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ON THE CONVERGENCE OF EIGENFUNCTION EXPANSION IN THE NORM OF SOBOLEFF SPACES

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1. Let $S_k \subset R^n$ $(n \geqq 3;\ k=1, \dots, l)$ be manifolds of dimension $\dim S_k = m_k \leqq \leqq n-3$ having smooth projection to R^{m_k}, i.e. there exist coordinates $(\xi, y) = =(\xi_1, \dots, \xi_{m_k};\ y_1, \dots, y_{n-m_k})$ and functions $\varphi_j^k \in C^1(R^{m_k} \to R^{n-m_k})$ such that

$$S_k = \{(\xi, y) \in R^n\colon\ y_j = \varphi_j^k(\xi),\ |\nabla \varphi_j^k(\xi)| \leqq C_j^k\}, \quad S = \bigcup_{k=1}^{l} S_k.$$

Let $q \in C^\infty(R^n \setminus S)$ be a real-valued function, for which

$$|D^\alpha q(x)| \leqq C[\operatorname{dist}(x, S)]^{-\tau-|\alpha|}, \quad (x \in R^n,\ 0 \leqq |\alpha| \leqq 1), \tag{1}$$

holds, for some $\tau \geqq 0$.

Consider the Schrödinger operator $L_0 = -\Delta + q(x)\,\cdot$, $D(L_0) = C_0^\infty(R^n)$. Such operators occur as the Hamiltonian of molecules [6—12]. E.g., in the case of Li (or H_2) molecule we have $n=6$, $m=3$, $x \in R^3$, $y \in R^3$, $q(x, y) = c_1|x|^{-1} + c_2|y|^{-1} + +c_3|x-y|^{-1}$, $H = -\Delta + q(x, y)\,\cdot$. In the case of homogeneous and isotropic space the manifolds S_k are subspaces in R^n.

It is easy to see that for $\dim S \leqq n-3$ we have $q \in L^2_{\mathrm{loc}}(R^n)$ if $\tau < 3/2$. Indeed, taking into account

$$l^{-1} \sum_{k=1}^{l} [\operatorname{dist}(x, S_k)]^{-1} \leqq [\operatorname{dist}(x, S)]^{-1} \leqq \sum_{k=1}^{l} [\operatorname{dist}(x, S_k)]^{-1},$$

it is enough to prove this for $S = S_k$, $\dim S = m \leqq n-3$,

$$S = \{(\xi, y) \in R^n\colon\ y_j = \varphi_j(\xi),\ |\nabla \varphi_j(\xi)| \leqq C_j;\ j = 1, \dots, n-m\}.$$

Using the coordinate transformation $(\xi, y) \to (\xi, z)$, $z_j = y_j - \varphi_j(\xi)$ we have for the Jacobian $D(\xi, z)/D(\xi, y) = 1$ and for any $0 \leqq \eta \in C_0^\infty(R^n)$

$$\int_{R^n} |q(x)|^2 \eta(x)\, dx = \int_{R^m} d\xi \int_{R^{n-m}} |q(\xi, z+\varphi(\xi))|^2 \eta(\xi, z+\varphi(\xi))\, dz, \tag{2}$$

where $\varphi = (\varphi_1, \dots, \varphi_{n-m}) \in C^1(R^m \to R^{n-m})$. On the other hand for any $x = (\xi, y) \in R^n$ and $u = (\tilde{\xi}, \varphi(\tilde{\xi})) \in S$ we have

$$|y - \varphi(\xi)| \leqq |y - \varphi(\tilde{\xi})| + |\varphi(\tilde{\xi}) - \varphi(\xi)| \leqq |y - \varphi(\tilde{\xi})| + |\nabla \varphi(\xi^*)| \cdot |\tilde{\xi} - \xi| \leqq$$
$$\leqq C(|y - \varphi(\tilde{\xi})| + |\tilde{\xi} - \xi|),$$

hence

$$|y - \varphi(\xi)|^2 \leqq 2C^2(|y - \varphi(\tilde{\xi})|^2 + |\tilde{\xi} - \xi|^2) = 2C^2|x - u|^2,$$

i.e. $|y-\varphi(\xi)|\leqq C\operatorname{dist}(x,S)$, consequently

$$|q(\xi,z+\varphi(\xi))|\leqq C[\operatorname{dist}((\xi,z+\varphi(\xi)),S)]^{-\tau}\leqq C|z|^{-\tau}.$$

According to (2) we have

$$\int_{R^n}|q(x)|^2\eta(x)\,dx\leqq C\int_{R^m}d\xi\int_{R^{n-m}}|z|^{-2\tau}\eta(\xi,z+\varphi(\xi))\,dz<\infty \tag{3}$$

if $2\tau<n-m$. But we assume in this work that $m\leqq n-3$, i.e. $n-m\geqq 3$ and hence for $\tau<3/2$ we get $2\tau<3\leqq n-m$. It follows from Lemma 3 of the present work that the operator L_0 is bounded below, i.e. $(L_0f,f)=(-\Delta f,f)+(gf,f)=(\nabla f,\nabla f)+(qf,f)\geqq -c(f,f)$ for every $f\in C_0^\infty(R^n)$ and hence, by a theorem of K. O. Friedrichs [3] the operator L_0 has a selfadjoint extension L with $L\geqq -cI$. Denote $L=\int_{-c}^{\infty}\lambda\,dE_\lambda$ the spectral expansion of L and consider for any $f\in L_2(R^n)$ the expansion $E_\lambda f$.

It is proved in [5]: if $\tau=1$ and $0\leqq s\leqq 1$, then $\|E_\lambda f-f\|_{H^s(R^n)}\to 0$ as $\lambda\to\infty$. $H^s(R^n)$ denotes the space of functions from $L_2(R^n)$, with the norm [6, 2.3.3]

$$\|f\|_{H^s(R^n)}:=\|(I-\Delta)^{s/2}f\|_{L_2(R^n)}=\|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L_2(R^n)}.$$

Later on this theorem was extended in [4] for $\tau=1$ and $0\leqq s\leqq 2$. The localization of E_λ was investigated in [8]. Our aim is to prove the following

THEOREM. *Suppose* $\tau\in[0,3/2)$ *and* $0\leqq s\leqq 2$ *or* $\tau\in[0,1/2)$ *and* $0\leqq s<\frac{7}{2}-\tau$. *Then, for any* $f\in H^s(R^n)$ *we have*

$$\|E_\lambda f-f\|_{H^s(R^n)}\to 0\quad\text{as}\quad\lambda\to\infty. \tag{4}$$

It follows from Lemma 3 below — among others — taking into account the Kato—Rellich theorem [11, X.2] that the operator L_0 is essentially selfadjoint, further $D(\bar{L}_0)=D(L)=H^2(R^n)$. Our theorem seems to be true for arbitrary $\tau\in[0,3/2)$ and $0\leqq s<\frac{7}{2}-\tau$ but our Lemma 9 is not enough to prove this. According to the ideas of L. L. Stachó [15] this last result does not seem to be refinable, namely we can not replace $\tau=3/2$ or $s=\frac{7}{2}-\tau$.

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2. For the proof we need some lemmas.

LEMMA 1. *Let* $k\geqq 3$, $1\leqq p<k$, $0\leqq s<k/p$. *Then for any* $f\in L_p^s(R^k)$

$$\||x|^{-s}f(x)\|_{L_p(R^k)}\leqq C\|f\|_{L_p^s(R^k)} \tag{5}$$

holds.

Here and below in this work C is a constant independent of f and not necessarily the same in each occurrences.

PROOF. Using the notation $I:=\| |x|^{-s}f(x)\|^p_{L_p(R^k)}$ we get by Hölder's inequality

$$I=\int_{R^k}|x|^{-sp}|f(x)|^p\,dx \leqq p\int_\theta d\theta\int_0^\infty r^{k-1-sp}\int_r^\infty |f|^{p-1}\left|\frac{\partial f}{\partial t}\right| dt\,dr=$$

$$=\frac{p}{k-sp}\int_{R^k}|x|^{-sp+1}|f|^{p-1}\left|\left(\nabla_x f, \frac{x}{|x|}\right)\right| dx \leqq$$

$$\leqq C\Big(\int_{R^k}(|x|^{-s}|f(x)|)^p\,dx\Big)^{(p-1)/p}\Big(\int_{R^k}|\nabla f|^p|x|^{(-s+1)p}\,dx\Big)^{1/p}=$$

$$=CI^{(p-1)/p}\Big(\int_{R^k}|\nabla f|^p|x|^{(-s+1)p}\,dx\Big)^{1/p},$$

hence

(6) $$\| |x|^{-s}f(x)\|_{L_p(R^k)} \leqq C\| |x|^{(-s+1)}\nabla f(x)\|_{L_p(R^k)}.$$

If s is an integer, then iterating (6) s times we get (5).

Now define

$$s_0:=\begin{cases}\dfrac{k}{p}-1, & \text{when } \dfrac{k}{p} \text{ is an integer}\\ \left[\dfrac{k}{p}\right] & \text{otherwise.}\end{cases}$$

Taking into account Theorem 4.3.2/2 of Triebel [6]:

$$L^s_p(R^k)=\big(L_p(R^k), W^{s_0}_p(R^k)\big),\quad s=\theta s_0,\quad 0<\theta<1;$$

we obtain

(7) $$\| |x|^{-s}f(x)\|_{L_p(R^k)} \leqq C\|f\|_{L^s_p(R^k)}\quad (0\leqq s\leqq s_0,\ p<k/s).$$

Now let $s\in(s_0, k/p)$. It follows from (7) that for $1\leqq p_0<k/s_0$

(8) $$\| |x|^{-s_0}f(x)\|_{L_{p_0}(R^k)} \leqq C\|f\|_{L^{s_0}_{p_0}(R^k)}$$

holds. On the other hand, for any $1\leqq p_1<k/(s_0+1)$ we get from (7)

(9)

$$\| |x|^{-s_0}f(x)\|_{L^1_{p_1}(R^k)}\leqq C[\| |x|^{-s_0}f(x)\|_{L_{p_1}(R^k)}+\| |x|^{-s_0}\nabla f(x)\|_{L_{p_1}(R^k)}+\| |x|^{-s_0-1}f(x)\|_{L_{p_1}(R^k)}]\leqq$$

$$\leqq C[\|f\|_{L^{s_0}_{p_1}(R^k)}+\|f\|_{L^{s_0+1}_{p_1}(R^k)}]\leqq C\|f\|_{L^{s_0+1}_{p_1}(R^k)}.$$

Taking into account $(L_{p_0}, L^1_{p_1})_\delta=L^\delta_p$ $(0<\delta<1,\ p^{-1}=(1-\delta)p_0^{-1}+\delta p_1^{-1})$ (cf. Triebel [6], 2.4.2/1) we obtain from (8) and (9) the estimate

(10) $$\| |x|^{-s_0}f(x)\|_{L^\delta_p(R^k)} \leqq C\|f\|_{L^{s_0+\delta}_p(R^k)}\quad (\forall\, 0<\delta<1).$$

Now, using (10) we prove (5) for $s_0<s<k/p$. Define $\delta=s-s_0$. It is easy to see that $\delta\in(0,1)$. Indeed, if k/p is an integer, then $\delta=s-s_0=\left(\frac{k}{p}-\varepsilon\right)-\left(\frac{k}{p}-1\right)=1-\varepsilon$

$\left(s=\frac{k}{p}-\varepsilon,\ 0<\varepsilon<1\right)$. If k/p is not an integer, then $\delta=\left(\frac{k}{p}-\varepsilon\right)-\left[\frac{k}{p}\right]<1-\varepsilon$. Consequently, from (10) we get

$$\| |x|^{-s} f(x)\|_{L_p(R^k)} = \left\| |x|^{-\delta}(|x|^{-s_0} f(x))\right\|_{L_p(R^k)} \leqq$$
$$\leqq C \| |x|^{-s_0} f(x)\|_{L_p^{\delta}(R^k)} \leqq C\|f\|_{L_p^s(R^k)}.$$

Lemma 1 is proved.

LEMMA 2. *For any natural number* $k\geqq 3$, $0\leqq s<3/2$ *and* $f\in C_0^\infty(R^k)$

(11) $$\| |x|^{-s} f(x)\|^2_{L_2(R^k)} \leqq C\|f\|_{H^1(R^k)}\|f\|_{H^2(R^k)}.$$

PROOF. First we prove (11) for $s\geqq 1$. Using (6) at $p=2$ and taking into account the inequality $|x|^{-2s+2}\leqq |x|^{-1}+1$ $(0\leqq 2s-2\leqq 1)$ we get

$$\| |x|^{-s} f(x)\|^2_{L_2(R^k)} \leqq C\| |x|^{-s+1}\nabla f(x)\|^2_{L_2(R^k)} \leqq$$
$$\leqq C[\| |x|^{-1/2}\nabla f(x)\|^2_{L_2(R^k)}+\|\nabla f(x)\|^2_{L_2(R^k)}].$$

Hence, taking into account the following estimate (cf. [4, Lemma 1])

(12) $$\| |x|^{-1/2} f(x)\|^2_{L_2(R^k)} \leqq C\|f\|_{H^1(R^k)}\|f\|_{L_2(R^k)} \quad (k\geqq 3, f\in C_0^\infty(R^k))$$

we obtain (11) for the case $1\leqq s<3/2$. If $0\leqq s\leqq 1$, then (11) follows from (5) immediately. Lemma 2 is proved.

LEMMA 3. *For any* $\tau\in[0, 3/2)$ *and* $\varepsilon>0$ *there exists* $C(\varepsilon)>0$ *such that for every* $f\in C_0^\infty(R^n)$ $(n\geqq 3)$ *the following estimate holds:*

(13) $$\|qf\|^2_{L_2(R^n)} \leqq \varepsilon\|f\|^2_{H^2(R^n)}+C(\varepsilon)\|f\|^2_{L_2(R^n)}.$$

PROOF. Using (3) for $\eta=|f|^2$, applying (11) for $k=n-m$ and taking into account the inequality

(14) $$ab \leqq \varepsilon a^2+\frac{1}{4\varepsilon}b^2 \quad (a, b, \varepsilon>0),$$

we get

$$\|qf\|^2_{L_2(R^n)} \leqq C\|f\|_{H^1(R^n)}\|f\|_{H^2(R^n)} \leqq \frac{\varepsilon}{2}\|f\|^2_{H^2(R^n)}+C(\varepsilon)\|f\|^2_{H^1(R^n)}.$$

Hence, taking into consideration the estimate

$$\|f\|^2_{H^1(R^n)} \leqq \varepsilon_1\|f\|^2_{H^2(R^n)}+C(\varepsilon_1)\|f\|^2_{L_2(R^n)},$$

we obtain

$$\|qf\|^2_{L_2(R^n)} \leqq \frac{\varepsilon}{2}\|f\|^2_{H^2(R^n)}+\varepsilon_1 C(\varepsilon)\|f\|^2_{H^2(R^n)}+C(\varepsilon_1)C(\varepsilon)\|f\|^2_{L_2(R^n)}.$$

If we choose ε_1 so that $\varepsilon_1 C(\varepsilon)<1/2$, then (13) follows. Lemma 3 is proved.

COROLLARY. *For any* $\tau\in[0, 3/2)$ *the operator* L_0 *is essentially selfadjoint and* $D(\bar{L}_0)=D(L)=H^2(R^n)$.

PROOF. From (13) we obtain for any $\varepsilon>0$ the estimate

$$\|qf\|_{L_2(R^n)} \leqq \varepsilon\|(I-\Delta)f\|_{L_2(R^n)}+C(\varepsilon)\|f\|_{L_2(R^n)}. \tag{15}$$

Since $I-\Delta$ is essentially selfadjoint and $D(\overline{I-\Delta})=H^2(R^n)$, the Corollary follows by Kato—Rellich's theorem [11, X.2].

REMARK. For the essential selfadjointness of L_0 it is enough to prove the estimate

$$\|qf\|_{L_2(R^n)} \leqq C\|f\|_{H^2(R^n)}\|f\|_{H^{2-\delta}(R^n)},$$

for some $\delta>0$, because

$$\|qf\|_{L_2(R^n)} \leqq \varepsilon\|f\|_{H^2(R^n)}+C(\varepsilon)\|f\|_{H^{2-\delta}(R^n)} \leqq$$
$$\leqq \varepsilon\|f\|_{H^2(R^n)}+\varepsilon_1 C(\varepsilon)\|f\|_{H^2(R^n)}+C(\varepsilon_1)C(\varepsilon)\|f\|_{L_2(R^n)}.$$

LEMMA 4. *For any* $f\in H^2(R^n)$

$$\|Lf\|_{L_2(R^n)} \leqq C\|f\|_{H^2(R^n)}. \tag{16}$$

PROOF. Using (13) we obtain for any $f\in H^2(R^n)$

$$\|Lf\|_{L_2(R^n)} = \|-\Delta f+qf\|_{L_2(R^n)} \leqq \|\Delta f\|_{L_2(R^n)}+\|qf\|_{L_2(R^n)} \leqq$$
$$\leqq C[\|f\|_{H^2(R^n)}+\|f\|_{L_2(R^n)}] \leqq C\|f\|_{H^2(R^n)}.$$

Lemma 4 is proved.

LEMMA 5. *There exist constants* $C_1>0$ *and* $C_2>0$ *such that for every* $f\in H^2(R^n)$

$$\|Lf\|^2_{L_2(R^n)} \geqq C_1\|f\|^2_{H^2(R^n)}-C_2\|f\|^2_{L_2(R^n)}. \tag{17}$$

PROOF. Using (14), applying the Cauchy—Bunyakovsky inequality, further taking into account the identity

$$\|Lf\|^2_{L_2(R^n)} = \|\Delta f\|^2_{L_2(R^n)}-2(qf, \Delta f)+\|qf\|^2_{L_2(R^n)},$$

we obtain

$$|(qf, \Delta f)| \leqq \|qf\|_{L_2(R^n)}\|\Delta f\|_{L_2(R^n)} \leqq \varepsilon\|\Delta f\|^2_{L_2(R^n)}+C(\varepsilon)\|qf\|^2_{L_2(R^n)}$$

and

$$\|Lf\|^2_{L_2(R^n)} \geqq \|\Delta f\|^2_{L_2(R^n)}-2|(qf, \Delta f)|+\|qf\|^2_{L_2(R^n)} \geqq$$
$$\geqq \|\Delta f\|^2_{L_2(R^n)}-\varepsilon\|\Delta f\|^2_{L_2(R^n)}-C(\varepsilon)\|qf\|^2_{L_2(R^n)} \geqq$$
$$\geqq (1-\varepsilon)\|\Delta f\|^2_{L_2(R^n)}-C(\varepsilon)\|qf\|^2_{L_2(R^n)}.$$

Now applying (13) for some $\varepsilon_1>0$, it follows

$$\|Lf\|^2_{L_2(R^n)} \geqq (1-\varepsilon)\|\Delta f\|^2_{L_2(R^n)}-\varepsilon_1 C(\varepsilon)\|f\|^2_{H^2(R^n)}-C(\varepsilon, \varepsilon_1)\|f\|^2_{L_2(R^n)}.$$

On the other hand

$$\|\Delta f\|_{L_2(R^n)} = \|\Delta f-f+f\|_{L_2(R^n)} \geqq \|(\Delta-I)f\|_{L_2(R^n)}-\|f\|_{L_2(R^n)}$$

consequently

$$\|Lf\|^2_{L_2(R^n)} \geqq \big(1-\varepsilon-\varepsilon_1 C(\varepsilon)\big)\|f\|^2_{H^2(R^n)}-C(\varepsilon, \varepsilon_1)\|f\|^2_{L_2(R^n)}$$

and hence (17) follows if we set $\varepsilon=1/2$ and ε_1 is small enough. Lemma 5 is proved.

LEMMA 6. *There exists* $\mu_0>0$ *such that for any* $\mu\geqq\mu_0$ *and* $f\in C_0^\infty(R^n)$ *we have*

$$\|L_\mu f\|_{L_2(R^n)}\geqq C_\mu\|f\|_{H^2(R^n)}\quad (L_\mu:=L+\mu I). \tag{18}$$

The constant C_μ *does not depend on* f.

PROOF. It follows from (17) using the spectral theorem that

$$\|f\|^2_{H^2(R^n)}\leqq C_1\|Lf\|^2_{L_2(R^n)}+C_2\|f\|^2_{L_2(R^n)}\leqq$$

$$\leqq C\int_{-C_0}^{\infty}(\lambda^2+1)\,d(E_\lambda f,f)\leqq C\int_{-C_0}^{\infty}(\lambda+\mu)^2\,d(E_\lambda f,f)=C\|L_\mu f\|^2_{L_2(R^n)},$$

if $\mu\geqq\mu_0$ and μ_0 is large enough, because in this case we have $\lambda^2+1\leqq(\lambda+\mu)^2$ $(\lambda\geqq-C_0,\ \mu\geqq\mu_0)$. Lemma 6 is proved.

LEMMA 7 [4, Lemma 6]. *Let A and B be strongly positive selfadjoint operators in the Hilbert space H. Suppose that the conditions*

$$D(B)\subset D(A), \tag{19}$$

$$\|Af\|_H\leqq C\|Bf\|_H\quad (f\in D(B)), \tag{20}$$

are fulfilled. Then for any $\theta\in[0,1]$ *we have*

$$\|A^\theta f\|_H\leqq C_\theta\|B^\theta f\|_H\quad (f\in D(B)). \tag{21}$$

LEMMA 8. *For any* $\mu\geqq\mu_0$, $s\in\left[0,\frac{7}{2}-\tau\right)$ *and* $f\in H^s(R^n)$

$$\|L_\mu^{s/2}f\|_{L_2(R^n)}\leqq C_s\|f\|_{H^s(R^n)}. \tag{22}$$

PROOF. First we prove (22) for $0\leqq s\leqq 2$. It is trivial for $s=0$ and it was proved in Lemma 4 for $s=2$. Now apply Lemma 7 for $A=L_\mu$, $B=I-\Delta$, $D(B)=H^2(R^n)$. We obtain:

$$\|L_\mu^\theta f\|_{L_2(R^n)}\leqq C\|f\|_{H^{2\theta}(R^n)}\quad (0\leqq\theta\leqq 1). \tag{23}$$

Now let $2<s<\frac{7}{2}-\tau$. Using Lemma 1 we obtain for any $p_0<3/\tau$ the estimate

$$\|L_\mu f\|_{L_{p_0}(R^n)}\leqq C[\|f\|_{L_{p_0}(R^n)}+\|f\|_{L^2_{p_0}(R^n)}+\|qf\|_{L_{p_0}(R^n)}]\leqq \tag{24}$$

$$\leqq C[\|f\|_{L^2_{p_0}(R^n)}+\|f\|_{L^\tau_{p_0}(R^n)}]\leqq C\|f\|_{L^2_{p_0}(R^n)}.$$

On the other hand, using Lemma 1 once again, we obtain for any $p_1<3/(\tau+1)$ and $f\in L^3_{p_1}(R^n)$ the estimate

$$\|\nabla L_\mu f\|_{L_{p_1}(R^n)}\leqq C[\|f\|_{L^3_{p_1}(R^n)}+\|(\nabla q)f\|_{L_{p_1}(R^n)}+ \tag{25}$$

$$+\|q\nabla f\|_{L_{p_1}(R^n)}]\leqq C[\|f\|_{L^3_{p_1}(R^n)}+\|f\|_{L^{1+\tau}_{p_1}(R^n)}]\leqq C\|f\|_{L^3_{p_1}(R^n)}.$$

Using (24), (25), the equality $(L_{p_0}, L^1_{p_1})_\delta = L^\delta_p$ $(0<\delta<1,\ p^{-1}=(1-\delta)p_0^{-1}+\delta p_1^{-1})$ of Triebel [6, 2.4.2/1] and taking into account that in our case $p<3/(\tau+\delta)$, we obtain for any $\delta\in(0,1)$ and $f\in L^{2+\delta}_p(R^n)$ the estimate

$$\|L_\mu f\|_{L^\delta_p(R^n)} \leqq C\|f\|_{L^{2+\delta}_p(R^n)}. \tag{26}$$

Now we are in the position to prove (22) for $2<s<\frac{7}{2}-\tau$. Set $\delta:=s-2$. Then $\delta<\frac{7}{2}-\tau-2<\frac{3}{2}$, further we obtain from (26) that for any $f\in H^s(R^n)$ we have $L_\mu f\in H^\delta(R^n)$. Using (23) and then (26) we obtain

$$\|L_\mu^{s/2}f\|_{L_2(R^n)} = \|L_\mu^{\delta/2}(L_\mu f)\|_{L_2(R^n)} \leqq C\|L_\mu f\|_{L^\delta_2(R^n)} \leqq C\|f\|_{L^{2+\delta}_2(R^n)} = C\|f\|_{H^s(R^n)}.$$

Lemma 8 is proved.

LEMMA 9. *Suppose* $0\leqq s\leqq 2$, $0\leqq\tau<3/2$ *or* $0\leqq\tau<1/2$ *and* $0\leqq s<\frac{7}{2}-\tau$. *Then for any* $\mu\geqq\mu_0$ *and* $g\in H^s(R^n)$ *we have*

$$\|g\|_{H^s(R^n)} \leqq C\|L_\mu^{s/2}g\|_{L_2(R^n)}. \tag{27}$$

PROOF. (27) is trivial for $s=0$ and it was proved in Lemma 6 for $s=2$. Hence, using Lemma 7 for $B=L_\mu$, $A=I-\Delta$, $D(A)=H^2(R^n)$, we obtain

$$\|g\|_{H^s(R^n)} \leqq C\|L_\mu^{s/2}g\|_{L_2(R^n)} \quad (0\leqq s\leqq 2,\ 0\leqq\tau<7/2-\tau). \tag{28}$$

Now suppose $0\leqq\tau<1/2$ and $2<s<\frac{7}{2}-\tau$. Let $\delta:=s-2$. For any $g\in C_0^\infty(R^n)$ we have obviously by (28)

$$\begin{aligned}\|g\|_{H^s(R^n)} &= \|(I-\Delta)g\|_{H^\delta(R^n)} \leqq C\|L_\mu^{\delta/2}(I-\Delta)g\|_{L_2(R^n)} \leqq \\ &\leqq C[\|L_\mu^{\delta/2}g\|_{L_2(R^n)}+\|L_\mu^{\delta/2}(L_\mu-q)g\|_{L_2(R^n)}] \leqq C[\|L_\mu^{-1}(L_\mu^{s/2}g)\|_{L_2(R^n)}+\|L_\mu^{\delta/2}(qg)\|_{L_2(R^n)}+ \\ &+\|L_\mu^{s/2}g\|_{L_2(R^n)}] \leqq C[\|L_\mu^{s/2}g\|_{L_2(R^n)}+\|L_\mu^{\delta/2}(qg)\|_{L_2(R^n)}].\end{aligned} \tag{29}$$

Now we estimate $\|L_\mu^{\delta/2}(qg)\|_{L_2}$. We obtain from (3) and (5)

$$\|qg\|_{L_2(R^n)} \leqq C\|g\|_{H^\tau(R^n)} \quad (0\leqq\tau<3/2) \tag{30}$$

and

$$\|gq\|_{H^1(R^n)} \leqq \|q\nabla g\|_{L_2}+\|\nabla qg\|_{L_2}+\|gq\|_{L_2} \leqq C\|g\|_{H^{\tau+1}(R^n)} \quad (0\leqq\tau<1/2). \tag{31}$$

We apply the interpolation theorem of Stein [13]. To this suppose δ is such that $\tau+\delta<3/2$ and choose $\varepsilon>0$ so that $\tau(\delta)=\tau$, where $\tau(z):=z(0{,}5-\varepsilon)+(1{,}5-\varepsilon)(1-z)$. Define the operators A_z and T_z as follows:

$$A_z g := |q(x)|^{\tau(z)/\tau}(\operatorname{sgn} q(x))g(x), \quad T_z g := (I-\Delta)^{z/2}A_z g.$$

From (30) and (31) we obtain for any $g\in C_0^\infty(R^n)$:

$$\|T_z g\|_{L_2(R^n)} = \|A_z g\|_{L_2(R^n)} \leqq C\|g\|_{H^{3/2-\varepsilon}(R^n)} \quad (\operatorname{Re} z=0),$$

and

$$\|T_z g\|_{L_2(R^n)} \leqq \|A_z g\|_{H^1(R^n)} \leqq C\|g\|_{H^{3/2-\varepsilon}(R^n)} \quad (\operatorname{Re} z=1),$$

hence by Stein's interpolation theorem [13] we get for $z=\delta$:

$$\|T_\delta g\|_{L_2} \leqq C\|g\|_{H^{3/2-\varepsilon}}, \quad \|A_\delta g\|_{L_2} \leqq C\|g\|_{H^{3/2-\varepsilon}},$$

i.e. using also (14) we obtain

$$\|qg\|_{H^\delta(R^n)} \leqq \varepsilon\|g\|_{H^2(R^n)} + C(\varepsilon)\|g\|_{L_2(R^n)}.$$

Hence and from (29) the desired estimate (27) follows. Lemma 9 is proved.

PROOF OF THE THEOREM. Using (22) and (27) we obtain for $f\in H^s(R^n)$:

$$\|f-E_\lambda f\|_{H^s} = \|L_\mu^{-s/2}L_\mu^{s/2}(I-E_\lambda)f\|_{H^s} \leqq$$
$$\leqq C\|L_\mu^{s/2}(I-E_\lambda)f\|_{L_2} = C\|(I-E_\lambda)(L_\mu^{s/2}f)\|_{L_2} \to 0 \quad (\lambda\to\infty).$$

The Theorem is proved.

REMARK. If the S_k's are subspaces, then we can state the Theorem for any $\tau\in[0, 3/2)$ and $s\in\left[0, \frac{7}{2}-\tau\right)$ because in this case we can prove Lemma 9 in a more general form. This follows from the following fact: if $g(z)\in C_0^\infty(R^k\setminus\{0\})$ is a function for which $|D^\alpha g(z)|\leqq C|z|^{-\tau-|\alpha|}$ $(z\neq 0)$ holds, then $g\in H^s(R^n)$ for any $s<\frac{k}{2}-$ $-\tau=:\delta$. For the proof of this fact it is enough to show that $g\in H_1^{k-\tau}(R^k)$ (here H denotes the Nikol'skiĭ's class of functions), because taking into account the well known imbeddings $H_1^{k-\tau}\subset H_2^\delta\subset B_{2,2}^{\delta-\varepsilon}\subset L_2^{\delta-\varepsilon}$ our statement follows. We use here the notations of [14]. For the proof we must show the estimate

$$I := \omega_2^{(2)}(D^\alpha g, t) := \sup_{|h|\leqq t}\int_\Omega |\Delta_h^2 D^\alpha g|\,dz = O(t^{s-|\alpha|}) \quad (\operatorname{supp} g\subset\Omega).$$

The desired estimate follows immediately for $|z|<2h$ and $|z|\geqq 2h$, resp. from the following estimates:

$$\sup_{|h|\leqq t}\int_{\Omega'}|\Delta_h^2 D^\alpha g(z)|\,dz \leqq 4\sup_{|h|\leqq t}\int_{\Omega'}|D^\alpha g(z)|\,dz \leqq$$
$$\leqq C\sup_{|h|\leqq t}\int_0^{2|h|}|z|^{-\tau-|\alpha|+k-1}\,dz = \sup_{|h|\leqq t} O(|h|^{k-\tau-|\alpha|}) =$$
$$= O(t^{s-|\alpha|}), \quad \Omega' := \{z\in\Omega:\ |z|<2|h|\};$$
$$|\Delta_h^2 D^\alpha g| = \left|\sum_{i,j=1}^k \frac{\partial^2}{\partial z_i\partial z_j}(D^\alpha g)(z^*)h_ih_j\right| \leqq$$
$$\leqq C\sum_{|\beta|=2}(D^{\alpha+\beta}g)(z^*)|h|^2, \quad z^*\in[z-h, z+h],$$

hence

$$I \leqq \sup_{|h|\leqq t}|h|^2\sum_{|\beta|=2}\int_{\Omega''}|D^{\alpha+\beta}g|\,dz + O(t^{s-|\alpha|}) \leqq$$
$$\leqq \sup_{|h|\leqq t}\int_{2|h|}^A |z|^{-\tau-|\alpha|-2+k-1}\,dz + O(t^{s-|\alpha|}) = O(t^{s-|\alpha|}), \quad \Omega'' := \{z\in\Omega;\ |z|\geqq 2|h|\}.$$

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