

The ideas for the proof of the converse statement are extracted from the end of the proof of Hadwiger's characterisation theorem ([4, Section 6.1.10]). For $i \in \{1, \ldots, d\}$ and $R \in \mathbb{R}_0^+$ let

$$B_R^i := \{(x_1, \ldots x_i, 0 \ldots, 0) \in \mathbb{R}^d \,|\, \|(x_1, \ldots, x_i)\| \leq R\}.$$

Then by using the dimension invariance of the intrinsic volumes and the Steiner formula in $\mathbb{R}^{\max\{i,j\}}$ one gets

$$V_j(B_R^i) = \begin{cases} R^j \binom{i}{j} \frac{\kappa_i}{\kappa_{i-j}} & \text{if } j \leq i, \\ 0 & \text{if } j > i. \end{cases}$$

The function

$$\int_{\mathbb{R}_0^+} V_d(B_R^i + \lambda B^d)\, d\rho(\lambda) = \sum_{j=0}^{d} \kappa_{d-j}\mu_{d-j} V_j(B_R^i) = \sum_{j=0}^{i} \binom{i}{j} \frac{\kappa_{d-j}\,\kappa_i}{\kappa_{i-j}} \mu_{d-j} R^j$$

cannot tend to $-\infty$ as R tends to ∞, since we assumed the ρ-weighted parallel volume to be monotone. Therefore μ_{d-i} is non-negative. □

As mentioned before, this proposition yields that a necessary condition for a weighted parallel volume to be monotone is that the 0-th to $(d-1)$-th moments of the signed measure are non-negative. However, this condition is not sufficient. In fact, obviously there is a signed measure ρ whose restriction to $[0,1]$ is -1 times the Lebesgue measure and whose 0-th to $(d-1)$-th moments are non-negative. Since the parallel bodies of the sphere S^{d-1} and the ball B^d at distance greater than 1 coincide, the ρ-weighted parallel volume of the sphere is greater than that of the ball. So the ρ-weighted parallel volume is not monotone.

On the other hand an obvious sufficient condition for the parallel volume to be monotone, is that the signed measure ρ is a measure. We think that this condition is far from being necessary. In order to construct an explicit example of a signed measure which is not a measure but whose weighted parallel volume is monotone, we introduce some notion:

Let $K \in \mathcal{C}'$ and $x \in \mathbb{R}^d$. Then there is $p \in K$ with $d(p,x) = d(K,x)$. If p is unique, we call it the *metric projection* from x onto K, $p(K,x) := p$. The *exoskeleton* $\operatorname{exo}(K)$ of K is the set of all points x for which p is not unique.

Lemma 18. *Let $K, M \subseteq \mathbb{R}^d$ be two compact sets satisfying $M \subseteq K$. Let $r < s$. Then*

$$\frac{1}{s^d} V_d((K + sB^d) \setminus (M + sB^d)) \leq \frac{1}{r^d} V_d((K + rB^d) \setminus (M + rB^d)).$$

Proof. The special case, where $M = \emptyset$, is already proven in [7, Hilfssatz 7]. In the present, more general, setting only one additional idea is needed. Let f be the function from the proof of [7, Hilfssatz 7], which satisfies

$$f(x) - p(K,x) = \frac{s}{r}(x - p(K,x)),$$

and is defined on some set $L \subseteq \mathbb{R}^d$ with $L \cap \operatorname{exo}(K) = \emptyset$. We have to show that

$$f((M + rB^d) \cap L) \subseteq M + sB^d.$$

Let $x \in (M + rB^d) \cap L$. Then $d(p(K,x), x) \leq d(M,x) \leq r$. Hence $d(x, f(x)) \leq s - r$ and thus

$$d(M, f(x)) \leq d(M,x) + d(x, f(x)) \leq r + (s - r) = s.$$

So $f(x) \in M + sB^d$. □

Stacho [13, Section 4] has proven the case, where $M = \emptyset$, of this lemma in general Minkowski spaces. However, it does not seem to be possible to extend his proof to more general sets M.

Now we are ready to show that there is a signed measure ρ which is not a measure, but whose weighted parallel volume is monotone.

Example 19. Choose two real numbers $r < s$ and set

$$\rho := \frac{1}{r^d}\delta_r - \frac{1}{s^d}\delta_s,$$

where δ_r and δ_s denote the Dirac-measures in r and s. Now let $K, L \subseteq \mathbb{R}^d$ be two compact sets. If $L \subseteq K$, then by Lemma 18

$$\begin{aligned}\int_{\mathbb{R}_0^+} V_d(K+\lambda B^d)\, d\rho(\lambda) - \int_{\mathbb{R}_0^+} & V_d(L+\lambda B^d)\, d\rho(\lambda) \\ &= \int_{\mathbb{R}_0^+} V_d((K+\lambda B^d)\setminus(L+\lambda B^d))\, d\rho(\lambda) \\ &= \frac{1}{r^d} V_d((K+rB^d)\setminus(L+rB^d)) - \frac{1}{s^d} V_d((K+sB^d)\setminus(L+sB^d)) \\ &\geq 0.\end{aligned}$$

So the ρ-weighted parallel volume is monotone.

We like to take this opportunity to point out the connection between Lemma 18 and the Poulson-Kneser-Conjecture. This conjecture deals with extensions, where a tuple $(q_1, \ldots, q_N)$ of points in $\mathbb{R}^d$ is called *extension* of a tuple $(p_1, \ldots, p_N)$ of points in $\mathbb{R}^d$, if

$$\|p_i - p_j\| \leq \|q_i - q_j\|, \quad i, j = 1, \ldots, N.$$

Recall that $B_R(p)$ is the ball with radius R and midpoint p. The Kneser-Poulson-Conjecture is the following:

Conjecture 20. *Let $(q_1, \ldots, q_N)$ be an extension of $(p_1, \ldots, p_N)$ and $R > 0$. Then*

$$V_d(\bigcup_{i=1}^N B_R(p_i)) \leq V_d(\bigcup_{i=1}^N B_R(q_i)).$$

For further information and a partial proof, see [1].
From Lemma 18 we get the following corollary that is, if K is finite, a special case of the Kneser-Poulson-Conjecture:

Corollary 21. *Let $K \subseteq \mathbb{R}^d$ be compact. Then $V_d(rK + B^d)$ is monotonically increasing in r.*

Proof. Let $r < r'$. Then $\frac{1}{r'} < \frac{1}{r}$. If $M = \emptyset$, then $M + \frac{1}{r}B^d = M + \frac{1}{r'}B^d = \emptyset$. Hence Lemma 18 gives

$$V_d(rK + B^d) = r^d V_d(K + \frac{1}{r}B^d) \leq (r')^d V_d(K + \frac{1}{r'}B^d) = V_d(r'K + B^d). \quad \square$$

The connection between Lemma 18 and the Kneser-Poulson-Conjecture has not been realized yet. Stacho [13] does not mention the Kneser-Poulson-Conjecture, which had already been formulated at that time. Later Rehder [9] proved the special case of the Kneser-Poulson-Conjecture, where the extension was obtained by a dilatation, for an arbitrary compact and convex set instead of the unit ball. So Stacho's and Rehder's results generalize the same special case of the Kneser-Poulson-Conjecture, but Rehder does not quote Stacho. Still in [1] Rehder is quoted and not Stacho.

Now we deal with the question, whether the signed measure ρ is uniquely determined by the ρ-weighted parallel volume, or, more generally, whether a signed measure μ on $\mathcal{K}$ is uniquely determined by the functional f_μ. However, it is quite easy to see that the answer to the latter question is negative. In fact, let μ and η be two Dirac-measures on convex bodies that are translates of each other. Then by the translation invariance of the volume we have $f_\mu = f_\eta$.

We want to exclude such trivial counter-examples. For this we choose a *center map* c, i.e. a measurable map $c : \mathcal{K} \to \mathbb{R}^d$ such that $c(K + x) = c(K) + x$ for all $K \in \mathcal{K}$, $x \in \mathbb{R}^d$, for instance, we let $c(K)$ denote the midpoint of the smallest ball in which K is contained. Let

$$\mathcal{K}^* := \{K \in \mathcal{K} | c(K) = 0\} \tag{24}$$

be the set of all centered convex bodies. Now we will only consider signed measures μ which are concentrated on $\mathcal{K}^*$.

Theorem 22. *Let μ and η be two signed measures on $\mathcal{K}^*$ satisfying (13). Then $\mu = \eta$, iff $f_\mu = f_\eta$.*

Proof. Obviously $f_\mu = f_\eta$, if $\mu = \eta$.

So assume $f_\mu = f_\eta$. We put

$$\bar{\mu} := \mu + \mu^- + \eta^- \qquad \text{and} \qquad \bar{\eta} := \eta + \mu^- + \eta^-.$$

For a measure ρ on $\mathcal{K}^*$ we define a measure M_ρ on $\mathcal{K}$ by

$$M_\rho(\mathcal{A}) = \int_{\mathcal{K}^*} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{A}}(A + x)\, dx\, d\rho(A)$$

for any measurable set $\mathcal{A}$. Moreover we put

$$\mathcal{K}_C := \{K \in \mathcal{K} | C \cap K \neq \emptyset\},\ C \in \mathcal{C}.$$

Let $A^* = \{-x | x \in A\}$, $A \in \mathcal{K}$. Now

$$\begin{aligned} M_{\bar{\mu}}(\mathcal{K}_C) &= \int_{\mathcal{K}^*} \int_{\mathbb{R}^d} \mathbf{1}_{(A+x)\cap C \neq \emptyset}\, dx\, d\bar{\mu}(A) \\ &= \int_{\mathcal{K}^*} V_d(C + A^*)\, d\bar{\mu}(A) \\ &= f_\mu(C^*) + \int_{\mathcal{K}^*} V_d(C + A^*)\, d(\mu^- + \eta^-)(A) \\ &= f_\eta(C^*) + \int_{\mathcal{K}^*} V_d(C + A^*)\, d(\mu^- + \eta^-)(A) \\ &= M_{\bar{\eta}}(\mathcal{K}_C). \end{aligned}$$

Just like [11, Lemma 1.3.1] one can show that $\{\mathcal{K}_C | C \in \mathcal{C}\}$ is a generating system of the Fell-Matheron-σ-algebra. Since it is intersection-stable, we conclude $M_{\bar{\mu}} = M_{\bar{\eta}}$. Now let

$\mathcal{A} \subseteq \mathcal{K}^*$ be measurable. Because $A + x = A' + x'$, $A, A' \in \mathcal{K}^*$, $x, x' \in \mathbb{R}^d$, obviously implies $A = A'$ and $x = x'$, we have

$$M_{\bar{\mu}}(\{A + x | A \in \mathcal{A},\, x \in [0,1]^d\}) = \int_{\mathcal{K}^*} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{A}}(A) \mathbf{1}_{[0,1]^d}(x)\, dx\, d\bar{\mu}(A) = \bar{\mu}(\mathcal{A}).$$

As this equation holds also for $\bar{\eta}$ instead of $\bar{\mu}$, we conclude $\bar{\eta} = \bar{\mu}$ and hence $\eta = \mu$. $\square$

Corollary 23. *Let B be a convex body consisting of more than one point and let ρ and η be two signed measures on $\mathbb{R}_0^+$ with finite 0-th to d-th moments. If the ρ-weighted B-parallel volume and the η-weighted B-parallel volume coincide, then $\rho = \eta$.*

Proof. This is an immediate consequence of Theorem 22. $\square$

4 Application to Boolean models

In this section we want to apply the results of the previous sections to the capacity functional of Boolean models used in stochastic geometry (see e.g. [11]).

A (stationary) *Boolean model* (with convex grains) is a random closed set Z, which can be constructed in the following way: Let $X = \{(X_i, Z_i) | i \in \mathbb{N}\}$ be a marked stationary Poisson process with intensity γ on $\mathbb{R}^d$, where the mark space is the set $\mathcal{K}^*$ defined by (24) and the mark distribution is a probability distribution $\mathbb{Q}$ on $\mathcal{K}^*$ satisfying

$$\int_{\mathcal{K}^*} V_d(K + A)\, d\mathbb{Q}(A) < \infty, \quad K \in \mathcal{C}.$$

Then let

$$Z = \bigcup_{i=1}^{\infty} X_i + Z_i.$$

Now γ is called the *intensity*, $\mathbb{Q}$ is called the *grain distribution* of the Boolean model Z and the sets $X_i + Z_i$, $i \in \mathbb{N}$, are called the *grains* of Z. For further information, see [11, section 4.4].

The *capacity functional* of a random closed set $Z \subseteq \mathbb{R}^d$ is

$$T_Z : \mathcal{C} \to [0,1],\, C \mapsto \mathbb{P}(Z \cap C \neq \emptyset).$$

For further information, see [11, section 1.4].

Recall the definition (14) of f_μ and let log denote the logarithm to base e.

Lemma 24. *Let Z be a Boolean model. Then Z has intensity γ and grain distribution $\mathbb{Q}$, iff*

$$-\log(1 - T_Z) = f_{\gamma \cdot \mathbb{Q}}.$$

Proof. The necessacity is an immediate consequence of [11, Satz 4.4.4].
Therefore by [11, Satz 1.4.2 and Satz 4.2.2] we have equivalence. $\square$

Denote the set of all centered convex bodies with interior points by $\mathcal{K}_0^*$.

Corollary 25. *Let Z be a Boolean model with intensity γ and grain distribution $\mathbb{Q}$. Then T_Z is continuous (w.r.t. the Hausdorff topology), iff $\mathbb{Q}$ is concentrated on $\mathcal{K}_0^*$.*

Proof. First assume that T_Z is continuous. Then by Lemma 24 $f_{\gamma\cdot\mathbb{Q}}$ is continuous, too. As a consequence of Theorem 10 we obtain

$$\gamma \cdot \int_{\mathcal{K}^*\setminus\mathcal{K}_0^*} V_d(B^d + A)\, d\mathbb{Q}(A) = 0.$$

Since $V_d(B^d + A) > 0$ for all $A \in \mathcal{K}^* \setminus \mathcal{K}_0^*$ and $\mathbb{Q}$ is a measure, we conclude that $\mathbb{Q}$ is concentrated on $\mathcal{K}_0^*$.
Now assume that $\mathbb{Q}$ is concentrated on $\mathcal{K}_0^*$. Then $f_{\gamma\cdot\mathbb{Q}}$ is continuous by Theorem 10 and from Lemma 24 we obtain that T_Z is continuous, too. □

Corollary 26. *Let Z be a Boolean model with intensity γ and grain distribution $\mathbb{Q}$. Then we have for all $K, L \in \mathcal{C}$*

$$\begin{aligned}
(1 - T_Z(K \cup L)) \cdot (1 - T_Z(K \cap L)) &\geq (1 - T_Z(K)) \cdot (1 - T_Z(L)) && (25)\\
\mathbb{P}(Z \cap (K \setminus L) = \emptyset | Z \cap L = \emptyset) &\geq \mathbb{P}(Z \cap (K \setminus L) = \emptyset | Z \cap (K \cap L) = \emptyset). && (26)
\end{aligned}$$

Proof. From Proposition 15 we have

$$f_{\gamma\cdot\mathbb{Q}}(K \cup L) + f_{\gamma\cdot\mathbb{Q}}(K \cap L) \leq f_{\gamma\cdot\mathbb{Q}}(K) + f_{\gamma\cdot\mathbb{Q}}(L). \tag{27}$$

So by Lemma 24 we get (25).
Since

$$1 - T_Z(M) = \mathbb{P}(Z \cap M = \emptyset), \quad M \in \mathcal{C},$$

we conclude from (25)

$$\frac{\mathbb{P}(Z \cap (K \cup L) = \emptyset)}{\mathbb{P}(Z \cap L = \emptyset)} \geq \frac{\mathbb{P}(Z \cap K = \emptyset)}{\mathbb{P}(Z \cap (K \cap L) = \emptyset)}.$$

Hence

$$\mathbb{P}(Z \cap (K \setminus L) = \emptyset | Z \cap L = \emptyset) \geq \mathbb{P}(Z \cap (K \setminus L) = \emptyset | Z \cap (K \cap L) = \emptyset). \quad \square$$

There is only one centered convex body which consists of one point only. In the following we assume w.l.o.g. that this convex body is $\{0\}$.

Corollary 27. *Let Z be a Boolean model with intensity γ and grain distribution $\mathbb{Q}$. Then equality holds in the inequalities of Corollary 26, iff $\mathbb{Q}$ is the Dirac measure $\delta_{\{0\}}$ on $\{0\}$.*

Proof. All one has to show is that equality holds in (27), iff $\mathbb{Q} = \delta_{\{0\}}$, since from there on all arguments in the proof of Corollary 26 hold with inequality replaced by equality and the direction of conclusion can be reversed.

If $\mathbb{Q} = \delta_{\{0\}}$, then $f_{\gamma\cdot\mathbb{Q}}$ is a multiple of Lebesgue measure and hence equality holds in (27). On the other hand, if equality holds in (27) for arbitrary compact sets, then Theorem 13 implies that $f_{\gamma\cdot\mathbb{Q}}$ is a multiple of Lebesgue measure. Then

$$\gamma \cdot \int_{\mathcal{K}^*} V_d(S^{d-1} + A)\, d\mathbb{Q}(A) = 0,$$

where S^{d-1} denotes the sphere, since the sphere has Lebesgue measure zero. However $V_d(S^{d-1} + A) > 0$ for every convex body A containing more than one point and hence $\mathbb{Q} = \delta_{\{0\}}$. $\square$

It is easy to see that the intersection Z' of a Boolean model Z with an affine subspace E is a Boolean model in E.

Corollary 28. *(i) Let $Z \subseteq \mathbb{R}^n$ be a Boolean model with grains that are a.s. balls such that the distribution function of the radii is $1 - e^{-\pi t^2}$. For an affine subspace $E \subseteq \mathbb{R}^n$ the grains of $Z' := Z \cap E$ are a.s. balls and the distribution function of their radii is again $1 - e^{-\pi t^2}$. Moreover Z and Z' have the same intensity.*
(ii) Let $\mathbb{P}$ be a probability distribution on $\mathbb{R}_0^+$. Assume that there is $m \in \mathbb{N}^+$ such that for all Boolean models $Z \subseteq \mathbb{R}^n$, $n > m$, whose grains are a.s. balls with radii distributed according to $\mathbb{P}$ the following holds: For all m-dimensional affine subspaces $E \subseteq \mathbb{R}^n$ the grains of the Boolean model $Z' := Z \cap E$ are a.s. balls with radii distribution $\mathbb{P}$ and Z' has the same intensity as Z. Then the distribution function of $\mathbb{P}$ is $1 - e^{-\pi t^2}$.

Proof. For both parts of this corollary one has to observe that the capacity functional $T_{Z'}$ is the restriction of T_Z to the set of all compact sets contained in E, which is an immediate consequence of the definition of T_Z.
(i) The grains of Z' are obviously balls. For the computation of the radii distribution, recall that $W(K) = \mathbb{E}\, V_d(K + \Lambda B^d)$, where Λ is a random variable with distribution function $1 - e^{-\pi t^2}$. Let γ denote intensity of Z. By Lemma 24 we get $\log(1 - T_Z) = -\gamma \cdot W$. So by Theorem 4 (i) we have $\log(1 - T_{Z'}) = -\gamma \cdot W$. Hence from Lemma 24 it follows that the distribution function of the radii of the grains of Z' is again $1 - e^{-\pi t^2}$ and Z' has intensity γ.
(ii) For all compact sets $K \subseteq E$ we have $T_Z(K) = T_{Z'}(K)$ and hence, by Lemma 24,

$$\gamma \cdot \int_{\mathbb{R}_0^+} V_m(K + \lambda B^m)\, d\mathbb{P}(\lambda) = \gamma \cdot \int_{\mathbb{R}_0^+} V_n(K + \lambda B^n)\, d\mathbb{P}(\lambda),$$

where γ is the intensity of Z. Hence Theorem 4 (iii) implies that $\mathbb{P}$ has distribution function $1 - e^{-\pi t^2}$. $\square$

We have to confess that part (ii) of Corollary 28 is unsatisfactory for two reasons. We fix m and then let n run over all numbers greater than m. It would be more natural to do it the other way round. Moreover, in its proof we only work with the restriction of T_Z to $\mathcal{K}$ and not with T_Z on the whole of $\mathcal{C}$. So the result probably holds under much weaker assumptions - however under this weaker assumptions it cannot be obtained as a corollary of Theorem 4 in a reasonable way.

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