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On the Univalency of Regular Functions (*).

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Summary. – *This paper is devoted to the various criteria of the univalency of regular functions. We establish a new criterion by means of the methods of the infinite dimensional algebraic geometry. It is formulated explicitely in terms of the coefficients of the Taylor expansion of a regular function in contrast to the known ones. To do it we need to use the coefficients of the Shapovalov form for the Virasoro algebra which are well-known in the representation theory of the infinite-dimensional algebras and groups. Also we find the expression for the univalency radius of a regular function which is analogous to Cauchy-Hadamard formula for the regularity radius (the radius of the convergence). Besides that we discuss some questions of the numerical computation of the univalency radius.*

1. – Some preliminary remarks and historical review.

The class S, one of the most mysterious object of the geometrical theory of functions of a complex variable, consists of functions $f(z)$, univalent and regular (holomorphic) in the unit disk $D_+ = \{z \in C\colon |z| \leq 1\}$, normalized by the conditions $f(0)=0$, $f'(0) = 1$ [1]. Each such function realizes a conformal mapping of the disk D_+ onto a domain in the complex plane C with the unit conformal radius. Transversely, for each domain of such kind there exists exactly one function $f(z)$ from the class S, whose image coincides with it. For a visuality one might consider elements of the set S as the equiangular maps for the southern hemisphere of the Riemann sphere [2].

The more detail information about the class S and conformal mappings one should get in the monographs [1, 3, 4]. The applications are discussed in [5].

The family of functions $f(z)$ which belong to S may be regarded as an infinite dimensional complex manifold, The alternative point of view is stated, for example, in the paper [6]. The arguments of that paper are very similar to ones for the negative answer on the question «Are the diffeomorphism groups the Lie groups?» whereas the results of the paper [7]. Nevertheless, we hold to the point of view that in spite of a great number of papers, devoted to the definition of the infinite dimensional mani-

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fold (P. DE LA HARP, L. A. HARRIS, W. KAUP, J. M. ISIDRO, J. A. LESLIE, H. OMORI, J. M. SOURIAU, B. I. ROSENFELD et al.), our idea about it is very diffuse. So we think that the class S satisfies our intuitive definition of the infinite-dimensional manifold.

The coordinate system upon S is determined by the coefficients $c_1, c_2, c_3, \dots c_n, \dots$ of the Taylor expansion of a function $f(z)$

$$f(z) = z + c_1 z^2 + c_2 z^3 + c_3 z^4 + \dots + c_n z^{n+1} + \dots . \tag{1}$$

The expansion defines the imbedding of the class S into the space $C_0[\![z]\!]$ of formal series of the form (1). Hence S may be identified with a domain in $C_0[\![z]\!]$. Because of the Bieberbach-de Branges inequalities [8, 9]

$$|c_n| < n + 1$$

one might also regard the domain S as a bounded one.

In the thirties of this century there were discovered the variational formulae for the class S. There exists many variants of them, but all are based on the simplest one, which belongs to M. SCHIFFER [10]. More general case was considered by G. M. GOLUZIN [3]. Recently, there was discovered in the theory of representations of the infinite-dimensional groups that some kind of the variational formulae is closed, i.e. the corresponding variations form the Lie algebra, and, moreover, they canbe exponentiated to the action of an infinite-dimensional group. Namely, there was proved in the paper [11] that the group $\mathrm{Diff}_+ S^1$ of diffeomorphisms of a circle, preserving an orientation, transitively acts on S. The stabilizer of an arbitrary point is isomorphic to S^1. The infinitesimal action, counted in [12], gives the variational formulae on the class S, analogous to ones of M. SCHIFFER, as it was shown in [13]. Moreover, the formulae of boundary variations correspond to the action of singular fields. These formulae can be extended to the space $C_0[\![z]\!]$ by an analyticity. This fact of the complex analysis should be related to the recent discovery of the Virasoro algebra action on the universal Teichmüller space [14], obtained by M. L. KONCEVICH [15] and provoking in 1987-1988 a serie of investigations (YU. I. MANIN, A. A. BEILINSON, V. A. SCHECHTMAN, E. ARBARELLO, DE CONCINI, V. KAC, C. PROCESI, E. WITTEN, L. ALVAREZ-GAUMÉ, C. GOMEZ, C. REINA, G. MOORE, C. VAFA, A. MOROSOV, A. S. SCHWARZ et al.).

So the domain S is an infinite dimensional homogeneous domain. As an abstract homogeneous space $\mathrm{Diff}_+ S^1/S^1$ appeared in the mathematical literature at the end of the seventies—the beginning of the eighties.It appeared in the paper [16], devoted to the theory of the Hill's equation [17], as a coadjoint orbit of the Virasoro-Bott group. As such it is the symplectic manifold (see [18]). The various modes of such representation define the two-parameter family of symplectic structures $\omega_{h,c}$ [19, 20]. Coupled with the complex structure $\{\omega_{h,c}\}$ forms the family of the Kähler metrics $w_{h,c}$ [11]. This construction is reminiscent of the Cayley-Klein realization of the Bolyai-Lobachevsky space. One can get more detail information from the papers [12, 13].

The Kähler metrics, as it will be proved below, are the infinite dimensional analogs of Bergman metrics on bounded homogeneous domains (see [21]). It should be also mentioned that there exists the unique invariant Kähler-Einstein metric on $\mathrm{Diff}_+ S^1/S^1$ [22, 23] with the ratio of parameters $c/h = 26$. If the metric $w_{h,c}$ isn't einsteinian, the group of all biholomorphic isometries of S coincides with $\mathrm{Diff}_+ S^1$ [24]. Moreover, we suppose that S is very like as the infinite dimensional circular domains which were investigated intensively in the seventies by N. WALLACH, R. GREENFIELD, J.-P. VIGUÉ, E. VESENTINI, H. UPMEIER, W. KAUP, R. BRAUN, J. M. ISIDRO et al. [25], so a biholomorphic automorphism of S, stabilizing the point $f_0(z) = z$, is linear in $C_0[\![z]\!]$, preserves the modules of the coefficients and coincides with a rotation of S^1. So $\mathrm{Diff}_+ S^1$ is also the group of all biholomorphic automorphisms of S and, therefore, the group of all biholomorphic isometries for the Einstein metric, too. Note that S is an example of an infinite dimensional homogeneous bounded domain that isn't a symmetric one.

Independently, the manifold $\mathrm{Diff}_+ S^1/S^1$ have appeared in string theories [22, 26, 23] as a geometrical background for the string field theory (W. SIEGEL, T. BANKS, M. PESKIN, E. WITTEN et al.). There is a voluminous literature devoted to this manifold. More detail information about the recent achievements in this direction one should get from the paper of the author [27].

Unfortunately, we have a very uncomplete information about the structure of the class S, especially in terms of the coefficients $c_1, c_2, c_3, \dots c_k, \dots$ (see f.e. the monograph [28]). To advance in such description we shall establish the new criterion of the univalency in terms of $c_1, c_2, c_3, \dots c_k, \dots$ using the Kodaira imbedding of the infinite dimensional manifold $\mathrm{Diff}_+ S^1/S^1$ into the infinite dimensional projective spaces.

2. – The necessary conditions of the univalency.

This paragraph is devoted to the brief review of the necessary conditions of the univalency.

The simplest one is the condition of the local univalency. A regular function $f(z)$ is univalent in some neibourhood of the point a, iff $f'(a) \neq 0$. Hence, if a function $f(z)$ belongs to the class S, its derivative turn into zero nowhere in the unit disk. Of course, the local univalency doesn't guarantee the global univalence of the function in the disk. For example, the function $f(z) = \exp(Rz)$, where $R > \pi$, isn't univalent in the disk, though the condition of the local univalency holds in every point of the disk.

The next condition is the geometrical one on the image of the boundary S^1 of the disk D_+. If the function $f(z)$ is univalent in D_+, then the image of S^1 is the Jordan curve. This condition is also sufficient.

In general, a necessary condition of the univalency is a property, as a rule an inequality, which an univalent function satisfies.

The most famous ones are the Bieberbach inequalities [8] on the Taylor coefficients

$c_1, c_2, c_3, \dots c_k, \dots$ of a function $f(z) = z + c_1 z^2 + c_2 z^3 + c_3 z^4 + \dots + c_k z^{k+1} + \dots$

$$|c_k| < k + 1.$$

One should mention that our notation differs from the common one.

In 1907 KOEBE proved that $|c_1| <$ const; in 1916 BIEBERBACH proved that $|c_1| < 2$ and conjected that $|c_n| < n + 1$; in 1923 LÖWNER proved the conjecture for $n = 2$; in 1925 LITTLEWOOD obtained the inequality $|c_n| < e(n + 1)$; in 1955 GARABEDIAN and SCHIFFER proved the Bieberbach inequalities for $n = 3$ and HAYMAN proved that $|c_n| < n + 1$, if $n \geqslant n^0(f)$; in 1967-1968 PEDERSON and OZAWA obtained the Bieberbach inequalities for $n = 5$ and in 1972 PEDERSON and SCHIFFER for $n = 4$; in 1972 FITZ-GERALD proved that $|c_n| < 1.081\ (n + 1)$ and in 1978 HOROWITZ that $|c_n| < 1.0657\ (n + 1)$; finally, in 1984 DE BRANGES proved the Bieberbach inequalities for all n.

The Bieberbach conjecture had given the tremendous impetus to the theory of univalent functions. There were devised a lot of methods to attack it, which formed the recent look of the theory.

Besides the Bieberbach inequalities there exist many other interesting ones. One of them is the Robertson inequality. Let $S^{(2)}$ be the class of all odd functions from S; that is the class of functions

$$f_2(z) = (f(z^2))^{1/2} = z + c_1^{(2)} z^3 + c_2^{(2)} z^5 + \dots + c_k^{(2)} z^{2k+1} + \dots$$

where f belongs to S.

The naïve analog of the Bieberbach conjecture as it was shown in 1933 by FEKETE and SZEGÖ is false. But in 1936 M. ROBERTSON conjectured that the inequality [29]

$$\sum_{k=1}^{n} |c_k^{(2)}|^2 < n$$

holds for every f from $S^{(2)}$. The Robertson inequalities implies the Bieberbach ones, since

$$c_n = \sum_{k=0}^{n} c_k^{(2)} c_{n-k}^{(2)}, \qquad c_0^{(2)} = 1$$

and therefore

$$|c_n| < \sum_{k=1}^{n} |c_k^{(2)}|^2 + 1.$$

The Robertson inequalities was given much less attention then the Bieberbach ones. Before their proof by DE BRANGES in 1984 they was proved only for $n = 2$ and $n = 3$.

The next class of inequalities is connected with the logarithmic coefficients of a function f from S; that

$$\log(f(z)/z) = \sum_{k=1}^{\infty} 2g_k z^k.$$

In 1965-1967 N. A. LEBEDEV and I. M. MILIN established the following inequality [30, 31]

$$1 + \sum_{k=1}^{n} |c_k^{(2)}|^2 < (n + 1) \exp\left((n + 1)^{-1} \sum_{k=1}^{n} (n + 1 - k)(k|g_k|^2 - k^{-1})\right).$$

From this inequality the Bieberbach and the Robertson ones follow modulo the next

$$\sum_{k=1}^{n} (n + 1 - k) k |g_k|^2 < \sum_{k=1}^{n} (n + 1 - k)/k$$

it is called the Lebedev-Milin inequality. It was proved by DE BRANGES in 1984.

Our brief review on the necessary conditions is finished and we turn to the criteria of the univalency.

3. – Criteria of the univalency.

There exist many criteria of the univalency but in general they are equivalent to the simplest one, proposed by H. GRUNSKY in 1939 [32]. It was improved by many authors during the fourties—the sixties (G. M. GOLUZIN, Z. NEHARI, I. SCHUR, M. SCHIFFER, CH. POMMERENKE, J. JENKINS et al.).

In the modern notation the Grunsky criterion has the form

$$E - Z_f \overline{Z}_f > 0$$

where Z_f is so-called Grunsky matrix. Z_f is infinite and symmetric. Its matrix elements are defined by the next formula

$$(Z_f)_{mn} = \sqrt{m/n}\, \det \oint \begin{bmatrix} z^{-m} \\ I \\ z \\ z^2 \\ \vdots \\ z^{n-1} \end{bmatrix} [1/f \dots 1/f^{n+1}]\, dz.$$

An another way to define the matrix elements is the next expansion

$$\log((f(z) - f(w))/(z - w)) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} C_{jk} z^j w^k.$$

Terms with $j \neq 0$, $k \neq 0$ coincide with the matrix coefficients of Z_f. C_{j0} and C_{0k} are the logarithmic coefficients.

The geometrical interpretation of the Grunsky criterion was given in [13]. Let H be the space of the real-valued smooth I-forms $u(\exp(it))\,dt$ on the circle such as

$$\int u(\exp(it))\,dt = 0\,.$$

Let H^c be its complexification, $H^c_+ (H^c_-)$—its transversal subspaces, containing I-forms $u(\exp(it))\,dt$, which have the holomorphic extensions in $D_+ (D_-)$. $H^c \simeq \simeq O(S^1)/\text{Const}$, namely $f(z) \in O(S^1) \to df(z) \in H^c$. $H^c_\pm \simeq O(D_\pm)$.

There are defined the symplectic and the pseudohermitean structures on H^c:

$$(f(z), g(z)) = \oint f(z)\,dg(z)\,, \qquad f(z), g(z) \in O(S^1)\,.$$
$$\langle f(z), g(z)\rangle = \oint f(z)\,\overline{dg(z)}\,,$$

Let $\mathrm{Sp}(H^c, C)$ and $U(H^c_+, H^c_-)$ be the group of the invariance of these structures, $\mathrm{Sp}(H, R) = \mathrm{Sp}(H^c, C) \cap U(H^c_+, H^c_-)$.

Let's consider the grassmannian $\mathrm{Gr}(H^c)$—the set of all complex Lagrange subspaces in H^c. $\mathrm{Gr}(H^c)$ is the infinite dimensional homogeneous space with the group of transformations $\mathrm{Sp}(H^c, C)$. The infinite-dimensional grassmannians were intensively investigated in the eighties after the paper of M. SATO [33] (see also [34]) (M. SATO, Y. SATO, E. DATE, M. JIMBO, M. KASHIWARA, T. MIWA, G. SEGAL, G. WILSON, A. PRESSLEY et al.).

Let's consider the action of $\mathrm{Sp}(H,R)$ on $\mathrm{Gr}(H^c)$. There exist the well-improved theory of actions of real semisimple groups on the homogeneous spaces of their complexifications (I. I. PYATETSKII-SHAPIRO, F. I. KARPELEVICH, M. I. GRAEV, J. A. WOLF). The orbit of the point H^c_- is the open subspace R in $\mathrm{Gr}(H^c)$, which is isomorphic to $\mathrm{Sp}(H,R)/U$.

The manifold R is an infinite dimensional classical homogeneous domain of the IIIrd type [35]. Such domains were investigated during the seventies—the eighties by N. WALLACH, R. GREENFIELD, E. VESENTINI, J.-P. VIGUÉ, H. UPMEIER, W. KAUP, R. BRAUN, J. M. ISIDRO, L. L. STACHO, L. A. HARRIS et al. R can be imbedded in $\mathrm{Hom}(H^c_-, H^c_+)$ so that the elements of R are represented by the symmetric matrices Z such as $E - Z\overline{Z} > 0$.

The representation of $\mathrm{Diff}_+ S^1$ in H^c determines the monomorphism $\mathrm{Diff}_+ S^1 \to \to \mathrm{Sp}(H, R)$. Hence, $\mathrm{Diff}_+ S^1$ acts in R. This action's orbit of the initial point coincides with $\mathrm{Diff}_+ (S^1)/\mathrm{PSL}(2, R)$. Therefore, we has the mapping $M \to R$. There corresponds the subspace $V_f = \{zf'(z)\cdot f^{-1-k}(z)\,dt,\ k \geq 1\}$ to the point $f(z)$ of M.

The matrix corresponding to V_f is the Grunsky matrix. The mapping $f \to V_f$ is so-called Krichever mapping, constructed by him in 1977.

The Grunsky criterion immediately follows from the construction above. It seems after a modification it would be possible to obtain the Lebedev-Milin inequalities from it.

As it was said above there exist many other criteria besides the Grunsky one as Goluzin criterion or Nehari criterion, but all of them are the modifications and the generalisation of the Grunsky one.

4. – Some facts from the representation theory of the infinite dimensional groups, infinite dimensional geometry and harmonic analysis.

It seems to be convenient to list in this paragraph some necessary facts from the representation theory of the infinite dimensional groups, the infinite dimensional geometry and the harmonic analysis which are concerned the group of differomorphisms of a circle, its Lie algebra—the algebra of vector fields on a circle, their central extensions—the Virasoro-Bott group and the Virasoro algebra, the flag manifold of the Virasoro-Bott group $M = \mathrm{Diff}_+ S^1/S^1$.

As $\mathrm{Diff}_+ S^1$ we denote the group of all smooth diffeomorphisms of a circle, preserving an orientation. $\mathrm{Vect}\,S^1$—its Lie algebra consisted of all smooth vector fields $v(\exp(it))\,d/dt$ on the circle S^1 with the commutator

$$[v(\exp(it))\,d/dt,\ u(\exp(it))\,d/dt] =$$
$$= i\exp(it)(v(\exp(it))\,u'(\exp(it))) - v'(\exp(it))\,u(\exp(it))\,d/dt\,.$$

Let $\mathrm{CVect}\,S^1$ be its complexification. In the basis

$$e_k = i\exp(ikt)\,d/dt\,, \qquad k = \ldots -2,\ -1,\ 0,\ 1,\ 2,\ \ldots$$

the commutation relations have the form

$$[e_n, e_m] = (n - m)\,e_{n+m}\,.$$

I. M. GELFAND and D. B. FUCHS discovered in [36] that $\mathrm{CVect}\,S^1$ has the unique central extension. Independently, such result was obtained by M. VIRASORO [37]. The commutation relations for the Virasoro algebra have the form

$$[L_m, L_n] = (m - n)\,L_{m+n} + (m^3 - m)/12\cdot Z$$

where Z is the central element.

There corresponds the infinite dimensional group Vir (the Virasoro-Bott group) to the algebra vir as it was shown by R. BOTT [38]. There no correspond any groups to algebras $\mathrm{CVect}\,S^1$ and Cvir.

The algebras $\mathrm{CVect}\,S^1$ and Cvir are graded algebras. For them there were defined the Verma modules [39], investigated by many authors (V. G. KAC, B. L. FEIGIN, D. B. FUCHS et al.).

Now remind some facts from the Kähler geometry and the harmonic analysis on $\mathrm{Diff}_+ S^1/S^1$:

1) Each Kähler metric $w_{h,c}$ on $\mathrm{Diff}_+ S^1/S^1$ has the Kähler potential $K_{h,c}$, i.e. such real-valued function on $\mathrm{Diff}_+ S^1/S^1$ that

$$\partial_\mu \bar{\partial}_\nu K_{h,c} = (w_{h,c})_{\mu\bar{\nu}}.$$

The potential $K_{h,c}$ equals to $hK^1 + cK^2$, where K^1 and K^2 are the Kähler potentials of metrics $w_{1,0}$ and $w_{0,1}$, respectively. K^1 was calculated in [12]. It equals to $\log|g'(\infty)|$, where g is the conformal mapping from D_- onto $C \setminus f(D_+)$ such as $g(\infty) = \infty$. K^2 was calculated in [13]. It is equal to $\log\det(E - Z_f \bar{Z}_f)$.

2) There corresponds to each Kähler metric $w_{h,c}$ on $\mathrm{Diff}_+ S^1/S^1$ the line holomorphic bundle with the next properties [13]:

a) the algebra $\mathrm{CVect}\, S^1$ or Cvir acts holomorphically on the bundle $E_{h,c}$;

b) $E_{h,c}$ is the hermitean bundle with the metric $g_{h,c}(f) = \exp(-K_{h,c}(f))\, d\lambda\, d\bar{\lambda}$, where λ is the coordinate in a fiber;

c) the curvature of the hermitean connection $\nabla_{h,c}$ in $E_{h,c}$ is equal to $2\pi i w_{h,c}$, the algebra Cvir acts in $E_{h,c}$ by the covariant derivatives.

3) Let $O(E_{h,c})$ be the space of the polynomial sections of $E_{h,c}$ [13]. $O(E_{h,c})$ is the graded Cvir-module. Let $O'(E_{h,c})$ be the space of the linear functionals p on $O(E_{h,c})$ with the property; if $p(x) \neq 0$ then $\deg(x) < \infty$. $O'(E_{h,c})$ is the Verma module for Cvir [13].

4) Let's fixe the basis

$$e^{a_1, a_2, \ldots a_n} = c_1^{a_1} c_2^{a_2} \ldots c_n^{a_n}$$

in $O(E_{h,c})$ in the trivialisation of [13]. The Virasoro algebra acts in it by the formulae [13]

$$L_p = \partial_p + \sum_{k \geq 1} (k+1)\, c_k\, \partial_{k+p}, \qquad p > 0,$$
$$L_0 = \sum_{k \geq 1} k c_k \partial_k + h,$$
$$L_{-1} = \sum_{k \geq 1} ((k+2)\, c_{k+1} - 2c_1 c_k)\, \partial_k + 2hc_1,$$
$$L_{-2} = \sum_{k \geq 1} ((k+3)\, c_{k+2} - (4c_2 - c_1^2)\, c_k - b_k)\, \partial_k + h(4c_2 - c_1^2) + 0.5c(c_2 - c_1^2),$$

where b_k are the Laurent coefficients of $1/f$.

Let's also fixe a basis

$$e_{a_1, a_2, \ldots a_n} = c_1^{a_1} c_2^{a_2} \ldots c_n^{a_n},$$

in $O'(E_{h,c})$ dual to the basis $e^{a_1, a_2, \ldots a_n}$. The Virasoro algebra generators in it has the

form (13)

$$L_{-p} = c_p + \sum_{k \geq 1} (k+1)\, c_{k+p}\, \partial_k, \qquad p > 0,$$

$$L_0 = \sum_{k \geq 1} k c_k \partial_k + h,$$

$$L_1 = \sum_{k \geq 1} (k+2)\, c_k\, \partial_{k+1} - 2 \sum_{k \geq 1} c_k \partial_1 \partial_k + 2h\partial_1,$$

$$L_2 = \sum_{k \geq 1} (k+3)\, c_k\, \partial_{k+2} - 4 \sum_{k \geq 1} c_k \partial_2 \partial_k +$$
$$+ \sum_{k \geq 1} c_k \partial_1^2 \partial_k - \sum_{k \geq 1} c_k b_k (\partial_1 \ldots \partial_m \ldots) + h(4\partial_2 - \partial_1^2) + 0.5c(\partial_2^2 - \partial_1^2).$$

5) On the Verma module $O'(E_{h,c}) = V_{h,c}$ there exists the invariant hermitean form (the Shapovalov form [40]). The factor of $V_{h,c}$ by the kernel of the Shapovalov form is the irreducible module $L_{h,c}$ with the highest weight (B. L. FEIGIN, D. B. FUCHS). $V_{h,c}$ is unitarisable iff $c > 1$, $h > 0$. $L_{h,c}$ is unitarisable iff

$$c \geq 1, \quad h \geq 0 \quad \text{or} \quad c = 1 - \frac{6}{p(p+1)}, \qquad h = \frac{(Ap - B(p+1))^2 - 1}{4p(p+1)},$$

A, B, p-integers, $p \geq 2$, $1 \leq A \leq p$, $1 \leq B \leq p-1$ (YU. A. NERETIN, D. FRIEDAN, Z. QIU, S. SHENKER, P. GODDARD, A. KENT, D. OLIVE) (see f.e. [41]).

5. – The new criterium of the univalency.

LEMMA 1. – There exists the imbedding (the Kodaira imbedding) of $\mathrm{Diff}_+ S^1/S^1$ into $P(V_{h,c}^{\mathrm{form}})$, where $V_{h,c}^{\mathrm{form}}$ is the space of all functionals on $O(E_{h,c})$:

$$m \in \mathrm{Diff}_+ S^1/S^1 \to p \in V_{h,c}^{\mathrm{form}} : \forall x \in O(E_{h,c}) \quad ((x(m) = 0) \Rightarrow (p(x) = 0)).$$

The Kodaira imbedding is $\mathrm{Diff}_+ S^1$—equivariant and isometric.

PROOF. – The lemma is an infinite dimensional analog of the A. Borel and A. Weil theorem, proved in 1954 for the compact semisimple Lie groups, and the M. Atiyah and W. Schmid theorem, proved in 1977 for the non-compact semisimple real Lie groups. There are no any difference between their proof and proof of this lemma, so we doesn't repeat the classical arguments.

LEMMA 2. – The Kodaira imbedding in the basis $e_{a_1, a_2 \dots a_n}$ has the form

$$f(z) \to \sum_{n \geq 0} T_n(c_1, c_2, c_3, \dots c_k, \dots),$$

$$T_n(c_1, c_2, \dots c_n, \dots) = \sum_{a_1 + \dots + m a_m = n} \frac{c_1^{a_1} c_2^{a_2} \dots c_n^{a_n}}{a_1!\, a_2! \dots a_n!} e_{a_1 \dots a_n}.$$

The Kodaira imbedding can be extended to the holomorphic imbedding of $C_0[\![z]\!]$ in $P(V_{h,c}^{\text{form}})$.

PROOF. – The statement evidently follows from the formula of Lemma 1 and the definition of the basis $e_{a_1} \dots a_n$.

REMARK. – The mapping of the analogous form has appeared in mathematics many times in the finite dimensional case. Probably, the first example of such formula was written by G. VERONESE.

THEOREM (The criterion of the univalency). – Let $A_{a_1 \dots a_n \bar{b}_1 \dots \bar{b}_m}$ be the coefficients of the Shapovalov form for the unitarisable Verma module $V_{h,c}$ for the Virasoro algebra. The function

$$f(z) = z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$$

is univalent in the unit disk $|z| \leq 1$ iff

$$\sum_{k=0}^{\infty} \sum_{\substack{a_1 + \dots + n a_n = k \\ b_1 + \dots + m b_m = k}} A_{a_1 \dots a_n, \bar{b}_1 \dots \bar{b}_m} \frac{c_1^{a_1} \dots c_n^{a_n} \bar{c}_1^{b_1} \dots \bar{c}_m^{b_m}}{a_1! \dots a_n!\, b_1! \dots b_m!} < \infty .$$

PROOF. – One should mention that the sum of the left side of the inequality is the Bergman kernfunction for S, which coincides with $\exp(K_{h,c})$. It is evident that the kernfunction is finite on S. The statement of the theorem follows from the same arguments as for the theorem of the non-existence of the finite-dimensional unitary representations for a real semisimple non-compact Lie group. Namely, because of the infinite length of geodesics on S in the case $c > 1$, $h > 0$ the boundary value of the Kähler potential is infinite, hence the values of the kernfunction is infinite, too.

REMARK. – This statement remains correct after the modification of the proof for the Verma modules with the unitary factors. For example, the case $c = 1$ and $h = 0$ gives the Grunsky criterion.

REMARK. – In the case $c > 1$, $h > 0$ the criterion is also the condition of the regularity, in the case $c = 1$, $h = 0$ it isn't so.

6. – The radius of the univalency.

The problem of the determining of the regularity radius of the power series was solved in general by O. CHAUCHY in [42] and in full by J. HADAMARD in 1892 [43]. The answer is the next: the regularity radius of the series

$$z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$$

might be calculated by the formula

$$R = (\limsup_{n \to \infty} (|c_n|)^{1/n})^{-1}.$$

Certainly, the problem of the determining of the univalency radius is less known. Till the nowadays there wasn't obtained such clear formula for the univalency radius. In this paragraph we propose a variant of such formula which has the Cauchy-Hadamard-like type.

It should be convenient to mention the known results. The first belongs to E. LANDAU [44]: if $f(z) < M$ in D_+ and $M > 1$, then

$$R \geq M - (M^2 - 1)^{1/2}.$$

Some interesting estimates were obtained by T. UMEZAWA, Z. NEHARI, G. V. KUZMINA (see f.e. [45]) and other authors.

THEOREM 2. – The univalency radius of $z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$ equals to the regularity radius of $z + d_1 z^2 + d_2 z^5 + \dots + d_n z^{n+1}$, where

$$d_k = \sum_{\substack{a_1 + \dots + n a_n = k \\ b_1 + \dots + m b_m = k}} A_{a_1 \dots a_n \bar{b}_1 \dots \bar{b}_m} \frac{c_1^{a_1} \dots c_n^{a_n} \bar{c}_1^{b_1} \dots \bar{c}_m^{b_m}}{a_1! \dots a_n!\, b_1! \dots b_m!}.$$

PROOF. – Considering the normal coordinates on S: $c_n(t) = c_n t^n$ and using the first theorem we obtain the statement of the Theorem 2.

COROLLARY. – The radius of the univalency of the function

$$f(z) = z + c_1 x^2 + c_2 z^3 + c_3 z^4 + \dots + c_n z^{n+1} + \dots$$

is equal to

$$\left(\limsup_{n \to \infty} \left(\sum_{\substack{a_1 + \dots + m a_m = n \\ b_1 + \dots + k b_k = n}} A_{a_1 \dots a_m \bar{b}_1 \dots \bar{b}_k} \frac{c_1^{a_1} \dots c_m^{a_m} \bar{c}_1^{b_1} \dots \bar{c}_k^{b_k}}{a_1! \dots a_m!\, b_1! \dots b_k!} \right)^{1/n} \right)^{-1/2}.$$

7. – The numerical computation of the univalency radius.

All known methods of the univalency radius computations be are based on the algorithms which are executed for each individual function [45].

Our method contains two parts. The first one is the computation of the Shapovalov form. The second one is the formation of the series $z + d_1 z^3 + d_2 z^5 + \ldots + d_n z^{2n+1} + \ldots$ for an individual function and its regularity radius determining.

In principle the most complicated part is the first one. But it can be executed one time and the coefficients of the Shapovalov form may be tabulated. For this purpose one should explore the Shapovalov form invariance. Then, the formulae for the Virasoro algebra action in $V_{h,c}$ give the recurrent formulae for the Shapovalov form coefficients.

It is very difficult problem to investigate the asymptotic properties of the Shapovalov form for the determining of the rate of the convergence of the formula for the regularity radius of the series

$$z + d_1 z^3 + d_2 z^5 + \ldots + d_n z^{2n+1} + \ldots .$$

8. – Aknowledgments.

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P.S. – The Theorem 1 was announced without any proof in [46].

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Méthode de compacité et de décomposition applications: minimisation, convergence des martingales, lemma de Fatou multivoque (*).

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Summary. - *New results of decomposition for bounded sequences in L^1_E and in the space of integrably bounded multifunctions with non empty convex weakly compact values in a Banach space E and its applications to problems of Minimization, convergence of martingales, Multivalued Fatou lemma are presented.*

0. - Introduction.

On se propose de présenter dans ce papier quelques nouveaux resultats de décomposition pour une suite bornée dans L^1_E et dans l'espace $\mathcal{L}^1_{\mathrm{cfk}(E)}$ des multifonctions intégrablement bornées à valeurs convexes faiblement compactes non vides d'un espace de Banach séparable E. Ces résultats permettent d'obtenir de nouveaux résultats de minimization, de convergence des martingales, du lemme de Fatou multivoque.

1. - Notations.

Soit $(\Omega, \mathcal{F}, P)$ un espace probabilisé complet, E un espace de Banach séparable. On désigne par $\mathrm{cfk}(E)$ l'ensemble des parties convexes faiblement compactes non vides de E et par $\mathcal{L}^1_{\mathrm{cfk}(E)}(\mathcal{F})$ l'espace des multifonctions $\mathcal{F}$-mesurables X de Ω dans $\mathrm{cfk}(E)$ telles que $|X|: \omega \mapsto \sup\{\|u\|: u \in X(\omega)\}$ soit intégrable sur Ω. Un ensemble H dans $\mathcal{L}^1_{\mathrm{cfk}(E)}(\mathcal{F})$ est *bornée* (resp. *uniformément intégrable*) si l'ensemble $\{|X|: X \in H\}$ est *borné* (resp. *uniformément intégrable*) dans $L^1_{\mathbb{R}}(\mathcal{F})$. Enfin, on

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