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A general minimax theorem

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In the present note we prove a minimax theorem using the ideas of the papers [1–4]. Our theorem gives a generalization of the well-known theorem of Brezis – Nirenberg – Stampacchia [5] and also of a recent of G. Horváth [6]. To formulate our result we need the following

Definitions. A topological space Y is said to be an interval space [4] if there is a mapping $[.,.] : Y \times Y \to P(Y)$ such that $y_1, y_2 \in [y_1, y_2]$ and $[y_1, y_2]$ is connected and closed for each $y_1, y_2 \in Y$.

Let X be any nonempty set, Y an interval space and $f : X \times Y \to \mathbf{R}$ a function. f is said to be lower-semicontinuous on intervals of Y if for all $x \in X$ the restriction of the function $y \to f(x, y)$ to any interval $[y_1, y_2]$ of Y is lower-semicontinuous in the subspace topology of $[y_1, y_2]$. The upper-semicontinuity on intervals is defined similarly.

Let X be a topological space, Y an interval space and $f : X \times Y \to \mathbf{R}$ any function. f is said to be a Ky-Fan function if

(i) for every $y \in Y$, $x \to f(x, y)$ is lower-semicontinuous on X;

(ii) for every $x \in X$, $y \to f(x, y)$ is quasiconcave on Y, i.e. for every $c \in \mathbf{R}$ the set $\{y \in Y : f(x, y) \ge c\}$ is convex or empty.

In this paper we prove the following

Theorem. *Let X be any compact topological space, Y any interval space and $f : X \times Y \to \mathbf{R}$ any Ky-Fan function, which is upper-semicontinuous on every interval of Y and such that for each $y_1, y_2, \dots y_n \in Y$ and $a \in \mathbf{R}$ the set $\bigcap_{i=1}^{n} \{x \in X : f(x, y_i) \le a\}$ is connected or empty. Let $c \in \mathbf{R}$ be arbitrary. Then*

(i) if $c \ge c_*$ then there exists an $x_0 \in X$ such that $f(x_0, y) \le c$ for all $y \in Y$;

(ii) if $c < c_*$, then there exists $y_0 \in Y$ such that $f(x, y_0) > c$ for every $x \in X$, where $c_* := \sup_y \inf_x f(x, y)$.

Proof. (i) For each $c \geq c_*$ and $y \in Y$ define $H_y^c := \{x \in X : f(x, y) \leq c\}$. We shall prove that $\bigcap_{y \in Y} H_y^c \neq \emptyset$ for every $c \geq c_*$, which is equivalent to (i). To this consider first the case $c > c_*$. It is easy to see that for every $c > c_*$ and $y \in Y$ $H_y^c \neq \emptyset$. By the hypothesis, the sets H_y^c are closed, hence they are compact. Hence it is enough to prove that the family $\{H_y^c : y \in Y\}$ has the finite intersection property. Suppose indeed that for some $n \in N$ the intersection of any n sets is nonempty for each $c > c_*$, but there exist $c_1 > c_*$ and $y_1, y_2, \ldots, y_{n+1}$ for which we have: $\bigcap_{i=1}^{n+1} H_{y_1}^{c_1} = \emptyset$. Define $K(y) := H_y^{c_1} \cap H_{y_3}^{c_1} \cap \cdots \cap H_{y_{n+1}}^{c_1}$ (if $n = 1$ then let $K(y) := H_y^{c_1}$) for any $y \in Y$. According to our induction hypothesis $K(y) \neq \emptyset$ for every $y \in Y$ and on the other hand $K(y_1) \cap K(y_2) = \emptyset$.

We shall show that for all $y \in [y_1, y_2]$, $K(y) \subset K(y_1) \cup K(y_2)$. Suppose that there exists $x \in K(y)$ such that $x \notin K(y_1)$ and $x \notin K(y_2)$. Then $f(x, y_1) > c_1$, $f(x, y_2) > c_1$. Now define $c_0 := \min\{f(x, y_1),\ f(x, y_2)\}$. Clearly $c_0 > c_1$ and since the set $\{y : f(x, y) \geq c_0\}$ is convex, $f(x, y) \geq c_0 > c_1$. Thus we obtain a contradiction with $x \in K(y)$. Next we prove that $K(y) \subset K(y_1)$ or $K(y) \subset K(y_2)$ for all $y \in [y_1, y_2]$. Indeed, if the closed sets $A := K(y) \cap K(y_1)$ and $B := K(y) \cap K(y_2)$ are nonempty, then the relation $K(y) = A \cup B$ contradicts the connectedness of $K(y)$. (We have $A \cap B = \emptyset$).

Now define the sets $S_i := \{y \in [y_1, y_2] :\ K(y) \subset K(y_i)\}$ for $i = 1, 2$. It is easy to see that S_i is nonempty, $y_i \in S_i$ $(i = 1, 2)$, $S_1 \cup S_2 = [y_1, y_2]$ and $S_1 \cap S_2 = \emptyset$. We can show that both S_1 and S_2 must be closed in $[y_1, y_2]$, which contradicts the connectedness of $[y_1, y_2]$. Indeed, $y \in S_1 \iff K(y) \cap K(y_2) = \emptyset \iff \{x \in K(y_2) : f(x, y) \leq c_1\} = \emptyset$. Take a number c_2 such that $c_* < c_2 < c_1$. Then $\{x : f(x, y) \leq c_2\} \cap \left(\bigcap_{i=3}^{n+1} \{x : f(x, y_i) \leq c_2\}\right) \subset \{x : f(x, y) < c_1\} \cap \left(\bigcap_{i=3}^{n+1} \{x : f(x, y_i) \leq c_1\}\right) \subset K(y)$. The first set is nonempty since $c_2 > c_*$. Hence $K(y_2) \cap \{x : f(x, y) < c_1\} = \emptyset$ implies $K(y) \cap K(y_2) = \emptyset$, taking into account that we have $K(y) \subset K(y_1)$ or $K(y) \subset K(y_2)$. From this it follows that $\{x \in K(y_2) : f(x, y) \leq c_1\} = \emptyset \iff \{x \in K(y_2) : f(x, y) < c_1\} = \emptyset$. Then

$$S_1 = \{y \in [y_1, y_2] : f(x, y) \geq c_1 \text{ for all } x \in K(y_2)\} =$$

$$= \bigcap_{x\in K(y_2)} \big(\{y \in Y:\ f(x,y) \ge c_1\} \cap [y_1, y_2]\big)$$

which, by hypothesis, is closed. In the same way we get that S_2 is closed. This contradiction shows that for all $c > c_*$ $\bigcap_{y\in Y} H_y^c \neq \emptyset$.

Now consider the sets $H_y^{c_*} = \{x \in X : f(x,y) \le c_*\}$. Since X is compact and $x \to f(x,y)$ is lower-semicontinuous for each $y \in Y$, hence for each $y \in Y$ there exists an $x_y \in X$ such that $f(x_y, y) = \inf_x f(x,y)$, or in other words $x_y \in H_y^{c_*}$. Thus $H_y^{c_*} \neq \emptyset$ for each $y \in Y$. Observe that $H_y^{c_*} = \bigcap_{\varepsilon>0} H_y^{c_*+\varepsilon}$ and by the above statement $M_\varepsilon := \bigcap_{y\in Y} H_y^{c_*+\varepsilon} \neq \emptyset$ for all $\varepsilon > 0$. Since M_ε is compact, the intersection of the decreasing net $\{M_\varepsilon : \varepsilon > 0\}$ is nonempty, i.e. $\bigcap_{y\in Y} H_y^{c_*} \neq \emptyset$. This completes the proof of (i).

(ii) If $c < c_*$, then by definition there exists $y_0 \in Y$ such that $\inf_x f(x, y_0) > c$. Therefore $f(x, y_0) > c$ for all $x \in X$. The proof of the theorem is complete. □

Corollary 1. *Under the conditions of the theorem we have*

$$\sup_y \inf_x f(x,y) = \inf_x \sup_y f(x,y)$$

namely this follows from (i) *if we take* $c = c_*$.

Corollary 2. (C. Horváth [6], Theorem 1) *Let X be a compact topological space, Y a convex subset of a vector space and $f : X \times Y \to$ $\mathbf{R}$ a Ky-Fan function which is upper-semicontinuous on the segments of Y and such that for every finite subset F or Y and every $c \in \mathbf{R}$, $\bigcap_{y\in F} \{x \in X : f(x,y) \le c\}$ is connected or empty. Then*

(A) for every $c \in \mathbf{R}$ exactly one of the following statements holds:
  (i) there exists $x_0 \in X$ such that $f(x_0, y) \le c$ for every $y \in Y$,
  (ii) there exists $y_0 \in Y$ such that $f(x, y_0) > c$ for every $x \in X$.

(B) $\inf_x \sup_y f(x,y) = \sup_y \inf_x f(x,y)$.

Proof. In Y define the topology on the segments in the usual way. Then if $f : X \times Y \to \mathbf{R}$ is upper-semicontinuous on the segments of Y i.e. for every $x \in X$ and $y_1, y_2 \in Y$ the function

$$t \to f\big(x, (1-t)y_1 + t\,y_2\big)$$

is upper-semicontinuous on [0, 1], then f is upper-semicontinuous on intervals in the sense of our definition.

Remark 1. Our theorem generalizes Theorem 1 of [6] in two directions: the statement is established for classes of more general interval spaces and the alternative given by C. Horváth is precised.

Remark 2. In particular, our result contains the minimax theorem of H. Brézis, L. Nirenberg and G. Stampacchia [5] (see also [6]).

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