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ON SOME CONVEXITIES

I. JOÓ (Budapest)

The aim of the present paper is to investigate some convexities. First we introduce and investigate the "pseudoconvex" spaces and give a minimax theorem applying the ideas of [8].

The notion of convex spaces was introduced by Komiya [1] (see below for details). In [2] the authors proved a Nikaidó—Isoda-type theorem for convex spaces. An interesting convexity notion, not giving a convex space in the sense of [1] is obtained in [5]. The paper [4] contains investigations with respect to this convexity structure, analogous to that of in [2]. Now we give a common generalization of the convexities of [1] and [5], the so-called pseudoconvex space. It turns out that the compact pseudoconvex spaces have the fixed point property and a Nikaido—Isoda-type theorem holds. First we prove these assertions and after we prove an inequality between the Helly and Caratheodory number of pseudoconvex spaces. In connection with these investigations we mention the work of M. Horváth [3] where the Helly, Caratheodory and Radon numbers of a special convexity structure are calculated and whose proof is based on graph theoretical results. At last we continue the investigation of V. Komornik [10] and give a related minimax theorem in interval space.

1. DEFINITION 1. Let X be a point set. A mapping $\langle\cdot\rangle$: $P(X)\to P(X)$ is called convex hull operation if it satisfies the following conditions:

(1) $$\langle\emptyset\rangle=\emptyset,\quad \langle\{x\}\rangle=\{x\}\quad (x\in X)$$

(2) $$\langle A\rangle=\bigcup\{\langle F\rangle:\ F\subset A \text{ is a finite set}\},\quad (A\subset X)$$

(3) $$\langle\langle A\rangle\rangle=\langle A\rangle.$$

The convex hull operation defines a convexity structure on the set X; the set $A\subset X$ is called convex if $A=\langle A\rangle$.

DEFINITION 2 ([1]). A convex space is a triple $(X, \langle\cdot\rangle, \Phi)$, such that

a) X is a topological space and $\langle\cdot\rangle$ is a convex hull operation on it,

b) $\Phi=\{\varphi_F:\ F\subset A$ is finite$\}$, where φ_F: $\langle F\rangle\to\mathbf{R}^n$ $n:=\operatorname{card} F$ is a homeomorphic imbedding which is convex hull-preserving; that is $A\subset\langle F\rangle$ implies $\varphi(\langle A\rangle)=\operatorname{co}\varphi(A)$ where co denotes the usual convex hull in $\mathbf{R}^n$.

We shall considerably weaken this notion:

DEFINITION 3. A pseudoconvex space is a triple $(X, \langle\cdot\rangle, \Phi)$ such that

a) X is a topological space, and $\langle\cdot\rangle$ is a convex hull operation on it,

b) $\Phi=\{\varphi_F:\ F\subset A$ is finite$\}$, where φ_F: $\Delta^n\to\langle x_0, \dots, x_n\rangle$, $n=\operatorname{card}|F|-1$ is a continuous mapping of Δ^n onto $\langle F\rangle=\langle x_0, \ldots, x_n\rangle$; here Δ^n denotes the standard simplex of $\mathbf{R}^n$, i.e. $\Delta^n=(e_0, \ldots, e_n)$, where $e_0=(0, \ldots, 0)$, $e_1=(1, 0, \ldots, 0)$, $\ldots$, $e_n=(0, \ldots, 0, 1)$; further φ_F is convex hull-preserving in the (weakened) sense that for all subsimplex $(e_{i_0}, \ldots, e_{i_k})\subset(e_0, \ldots, e_n)$

$$\varphi_F\big((e_{i_0}, \ldots, e_{i_k})\big) = \langle x_{i_0}, \ldots, x_{i_k}\rangle. \tag{4}$$

In [5] the following convexity is defined on $\mathbf{R}^n$. Let $x=(x_0, \ldots, x_n)$ and $y=(y_0, \ldots, y_n)\in\mathbf{R}^n$. We shall give the interval $\langle x, y\rangle$ joining them as a polygon with at most $n+1$ pairwise orthogonal segments as follows. If $x_n\geqq y_n$ then let $I_n=\{(x_0, \ldots, x_{n-1}, t)\colon x_n\geqq t\geqq y_n\}$ and let $x'=(x_0, \ldots, x_{n-1}, y_n)$ be the other endpoint of I_n. If $x_n\leqq y_n$ then let $I_n=\{(y_0, \ldots, y_{n-1}, t)\colon x_n\leqq t\leqq y_n\}$ and $y'=(y_0, \ldots, y_{n-1}, x_n)$. In the first case we get I_{n-1} analogously to I_n: if, for example $x_{n-1}\leqq y_{n-1}$ then $I_{n-1}=\{(y_0, \ldots, y_{n-2}, t, y_n)\colon x_{n-1}\leqq t\leqq y_{n-1}\}$ and $y''=(y_0, \ldots, y_{n-2}, x_{n-1}, y_n)$; if $x_{n-1}>y_{n-1}$ then $I_{n-1}=\{(x_0, \ldots, x_{n-2}, t, y_n)\colon x_{n-1}\geqq t\geqq y_{n-1}\}$ and $x''=(x_0, \ldots, x_{n-2}, y_{n-1}, y_n)$. In the second case $(x_n\leqq y_n)$ we construct analogously I_{n-1} and in the third step I_{n-2} etc. Finally, the segments $I_0, \ldots, I_n$, parallel to the axis $x_0, \ldots, x_n$ resp. will join x and y (possibly not in the order of the indices). Now let a set $K\subset\mathbf{R}^n$ be convex if $x, y\in K$ implies $\langle x, y\rangle\subset K$. Then $\mathbf{R}^n$ endowed with this convexity is not a convex space; for example there are points $x, y, z\in\mathbf{R}^n$ such that $[x, y]$ and $[x, z]$ have common segments and no one is contained in the other. Nevertheless we can assert

LEMMA 1. *The space $\mathbf{R}^n$ with the convexity introduced in* [5] *is a pseudoconvex space.*

PROOF. Let $F=\{x_0, \ldots, x_k\}$; we have to give a mapping $\varphi_F\colon \Delta^k\to\langle x_0, \ldots, x_k\rangle$ with the desired properties. Let $(\lambda_0, \ldots, \lambda_k)\in\Delta^k$ and suppose that $\lambda_0=\ldots=\lambda_{j-1}=0$, $\lambda_j>0$. Let $y_j\in\mathbf{R}^n$ be the point of $\langle x_j, x_{j+1}\rangle$ which divides the length $|I_0|+\ldots+|I_n|$ of the polygon $\langle x_j, x_{j+1}\rangle$ going from x_j to x_{j+1} in proportion λ_{j+1}/λ_j; that is $|\langle x_j, y_j\rangle|\cdot\lambda_j=|\langle y_j, x_{j+1}\rangle|\cdot\lambda_{j+1}$. Continue the process inductively: if y_l is given then let $y_{l+1}\in\langle y_l, y_{l+2}\rangle$ be the point dividing the length of $\langle y_l, y_{l+2}\rangle$ in proportion $\lambda_{l+2}/(\lambda_0+\ldots+\lambda_{l+1})$. Finally set: $\varphi_F(\lambda_0, \ldots, \lambda_k):=y_{k-1}$. It can be seen from the construction that φ_F is continuous. Since $\langle x_0, \ldots, x_k\rangle=\bigcup\{[x_0, z]\colon z\in\langle x_1, \ldots, x_k\rangle\}$, we see that φ_F is convex hull-preserving, too. The Lemma 1 is proved.

LEMMA 2. (The Browder fixed point theorem.) *Let $(X, \langle\cdot\rangle, \Phi)$ be a compact T_2 pseudoconvex space and $T\colon X\to P(X)$ be a mapping for which*

$$Tx \neq \emptyset \quad \textit{and convex for all} \quad x\in X, \tag{5}$$

$$T^{-1}y = \{x\in X\colon y\in Tx\} \quad \textit{is open in } X \textit{ for all } y\in X. \tag{6}$$

Then T has a fixed point, i.e. there is a point $x_0\in X$ for which $x_0\in Tx_0$.

PROOF. The open covering $\{T^{-1}y\colon y\in X\}$ of X contains a finite subcovering $X=\bigcup_{i=0}^{n} T^{-1}y_i$. Denote $A_i=T^{-1}y_i\cap\langle y_0, \ldots, y_n\rangle$. Then $A_0, \ldots, A_n$ is an open covering the space $\langle y_0, \ldots, y_n\rangle$ which is compact (it is the range of the continuous mapping $\varphi_F\colon \Delta^n\to\langle y_0, \ldots, y_n\rangle$, $F:=\{y_0, \ldots, y_n\}$). Consequently there exists a partition of unity subordinate to this covering (see [7]). In other words, there are continuous mappings $\beta_0, \ldots, \beta_n \colon \langle y_0, \ldots, y_n \rangle \to \mathbf{R}$ such that $\beta_i \geqq 0$, $\sum \beta_i \equiv 1$ and $\operatorname{supp} \beta_i \subset A_i$. Now let $g \colon \langle y_0, \ldots, y_n \rangle \to \Delta^n$, $z \to \sum_{i=0}^{n} \beta_i(z) e_i$. Then g is continuous so has a fixed point $t \in \Delta^n$. Using the notation $\varphi_F(t) =: x_0$ we have $\varphi_F(g(x_0)) = x_0$. Between the coordinates of $g(x_0)$ let $\beta_{i_0}(x_0), \ldots, \beta_{i_k}(x_0)$ be nonvanishing, the other coordinates vanishing. Then $x_0 = \varphi_F(g(x_0)) \in \langle y_0, \ldots, y_{i_k} \rangle$. On the other hand $x_0 \in A_{i_0} \cap \ldots \cap A_{i_k}$ implies $y_{i_0}, \ldots, y_{i_k} \in T x_0$. But $T x_0$ is a convex set, hence $x_0 \in \langle y_{i_0}, \ldots, y_{i_k} \rangle \subset T x_0$ as we asserted.

Definition 4. A global pseudoconvex space is a pseudoconvex space $(X, \langle \cdot \rangle, \Phi)$ with the additional property that if $F' = \{x_{i_0}, \ldots, x_{i_k}\} \subset F = \{x_0, \ldots, x_n\}$, then

$$\varphi_{F'} = \varphi_F \circ j \tag{7}$$

where the map $j \colon \Delta^k \to \Delta^n$ is the linear extension of the mapping $j \colon e_s \to e_{i_s}$ $(s = 0, 1, \ldots, k)$. The product of two global pseudoconvex spaces $(X_1, \langle \cdot \rangle_1, \Phi_1)$ and $(X_2, \langle \cdot \rangle_2, \Phi_2)$ is defined to be $(X, \langle \cdot \rangle, \Phi)$ where

a) $X = X_1 \times X_2$,

b) if $F = \{(x_0^1, x_0^2), \ldots, (x_n^1, x_n^2)\}$, $F_i = \{x_0^i, \ldots, x_n^i\}$ $(i = 1, 2)$ then $\varphi_F := \varphi_{F_1} \times \times \varphi_{F_2} \colon \Delta^n \to X$ (i.e. $\varphi_F(t) = (\varphi_{F_1}(t), \varphi_{F_2}(t))$ if $t \in \Delta^n$) and $\Phi = \{\varphi_F \colon F \subset X \text{ is finite}\}$.

c) The convex hull of finite sets $F \subset X$ are defined by $\langle F \rangle := \varphi_F(\Delta^n)$ $(n+1 = \operatorname{card} F)$ and the convex hull of any set $A \subset X$ is determined by (2).

It is obvious that the product space is indeed a global pseudoconvex space. As a consequence of the Lemma 2 we get the following generalization of [2, Theorem 3].

Theorem 1. *Let $(X, \langle \cdot \rangle_1, \Phi_1)$ and $(Y, \langle \cdot \rangle_2, \Phi_2)$ be compact global pseudoconvex spaces and let $f, g \colon X \times Y \to \mathbf{R}$ be continuous functions such that*

(8) *the functions $x \to f(x, y)$ are $\langle \cdot \rangle_1$-quasiconcave (i.e. the sets $\{x \colon f(x, y) \geqq c\}$ are $\langle \cdot \rangle_1$-convex for all $y \in Y$, $c \in \mathbf{R}$),*

(9) *the functions $y \to g(x, y)$ are $\langle \cdot \rangle_2$-quasiconcave. Then there is a saddle point, i.e. a point $(x_0, y_0) \in X \times Y$ for which $f(x_0, y_0) \geqq f(x, y_0)$ for all $x \in X$ and $g(x_0, y_0) \geqq g(x_0, y)$ for all $y \in Y$.*

The proof of Theorem 1 is analogous to that of Theorem 3 in [2]; we omit the details.

2. Definition 5 ([3]). The Helly number H of a convexity structure $\langle \cdot \rangle$ is the smallest natural number n for which the following assertion holds.

(10) If $K_1, \ldots, K_N$ are convex sets and any $n+1$ sets have a common point, then $\bigcap_{i=1}^{n} K_i \neq \emptyset$.

The Caratheodory number C of $\langle \cdot \rangle$ is the smallest n for which we have for any $A \subset X$

$$\langle A \rangle = \cup \{\langle F \rangle \colon F \subset A, \ \operatorname{card} F \leqq n+1\}. \tag{11}$$

We shall prove

THEOREM 2. *Let* $(A, \langle\cdot\rangle, \Phi)$ *be a pseudoconvex space and suppose that for any* $a_0, \ldots, a_n \in A$ *the set* $\langle a_0, \ldots, a_n\rangle$ *is a zero set (this means that there exists a continuous mapping* $f\colon A \to [0, 1]$ *for which* $(f=0)=\langle a_0, \ldots, a_n\rangle$*). Then the Helly number is not greater than the Caratheodory number.*

For the proof we need

LEMMA 3. *If* $f\colon \Delta^n \to \Delta^n$ *is continuous and*

$$f((e_{i_0}, \ldots, e_{i_k})) \subset (e_{i_0}, \ldots, e_{i_k}), \tag{12}$$

i.e. if f maps any subsimplex into itself, then

$$f(\Delta^n) = \Delta^n. \tag{13}$$

PROOF. Use induction on n. The case $n=1$ is trivial. Indirectly suppose that Lemma 3 does not hold and take the smallest number n for which it fails. Then f maps the boundary $\partial\Delta^n$ of Δ^n onto itself, hence there exists a point $x_0 \in \operatorname{int} \Delta^n$ and a number $\delta > 0$ such that $f(\Delta^n)$ does not contain points in the δ-neighbourhood of x_0. Projecting from x_0 the values of $f(\Delta^n)$ onto $\partial\Delta^n$ we can suppose that

$$f\colon \Delta^n \to \partial\Delta^n. \tag{14}$$

Next we verify the following relations:

$$\partial\Delta^n \cap f^{-1}(\operatorname{int}(e_{i_0}, \ldots, e_{i_s})) \subseteqq \operatorname{int}(e_{i_0}, \ldots, e_{i_s}), \tag{15}$$

$$\partial\Delta^n \cap f^{-1}((e_{i_0}, \ldots, e_{i_s})) = (e_{i_0}, \ldots, e_{i_s}); \tag{16}$$

here "int" denotes the interior of the simplex $(e_{i_0}, \ldots, e_{i_s})$ i.e.

$$(e_{i_0}, \ldots, e_{i_s}) \setminus \partial(e_{i_0}, \ldots, e_{i_s}).$$

Prove (15) indirectly. If $x \in \partial\Delta^n$, $f(x) \in \operatorname{int}(e_{i_0}, \ldots, e_{i_s})$ but $x \notin \operatorname{int}(e_{i_0}, \ldots, e_{i_s})$ then $x \in (e_{j_0}, \ldots, e_{j_r})$, where $(e_{j_0}, \ldots, e_{j_r}) \not\supseteqq (e_{i_0}, \ldots, e_{i_s})$. Consequently $f(x) \in (e_{j_0}, \ldots, e_{j_r})$ which is in contradiction with $f(x) \in \operatorname{int}(e_{i_0}, \ldots, e_{i_s})$. Hence (15) is proved. Taking together the relations of type (15) we get

$$\partial\Delta^n \cap f^{-1}((e_{i_0}, \ldots, e_{i_s})) \subset (e_{i_0}, \ldots, e_{i_s}).$$

On the other hand, if $x \in (e_{i_0}, \ldots, e_{i_s})$ then $x \in \partial\Delta^n$ and $f(x) \in (e_{i_0}, \ldots, e_{i_s})$, so the other inclusion (16) is also proved.

Denote by Δ_i^{n-1} the side of Δ^n opposite to its i-th vertex and let $G_i := f^{-1}(\partial\Delta^n \setminus \Delta_i^{n-1})$. Then $(G_i)_{i=0}^n$ defines an open covering of Δ^n. By the compactness of Δ^n there exists an $\varepsilon > 0$ (the Lebesgue number of this covering) such that any subset of Δ^n of diameter not greater than ε is contained in one element G_i of the covering. Take a triangulation K of Δ^n (cf. [6]) which is fine enough to ensure that the star $\operatorname{st} b_s$ of any vertex $b_s \in K$ has diameter $< \varepsilon$. Then for any vertex $b_s \in K$ there exists an i such that

$$\operatorname{st} b_s \subset G_i. \tag{17}$$

Denote $\psi\colon b_s \to i$ with any i satisfying (17). We know that

(18) $$b_s \in (e_{i_0}, \ldots, e_{i_r}), \quad \text{st}\, b_s \subset G_i \quad \text{implies} \quad i \in \{i_0, \ldots, i_r\}.$$

Indeed, if $i \notin \{i_0, \ldots, i_r\}$, then $(e_{i_0}, \ldots, e_{i_r}) \subset \Delta_i^n$ so $\partial\Delta^n \cap f^{-1}((e_{i_0}, \ldots, e_{i_r})) \cap G_i = \emptyset$; now it follows from (16) that $(e_{i_0}, \ldots, e_{i_r}) \cap G = \emptyset$ in contradiction with $b_s \in (e_{i_0}, \ldots, e_{i_r})$ and $\text{st}\, b_s \subset G_i$. The property (18) enables us to make use the well-known Sperner lemma (cf. [6]) which asserts that if the colouring ψ of the vertices of the triangulation K satisfies (18) then K contains a simplex $G = (b_{s_0}, \ldots, b_{s_n}) \in K$ colored with different colours. This means $G \subset \text{st}\, b_{s_i} \subset G_i$ $(i = 0, \ldots, n)$, but this implies $G \subset \bigcap_{i=0}^{n} G_i = \emptyset$. The contradiction proves the Lemma 3.

PROOF OF THEOREM 2. It is also indirect. Suppose that $H > n \geqq C$ for some $n \in \mathbf{N}$. Then $H > n$ means that there exists $k \geqq n$ and convex sets $K_1, \ldots, K_{k+2} \subset A$ such that any $k+1$ of them have common points; say $a_i \in K_1 \cap \ldots \cap K_{i-1} \cap \cap K_{i+1} \cap \ldots \cap K_{k+2}$ but $\bigcap_{i=1}^{k+2} K_i = \emptyset$. This implies that

(19) $$\bigcap_{i=1}^{k+2} \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+2} \rangle = \emptyset.$$

On the other hand, it follows from $C \leqq n$ that

(20) $$\bigcup_{i=1}^{k+2} \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+2} \rangle = \langle a_1, \ldots, a_{k+2} \rangle.$$

By assumption there exist continuous functions $f_i\colon A \to [0, 1]$ such that

(21) $$(f_i = 0) = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+2} \rangle.$$

Then (19) implies $\sum_{i=1}^{k+2} f_i > 0$ so we may suppose $\sum_{i=1}^{k+2} f_i = 1$. Define the mapping

(22) $$F := (f_1, \ldots, f_{k+2})\colon \langle a_1, \ldots, a_{k+2} \rangle \to \partial\Delta^{k+1};$$

F maps indeed into $\partial\Delta^{k+1}$ by (20) and (21). Let $\varphi\colon \Delta^{k+1} \to \langle a_1, \ldots, a_{k+2} \rangle$ be the mapping from the definition of pseudoconvex spaces. Take the composition: $f := F \circ \varphi\colon \Delta^{k+1} \to \Delta^{k+1}$. It is clear from the construction of F that $F(\langle a_{i_0}, \ldots, a_{i_s} \rangle) \subset \subset (e_{i_0}, \ldots, e_{i_s})$; on the other hand φ satisfies $\varphi((e_{i_0}, \ldots, e_{i_s})) = \langle a_{i_0}, \ldots, a_{i_s} \rangle$. Consequently $f((e_{i_0}, \ldots, e_{i_s})) \subset (e_{i_0}, \ldots, e_{i_s})$, but then Lemma 3 asserts that $f(\Delta^{k+1}) = \Delta^{k+1}$ and this contradicts (22). The Theorem 2 is proved.

3. Since the appearance of von Neumann's classical minimax theorem [11] various generalizations had been published. The numerous results on this topic form already a whole theory and handbooks collect the most important theorems (see e.g. the very nice monograph of N. N. Vorobev [12]). It is well known that the bilinearity and the linear structure appearing in the original theorem of von Neumann [11] can be omitted; in fact topological tools as the Brouwer fixed point theorem and its relatives are enough to prove general minimax theorems (see e.g. [13], [14]). We have developed in [8] a method of proving minimax theorems which we call "the method of level sets" and it was further generalized by L. L. Stachó, A. Sövegjártó, M. Horváth, V. Komornik and others. This method is based only on some simple notions of the set theoretic topology: connectedness and the properties of compact sets. Further contributions to this approach are given by L. L. Stachó [9] and V. Komornik [10]. Finally we mention the works [2], [4], [5] which contain investigations concerning the Ky Fan's minimax theorem. Now we shall prove some new minimax theorems in order to illustrate how the just mentioned method works. Our investigations are motivated by the work [10] of V. Komornik.

For the formulation of our results we need some notions and notations. Let X, Y be arbitrary sets and f: $X\times Y\to\mathbf{R}$ be an arbitrary function. Define the following level sets where $(x\in X,\ y\in Y,\ c\in\mathbf{R})$:

$$H_c^y:=\{x: f(x,y)\geqq c\},\quad \tilde{H}_c^y:=\{x: f(x,y)>c\},$$
$$H_x^c:=\{y: f(x,y)\geqq c\},\quad \tilde{H}_x^c:=\{y: f(x,y)>c\}.$$

Introduce the notations

$$c_*:=\sup_x\inf_y f(x,y),\quad c^*:=\inf_y\sup_x f(x,y).$$

It is well known that

$$c_*\leqq c^*, \tag{23}$$

further it is proved in [15] the

LEMMA 4. a) $c^*=\inf\{c:\ H_x^c\neq\emptyset\ \forall x\in X\}=\inf\{c:\ \tilde{H}_x^c\neq\emptyset\ \forall x\in X\}$

b) $\sup\limits_x\inf\limits_y f(x,y)=\inf\limits_y\sup\limits_x f(x,y)$ iff $\bigcap\limits_{y\in Y}H_y^c\neq\emptyset$ for every $c<c^*$.

DEFINITION 6 ([9]). The pair $X, [.\,,.])$ is called an interval space if X is a topological space and $[.\,,.]$: $X\times X\to P(X)$ is a mapping such that

a) $x_1, x_2\in[x_1, x_2]$ $(x_1, x_2\in X)$,

b) $[x_1, x_2]$ is a connected set $(x_1, x_2\in X)$.

In this case the set $[x_1, x_2]$ is called the segment with endpoints x_1 and x_2.

The terminology shows how to determine a convexity structure on any interval space. Namely, the convex hull of a set $A\subset X$ is defined to be

$$\langle A\rangle:=\bigcup_{n=1}^{\infty}P_n(A),$$

where

$$P_1(A):=\cup\{[x_1, x_2]:\ x_1, x_2\in A\},$$
$$P_n(A):=\cup\{[x_1, x_2]:\ x_1, x_2\in P_{n-1}(A)\}.$$

We remark that in an interval space the segments are not necessarily convex. Now we generalize the classical notion of saddle function.

DEFINITION 7. Let $(X, \langle\cdot\rangle_1)$ and $(Y, \langle\cdot\rangle_2)$ be convexity structures. The function f: $X\times Y\to\mathbf{R}$ is called quasi-saddle function if the sets H_c^y are $\langle\cdot\rangle_1$-concave and the sets H_x^c are $\langle\cdot\rangle_2$-convex for all $c\in\mathbf{R}$.

Observe that H_c^y can be changed to $\tilde{H}_c^y$ and (or) H_x^c to $\tilde{H}_x^c$, the so defined class of quasi-saddle functions remains the same. The following lemma is a generalization of a theorem of L. L. Stachó [9] (the method of the proof goes back to [8], [16]).

LEMMA 5. *Let X and Y be interval spaces (we do not write the interval mappings $[\cdot, \cdot]_{1,2}$). Let $f\colon X\times Y\to\mathbf{R}$ be a function for which*

a) *$Y\setminus\tilde{H}_x^c$ is convex and its intersection with any segment is closed (we mean: closed in the subspace topology of the segment),*

b) *any finite intersection $\bigcap_{i=1}^{n} H_c^{y_i}$ is connected and its intersection with any segment is closed $(y_i\in Y,\ c<c^*)$.*

Then

$$\bigcap_{i=1}^{n} H_c^{y_i}\neq\emptyset \quad (n\geqq 1,\ y_i\in Y,\ c<c^*). \tag{24}$$

PROOF. We shall apply induction on n. The case $n=1$ follows from Lemma 4, a). Suppose indirectly that (24) holds for all $y_1, \dots, y_n\in Y$, $c<c^*$ but there exists $y_1, \dots, y_{n+1}\in Y$ and $c_0<c^*$ such that $\bigcap_{i=1}^{n+1} H_{c_0}^{y_i}=\emptyset$. Define $K:=\bigcap_{i=3}^{n+1} H_{c_0}^{y_i}$ (in case $n=1$ let $K=\emptyset$) and let $K(y):=H_{c_0}^{y}\cap K$. The induction hypothesis means that

$$K(y)\neq\emptyset \quad (y\in Y),\quad K(y_1)\cap K(y_2)=\emptyset. \tag{25}$$

From the assumption a) it follows that $y\in[y_1, y_2]$, $f(x, y_1)<c_0$, $f(x, y_2)<c_0$ imply $f(x, y)<c_0$. In other words

$$H_{c_0}^{y}\subset H_{c_0}^{y_1}\cup H_{c_0}^{y_2} \quad (y\in[y_1, y_2]) \tag{26}$$

and then

$$K(y)\subset K(y_1)\cup K(y_2) \quad (y\in[y_1, y_2]). \tag{27}$$

We assert that

$$K(y)\subset K(y_1) \quad\text{or}\quad K(y)\subset K(y_2) \quad (y\in[y_1, y_2]). \tag{28}$$

Suppose indirectly that there exist $x_1\in K(y)\cap K(y_1)$, $x_2\in K(y)\cap K(y_2)$. Then $[x_1, x_2]\subset K(y)\subset K(y_1)\cup K(y_2)$, $K(y_1)\cap K(y_2)=\emptyset$ gives a partition $[x_1, x_2]= =([x_1, x_2]\cap K(y_1))\cup([x_1, x_2]\cap K(y_2))$ of $[x_1, x_2]$ into disjoint and closed in $[x_1, x_2]$ sets. This contradicts the connectedness of $[x_1, x_2]$ and then (28) is proved. So we can define the partition

$$\begin{aligned} S_1 &:= \{y\in[y_1, y_2]\colon K(y)\subset K(y_1)\},\\ S_2 &:= \{y\in[y_1, y_2]\colon K(y)\subset K(y_2)\}, \end{aligned} \tag{29}$$

of the segment (the sets S_i are nonempty, $y_i\in S_i$). We shall prove that

$$S_1,\ S_2 \text{ are closed (in the topology of } [y_1, y_2]). \tag{30}$$

Indeed, we have by definition

$$y\in S_1 \text{ iff } K(y_2)\cap K(y)=\emptyset \text{ iff } K(y_2)\cap H^y_{c_0}=\emptyset \text{ iff }$$
$$\{x\in K(y_2)\colon f(x,z)\geqq c_0\}=\emptyset.$$

We show that

$$\{x\in K(y_2)\colon f(x,z)\geqq c_0\}=\emptyset \text{ iff } \{x\in K(y_2)\colon f(x,z)>c_0\}=\emptyset. \tag{31}$$

Take a number $c_0<c<c^*$, then by the inductional hypothesis we have

$$K(y)\supset K\cap\{x\colon f(x,y)>c_0\}\supset\{x\colon f(x,y)\geqq c\}\cap\bigcap_{i=3}^{n+1}\{x\colon f(x,y_i)\geqq c\}\neq\emptyset.$$

Consequently

$$K(y_2)\cap(K\cap\{x\colon f(x,y)>c_0\})=\emptyset \text{ implies } K(y_2)\cap K(y)=\emptyset,$$

which proves (31). Using (31) we can write

$$S_1=\{y\in[y_1,y_2]\colon f(x,y)\leqq c_0 \text{ for all } x\in K(y_2)\}=$$
$$=\bigcap_{x\in K(y_2)}\{y\in[y_1,y_2]\colon f(x,y)\leqq c_0\},$$

in other words

$$S_1=\bigcap_{x\in K(y_2)}([y_1,y_2]\cap(Y\setminus\tilde{H}^{c_0}_x)). \tag{32}$$

This means that S_1 is closed in $[y_1, y_2]$. The same assertion related to S_2 can be analogously proved. So we werified (30). But (30) contradicts the connectedness of the segment $[y_1, y_2]$, hence the inductional step works and the finite intersection property (24) holds. The Lemma 5 is proved.

LEMMA 6. *Take the assumptions of Lemma 5 and suppose that the sets H^y_c are closed ($c<c^*$) and one of them, say $H^{y_0}_{c_0}$ is compact. Then* $\sup_x \inf_y f(x,y)=\inf_y \sup_x f(x,y)$.

For the proof we have only to refer to Lemma 4, b). We get immediately

THEOREM 3. *Let X be a compact interval space, Y be an interval space, $f\colon X\times Y\to R$ be a quasi-saddle function such that*

a) *the functions $x\to f(x,y)$ are upper semicontinuous (u.s.c.) for all $y\in Y$,*

b) *the functions $y\to f(x,y)$ are lower semicontinuous (l.s.c.) for all $x\in X$.*

Then $\sup_x \inf_y f(x,y)=\inf_y \sup_x f(x,y)$.

V. Komornik proved in [10] the following theorem (by the method of level sets):

THEOREM (V. Komornik). *Let X be a compact interval space, Y a convex subset of some real topological vector space and $f\colon X\times Y\to\mathbf{R}$ be a quasi-saddle function such that*

a) *the functions* $f(.,y)$ *are u.s.c. on* X *for all* $y\in Y$,
b) *the functions* $f(x,.)$ *are u.s.c. on any interval of* Y *for all* $x\in X$.
Then $\inf_y \sup_x f(x,y)=\sup_x \inf_y f(x,y)$.

In the proof V. Komornik used essentially the linear ordering of the points of the segments of Y. If we take stronger continuity conditions, a result can be given for a larger class of convexities. Namely V. Komornik proved [10]:

THEOREM (V. Komornik). *Let* X *be a compact interval space,* Y *be an interval space,* $f\colon X\times Y\to\mathbf{R}$ *be an u.s.c. quasi-saddle function. Then* $\inf_y \sup_x f(x,y)=$ $=\sup_x \inf_y f(x,y)$.

REMARK. It is probable that the last theorem remains valid if the upper semicontinuity of f is changed to the upper semicontinuity of the partial functions $f(x,.)$ and $f(.,y)$ further f is quasiconcave in y.

If the closed intervals in Y are compact, then this follows easily. E.g. $H^{y_1}\cap H^{y_2}\neq$ $\neq\emptyset$ $(y_1,y_2\in Y)$ in this case. Indeed, suppose $H^{y_1}\cap H^{y_2}=\emptyset$ for some $y_1,y_2\in Y$, then let $[y_1',y_2']\subset[y_1,y_2]$ if $H^{y_1'}\cap H^{y_2'}=\emptyset$, $[y_1',y_2']\subset[y_1,y_2]$ and $H^{y_i'}\subset H^{y_i}$ $(i=1,2)$. Pick a maximal decreasing chain and a downwards cofinal well-ordering $[a_i,b_i]$. Let a be a condensation point of (a_i) and b that of (b_i). Then by compactness $\emptyset\neq\bigcap_i[a_i,b_i]$, this intersection is convex and $(a,b)\subset\bigcap_i[a_i,b_i]$. H^{a_i} is decreasing, $x\in\bigcap_i H^{a_i}$ implies $a_i\in H_x$ hence, because H_x is closed, $a\in H_x$ i.e. $x\in H^a$ i.e. $H^a\subset$ $\subset\bigcap_i H^{a_i}$. Similarly $H^b\subset\bigcap_i H^{b_i}$ i.e. $[a,b]$ is a smallest "wrong" interval. But for any $y\in(a,b)$ we have $H^y\subset H^a$ or $H^y\subset H^b$ and then $[y,b]$ or $[a,y]$ is smaller "wrong" interval than $[a,b]$; a contradiction. (Here $H^y=H_c^y$, $c<c^*$). (We mention that in one-dimensional case when $X=Y=[0,1]$, a quasi-saddle function f whose partial functions are u.s.c., must be globally u.s.c. Indeed, consider the level set $G:=\{(x,y)\colon f(x,y)\geqq c_0\}$. Its horizontal cuts $H_{c_0}^y$ are closed segments and the vertical cuts $Y\setminus H_x^{c_0}$ of $(X\times Y)\setminus G$ are open segments in $[0,1]$; such a set $G\subset[0,1]\times[0,1]$ must be closed indeed).

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LORÁND EÖTVÖS UNIVERSITY
MATHEMATICAL INSTITUTE
DEPARTMENT OF ANALYSIS
1088 BUDAPEST, MÚZEUM KRT. 6—8